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 On function spaces

The paper consists of two parts. In the first one we consider the mappings $f: J \times Y \rightarrow Z$ and $\varphi f: J \rightarrow E(X, Y)$, where $E(X, Y)$ is a family of mappings on X to Y , and $[\varphi f(t)](y) = f(t, y)$, $t \in J$, $y \in Y$ (φ - the exponential mapping). If f or φf is continuous, then we are interested in some information about the continuity of \hat{f} or $\varphi \hat{f}$, where $\hat{f}: J \times E(X, Y) \rightarrow E(X, Z)$ is defined by $\hat{f}(t, g)(x) = f(t, g(x))$, $t \in J$, $g \in E(X, Y)$, $x \in X$. The topology on the function spaces has a subbase consisted of the sets $M(A, U) = \{f \in E(X, Y) : f(A) \subset U\}$ or is induced by the quasi-uniformity with the subbase sets $W(A, U) = \{(f, g) \in E(X, Y) \times E(X, Y) : (f(x), g(x)) \in U \text{ for every } x \in A\}$ if Y is a quasi-uniform space. The results for $\varphi \hat{f}$, connected with the compact open topology are due to Lemma 1.10 which enables us to omit the problems of the structure of compact sets in function spaces. A similar situation is for the quasi-uniform spaces (Th. 1.22, Th. 1.23, Lemma 1.19). The results for \hat{f} are Theorems 1.27, 1.30, 1.32. The second part is devoted to the dependence between the compact open topology and the topology of uniform convergence on compacta. On the other hand, the structure of compact (or precompact) sets in function spaces is considered.

1. CONTINUITY

1.1. Definition (cp. [6, p. 6])

A quasi-uniformity (q-uniformity) on a set Y is the subfamily \mathcal{U} of subsets of $Y \times Y$, which satisfies the following conditions

- (Q1) $U \in \mathcal{U}$, $U \subset V$ implies $V \in \mathcal{U}$,
 (Q2) $U, V \in \mathcal{U}$ implies $U \cap V \in \mathcal{U}$,
 (Q3) for every $V \in \mathcal{U}$ there exists a $U \in \mathcal{U}$ such that $U^2 := U \circ U \subset V$,
 (Q4) $\Delta \subset \bigcap \mathcal{U}$.

If \mathcal{U} consists of symmetric sets and the above conditions hold for every symmetric $V \subset Y \times Y$, then \mathcal{U} is a uniformity. A set Y equipped with a q-uniformity (uniformity) \mathcal{U} will be called a q-space (uniform space) and it will be denoted by (Y, \mathcal{U}) (or Y if \mathcal{U} is understood). A q-space is uniformizable if (Y, \mathcal{V}) is uniform for a base \mathcal{V} of \mathcal{U} . By saying that a q-space (Y, \mathcal{U}) has a topological property we will mean that the topological space $(Y, \tau_{\mathcal{U}})$, where $\tau_{\mathcal{U}}$ is the topology induced by the q-uniformity \mathcal{U} , has this property.

For the continuity problems the above definition of uniformity is as good as the general one [1, II p.1] because every uniformity in the general sense has a base consisting of open symmetric entourages [4, Th. 6 p. 179].

We adopt Kelley's terminology concerning the separation axioms and the notion of compactness.

1.2. Proposition

For any uniform space (Y, \mathcal{U}) the following conditions are equivalent:

- (i) Y is a T_0 -space,
 (ii) Y is a $T_{3\frac{1}{2}}$ -space,
 (III) $\bigcap \mathcal{U} = \Delta^2$

Proof

The topological space $(Y, \tau_{\mathcal{U}})$ is completely regular [4, Corol. 17 p. 188]. The equivalence of Y being a T_2 -space and (iii) follows from [1, Prop. 3 II p. 5]. So it is enough to show that (i) implies (iii). Let $x, y \in Y$ and $y \notin V(x)$ for a $V \in \mathcal{U}$. Then, for U as in (Q3), we have $(x, y) \notin U^2$, $y \notin U^2(x)$ and $U(x) \cap U(y) = \emptyset$ U being symmetric, hence Y is a T_2 -space.

The above known property (see [4]) does not hold in general for q-spaces, as every topological space (Y, τ) is q-uniformizable, i.e. there exists a q-uniformity \mathcal{U} on Y which is compatible with τ (i.e. $\tau = \tau_{\mathcal{U}}$). The members of a subbase of such a q-uniformity (the Pervin q-uniformity [7]) are written in the form $U_G = (G \times G) \cup [(Y \setminus G) \times Y]$, $G \in \tau$.

The family of all mappings on X to Y will be denoted by $F(X, Y)$; $E(X, Y)$ is a subset of $F(X, Y)$. Let 2^X mean the family of all nonempty subsets of X . If $\emptyset \neq \delta \subset 2^X$ and X, Y are topological spaces (q-spaces), then C^δ (UC^δ) consists of all mappings $f: X \rightarrow Y$ continuous (uniformly continuous) on δ , i.e. such that $f|_A: A \rightarrow Y$ is continuous (uniformly continuous) on δ . The continuous mappings will be called maps.

Let $M(A, U) = \{f \in F(X, Y) : f(A) \subset U\}$ for arbitrary $A \subset X$, $U \subset Y$. If $\emptyset \neq \delta \subset 2^X$ and (Y, τ) is a topological space, then $\{M(A, U) \cap E(X, Y) : A \in \delta, U \in \tau\}$ is a subbase of a topology in $E(X, Y)$ (see [4, Th. 12 p. 47]). This topology will be denoted by $\tau|\delta$ (more precisely $(\tau|\delta)|E(X, Y)$) and $E_\delta(X, Y) = (E(X, Y), \tau|\delta)$. In what follows we write $M(A, U)$ in place of $M(A, U) \cap E(X, Y)$ if $E(X, Y)$ is understood.

Let $W(A, U) = \{(f, g) \in F(X, Y) \times F(X, Y) : (f(x), g(x)) \in U \text{ for every } x \in A\}$ for arbitrary $A \subset X$, $U \subset Y \times Y$. If $\emptyset \neq \delta \subset 2^X$ and (Y, \mathcal{U}) is a q-space, then $\{W(A, U) \cap E(X, Y) \times E(X, Y) : A \in \delta, U \in \mathcal{U}\}$ is a subbase of a q-uniformity [6, Th. Def. p. 69]. This q-uniformity will be denoted by $\mathcal{U}|\delta$ (more precisely $(\mathcal{U}|\delta)|E(X, Y)$) and $E_\delta(X, Y) = (E(X, Y), \mathcal{U}|\delta)$. If $E(X, Y)$ is fixed, then we write $W(A, U)$ in place of $W(A, U) \cap E(X, Y) \times E(X, Y)$. The family δ as above can be completed without changing $\mathcal{U}|\delta$ so as to satisfy the following conditions [2, p. 176]:

(F_I) every subset of any set $A \in \delta$ belongs to δ ,

(F_{II}) every finite sum of members of δ belongs to δ .

It must be stressed that $E_\delta(X, Y)$ is a topological space if Y is a topological space and $E_\delta(X, Y)$ is a q-space if Y is a q-space, i.e. the same symbol has two meanings.

The family of all points of a fixed set will be denoted by p . The family of all compact (relatively compact) subsets of a fixed topological space will be denoted by c (rc).

Similarly, pc is the family of all precompact subsets of the q-space under consideration. Besides, the following notations will be applied: $E[\delta, Y] = \{f(A) : f \in E(X, Y), A \in \delta \subset 2^X\}$, $\mathcal{P}[\delta] = \{\mathcal{A} \in \mathcal{P}c^2^{E(X, Y)}, A \in \delta \subset 2^X\}$.

Remark 1.3

For an arbitrary topological space (or q-space) X and any q-space (Y, \mathcal{U}) we may define $\tau_c := \tau_{\mathcal{U}|c}$ (the compact open topology) and $\tau_q := \tau_{\mathcal{U}|c}$ (the topology of uniform convergence on compacta). We have $\tau_c \subset \tau_q$ for $E(X, Y) \subset C^c(X, Y)$ (see [4, Th. 5 p. 223, Th. 10(e) p. 229]) and $(C^c(X, Y), \tau_c) = (C^c(X, Y), \tau_q)$ if (Y, \mathcal{U}) is a uniform space (cp. [4, Th. 11 p. 230]).

It is clear that the way the topology was defined on X , has no meaning for the dependence between $\tau_{\mathcal{U}|\delta}$ and $\tau_{\mathcal{U}|\delta}$ in $C^\delta(X, Y)$. Therefore we assume that X is a topological space (not a q-space) only for the sake of brevity.

Proposition 1.4

Let $X, (Y, \mathcal{C})$ be a set and a topological space respectively. Then $\tau|\delta = \tau_{\mathcal{U}|\delta} \subset \tau_{\mathcal{U}|\delta}$ on $E(X, Y)$ for the Pervin q-uniformity \mathcal{U} on Y and any $\delta \subset 2^X$.

Proof

Let $f \in M(A, G)$ for fixed $A \in \delta, G \in \mathcal{C}$. Then we have $(f(x), g(x)) \in (G \times G) \cap [(Y \setminus G) \times Y], x \in A$ for any $g \in W(A, U_G)(f)$ and therefore $g(A) \subset G$. It means $W(A, U_G)(f) \subset M(A, G)$ and $\tau_{\mathcal{U}|\delta} \subset \tau_{\mathcal{U}|\delta}$.

We have $\tau|\delta = \tau_{\mathcal{U}|\delta}$ in $E(X, Y)$ if $\tau|\delta$ is discrete, i.e. if for every $f \in E(X, Y)$ there exist $n \in \mathbb{N}, A_1, \dots, A_n \in \delta$ and $H_1, \dots, H_n \in \mathcal{C}$ such that $\{f\} = \bigcap \{M(A_i, H_i) : i=1, \dots, n\}$. Simple examples show that even for $\delta = p$ $M(\{x\}, H) \not\subset W(\{x\}, U_G)(f)$ can be satisfied for all $H \neq \emptyset, H \in \mathcal{C}$ if $f(x) \in \text{Fr } G$ for a $G \in \mathcal{C}$.

Remark 1.4

The previous considerations explain why the continuity problems in $E_\delta(X, Y)$ usually must be treated separately for Y being a topological space or a q-space as we consider $\tau_{\mathcal{U}|\delta}$ in the first case, and $\tau_{\mathcal{U}|\delta}$ in the second one.

If $f: J \times Y \rightarrow Z$ is a mapping, then $\varphi f: J \rightarrow F(Y, Z)$ is defined as follows $[\varphi f(t)](y) = f(t, y), t \in J, y \in Y$ (φ - the exponential mapping); $\hat{f}: J \times F(X, Y) \rightarrow F(X, Z)$ is defined by $\hat{f}(t, g)(x) = f(t, g(x)), x \in X, t \in J, g \in F(X, Y)$.

Proposition 1.5

Let X, Y, Z be topological spaces. If the following conditions are satisfied

$$\beta \subset 2^X, E[\beta, Y] \subset \mathcal{K} \quad (1)$$

$E(X, Y) \subset C^\beta(X, Y)$, $f \in C^\alpha(Y, Z)$, then $\hat{f}: E(X, Y) \rightarrow C^\beta(X, Z)$; if in addition $E(X, Y) \subset C(X, Y)$ and $f \in C(Y, Z)$, then $f: E(X, Y) \rightarrow C(X, Z)$.

Proof

If $g \in E(X, Y)$, then for any $A \subset \beta$ $g|_A$ is continuous, $g(A) \in E[\beta, Y]$ and therefore $f|_{g(A)}$ is continuous, i.e. $\hat{f}(g) \in C^\beta(X, Z)$.

Lemma 1.6

Let X be a set and let Y, Z be topological spaces for which

$$\delta \subset 2^X, \gamma \subset 2^{E(X, Y)}, \mathcal{K}, \mathcal{G} \subset 2^Y \text{ and } \gamma[\delta] \subset \mathcal{K} \cap \mathcal{G} \quad (2)$$

holds. If $f \in C^\alpha(Y, Z)$, then $\hat{f} \in C^\alpha(E_\delta(X, Y), F_\mathcal{G}(X, Z))$; if f is continuous, then \hat{f} is continuous.

Proof

If $A \in \delta$, $\mathcal{K} \in \gamma$ and $V \subset Z$ is open, then $\hat{f}^{-1}(M(A, V)) \cap \mathcal{K} = \{g \in \mathcal{K}: g(A) \subset f^{-1}(V)\} = M(A, \mathcal{K}(A) \cap f^{-1}(V)) \cap \mathcal{K} = M(A, \mathcal{K}(A) \cap U) \cap \mathcal{K}$ for an open set $U \subset Y$, as $f|_{\mathcal{K}(A)}$ is continuous. Hence $\hat{f}^{-1}(M(A, V)) \cap \mathcal{K} = M(A, U) \cap \mathcal{K}$ which proves the continuity of \hat{f} on $\mathcal{K} \in \gamma$. If f is continuous, then $f^{-1}(V)$ is open, $\hat{f}^{-1}(M(A, V)) = M(A, f^{-1}(V))$ and therefore \hat{f} is continuous.

Lemma 1.7

Let X, Y be sets and let J, Z be topological spaces. Assume (2) holds and $\varphi: J \rightarrow F_\mathcal{G}(Y, Z)$ is continuous. Then $\varphi \hat{f}: J \rightarrow F_\mathcal{G}(E(X, Y), F_\mathcal{G}(X, Z))$ is continuous and for each $\mathcal{K} \in \gamma$ $\{\hat{f}(\cdot, g) : g \in \mathcal{K}\}$ is evenly continuous family of mappings (or maps if $p \subset \gamma$).

Proof

Let $A \in \delta$, $\mathcal{K} \in \gamma$ and let $V \subset Z$ be open. Assume $\hat{f}(t, \cdot) \in M(\mathcal{K}, M(A, V))$, i.e. $\hat{f}(t, \mathcal{K}) \subset M(A, V)$, $(\hat{f}(t, \mathcal{K}))(A) \in V$ and $f(t, \mathcal{K}(A)) \subset V$ are satisfied. There exists a neighbourhood \mathcal{G} of t such that $f(\mathcal{G}, \mathcal{K}(A)) \subset V$ as $\mathcal{K}(A) \in \mathcal{G}$ and f is continuous. Therefore $\varphi \hat{f}(\mathcal{G}) \subset M(\mathcal{K}, M(A, V))$. It is obvious that we could take $\bigcap \{M(A_i, V_i) : i=1, \dots, n\}$ in the place of $M(A, V)$ and hence follows the continuity of $\varphi \hat{f}$. The continuity of $\varphi \hat{f}$ implies the even continuity of $\{f(\cdot, y) : y \in B\}$ for any $B \in \mathcal{G}$ and for $B = \{y\}$ we get the continuity of $f(\cdot, y)$.

Corollary 1.8

Let X be a set and let J, Y, Z be topological spaces. If (2) holds and $\varphi: J \rightarrow C_0^{\alpha\beta}(Y, Z)$ is continuous, then $\varphi: J \rightarrow C_0^{\alpha\beta}(E_\delta(X, Y), F_\delta(X, Z))$ is continuous. If in addition $\forall \epsilon \in \mathcal{E}$, then $E(X, Y) \in \mathcal{E}$.

Proof

In view of Lemma 1.6 $\varphi(t) \in C_0^{\alpha\beta}(E_\delta(X, Y), F_\delta(X, Z))$, $t \in J$ and we apply Lemma 1.7.

Corollary 1.9

Let J, X, Y, Z be topological spaces. If (1), (2) hold, $E(X, Y) \subset C^\beta(X, Y)$, and $\varphi: J \rightarrow C_0^{\alpha\beta}(Y, Z)$ is continuous, then $\varphi: J \rightarrow C_0^{\alpha\beta}(E_\delta(X, Y), C_0^\beta(X, Z))$ is continuous. If in addition $\varphi: J \rightarrow C_0^\beta(Y, Z)$ is continuous (i.e. $\forall \epsilon \in \mathcal{E}$) and $E(X, Y) \subset C(X, Y)$, then $\varphi: J \rightarrow C_0^\beta(E_\delta(X, Y), C_0^\beta(X, Z))$ is continuous ($E(X, Y) \in \mathcal{E}$).

Proof

In view of Proposition 1.5 $\varphi(t): E(X, Y) \rightarrow C^\beta(X, Z)$, $t \in J$ and we apply Corollary 1.8.

Lemma 1.10

Let X be a set and let Y be a topological space. If $\mathcal{A} \subset \mathcal{B} \subset \mathcal{C}$ in $E_\delta(X, Y)$, $A \in \mathcal{B}$ and $g(A) \in \mathcal{C}$ in Y for each $g \in \mathcal{B}$, then $\mathcal{A}(A) \in \mathcal{C}$ in Y . If $\mathcal{A} \in \mathcal{C}$ in $E_\delta(X, Y)$, $A \in \mathcal{B}$ and for every $g \in \mathcal{A}$ $g(A) \in \mathcal{C}$ in Y , then $\mathcal{A}(A) \in \mathcal{C}$ in Y .

Proof

Let $\{V_y\}_{y \in \mathcal{B}(A)}$ be an open cover of $\mathcal{B}(A)$. We have $g(A) \subset \bigcup \{V_{j, g} := \bigcup_{z \in g(A)} z, j \in T_g\} =: U_g$ for a finite set T_g . From $\mathcal{B} \subset U\{M(A, U_g) : g \in \mathcal{B}\}$ and the compactness of \mathcal{B} it follows that $\mathcal{B} \subset U\{M(A, U_{g_k}) : g_k \in \mathcal{B}, k \in K\}$ for a finite set K . Thus the set $\mathcal{B}(A) \subset \bigcup \{U_{g_k} := \bigcup_{z \in g_k(A)} z, j \in T_{g_k}, k \in K\}$ is compact and the inclusion $\mathcal{A}(A) \subset \mathcal{B}(A)$ proves the relative compactness of $\mathcal{A}(A)$. If $\mathcal{A} \in \mathcal{C}$ in $E_\delta(X, Y)$, then we follow the above proof for $\mathcal{B} = \mathcal{A}$.

Corollary 1.11

Let X, Y be topological spaces. If $\delta \subset \beta \cap \mathcal{C}$ in X and $E(X, Y) \subset C^\beta(X, Y)$, then for any $\mathcal{A} \in \mathcal{C}$ ($\mathcal{A} \in \mathcal{C}$) in $E_\delta(X, Y)$ we have $\mathcal{A}(A) \in \mathcal{C}$ ($\mathcal{A}(A) \in \mathcal{C}$) in Y (i.e. we have $\mathcal{A}[\delta] \subset \mathcal{C}$ for $\delta \subset \beta \cap \mathcal{C}$, $\mathcal{A} = \mathcal{C}$ (\mathcal{C}), $\delta = \mathcal{C}$ (\mathcal{C})).

Proof

As any $g \in \mathcal{A} \subset E(X, Y) \subset C^{\beta}(X, Y)$ is continuous on δ and $\delta \subset c$, we have $g(\delta) \in c$ in Y for any $A \in \delta$. Thus we can apply Lemma 1.10.

Theorem 1.12

Let J, X, Y, Z be topological spaces and let $\delta, \beta \subset 2^X$, $\delta \subset \beta \subset c$ in X , $\gamma \subset c$ ($\gamma \subset c \cap c$) in $E_{\delta}(X, Y)$, $\mathcal{K}, \delta \subset 2^Y$, $c \subset \delta$ ($c \subset \delta$) in Y . If $\varphi f: J \rightarrow C_{\delta}^{\mathcal{K}}(Y, Z)$ is continuous, (1) holds and $E(X, Y) \subset C^{\beta}(X, Y)$, then $\varphi f: J \rightarrow C_{\delta}^{\mathcal{K}}(E_{\delta}(X, Y), C_{\delta}^{\beta}(X, Z))$ is continuous; if $Y \in \beta$ and $E(X, Y) \subset C(X, Y)$, then $E(X, Y) \in \gamma$.

Proof

In view of Corollary 1.11 (2) holds and we can apply Corollary 1.9.

Theorem 1.13

Let J, X, Y, Z be topological spaces and let $\delta, \beta \subset 2^X$, $\delta \subset \beta \subset c$ in X , $c \in \mathcal{K}$ in Y and $E(X, Y) \subset C^{\beta}(X, Y)$. If $\varphi f: J \rightarrow C_{\delta}^{\mathcal{K}}(Y, Z)$ is continuous and (1) holds, then $\varphi f: J \rightarrow C_{\delta}^{\mathcal{K}}(E_{\delta}(X, Y), C_{\delta}^{\beta}(X, Z))$ is continuous. If in addition $Y \in \mathcal{K}$ and $E(X, Y) \subset C(X, Y)$, then $\varphi f: J \rightarrow C_c(E_{\delta}(X, Y), C_{\delta}(X, Z))$ is continuous. For each $\mathcal{A} \in c$ in $E_{\delta}(X, Y)$ $\{f(\cdot, g): g \in \mathcal{A}\}$ is evenly continuous family of maps. In particular the continuity of $\varphi f: J \rightarrow C_{\delta}^{\mathcal{K}}(Y, Z)$ ($\varphi f: J \rightarrow C_c(Y, Z)$) implies the continuity of $\varphi f: J \rightarrow C_{\delta}^{\mathcal{K}}(E_c(X, Y), C_{\delta}^{\mathcal{K}}(X, Z))$ for $E(X, Y) \subset C^c(X, Y)$ ($\varphi f: J \rightarrow C_c(E_c(X, Y), C_c(X, Z))$ for $E(X, Y) \subset C(X, Y)$).

Proof

The first part of 1.13 follows immediately from Theorem 1.12. The even continuity is a consequence of Lemma 1.7 (as $p \subset c$). Obviously for $\beta = c$ in Y , $E(X, Y) \subset C^{\beta}(X, Y)$ we have $E[\beta, Y] \subset c$ in Y , i.e. (1) holds, and hence the first part implies the third part of our theorem.

Definition 1.14

Let (Y, \mathcal{U}) , (Z, \mathcal{V}) be q -spaces and $\mathcal{B} \subset 2^Y$. Then $f: Y \rightarrow Z$ is strongly uniformly continuous on \mathcal{B} if for every $B \in \mathcal{B}$, $V \in \mathcal{V}$ there exists a $U \in \mathcal{U}$ such that $x \in B$, $(x, y) \in U$ imply $(f(x), f(y)) \in V$. If $Y \in \mathcal{B}$, then f is uniformly continuous.

Lemma 1.15

Let X be a set and let (Y, \mathcal{U}) , (Z, \mathcal{V}) be q -spaces for which (2) holds. If $f \in UC^{\mathcal{K}}(Y, Z)$, then $\hat{f} \in UC^{\mathcal{K}}(E_{\delta}(X, Y), F_{\delta}(X, Z))$. If f is strongly uniformly continuous on $E[\delta, Y]$ (or uniformly continuous), then $\hat{f} \in UC(E_{\delta}(X, Y), F_{\delta}(X, Z))$.

Proof

If $A \in \delta$, $A \in \gamma$ and $V \in \mathcal{V}$, then there exists a $U \in \mathcal{U}$ such that $(y, z) \in U \cap \{A(A) \times A(A)\}$ implies $(f(y), f(z)) \in V$. Thus $(g, h) \in W(A, U) \cap A^2$ implies $(g(x), h(x)) \in U \cap \{A(A) \times A(A)\}$, $x \in A$ and $(f(g(x)), f(h(x))) \in V$, $x \in A$, i.e. $(\hat{f}(g), \hat{f}(h)) \in W(A, V)$ which means the uniform continuity of \hat{f} on the members of γ . If f is strongly uniformly continuous on $E[\delta, Y]$, then $y \in g(A)$, $(y, z) \in U$ imply $(f(y), f(z)) \in V$ and from $(g, h) \in W(A, U)$, i.e. $(g(x), h(x)) \in U$, $g(x) \in g(A)$, $x \in A$ we obtain $(f(g(x)), f(h(x))) \in V$, $x \in A$, i.e. $(\hat{f}(g), \hat{f}(h)) \in W(A, V)$.

Lemma 1.16

Let X, Y be sets and let (J, \mathcal{J}) , (Z, \mathcal{V}) be q -spaces. Assume (2) holds and $\varphi f: J \rightarrow F_{\delta}(Y, Z)$ is uniformly continuous. Then $\varphi \hat{f}: J \rightarrow F_{\gamma}(E(X, Y), F_{\delta}(X, Z))$ is uniformly continuous and for each $A \in \gamma$ $\{\hat{f}(\cdot, g): g \in A\}$ is a uniformly equicontinuous family of maps.

Proof

Let $A \in \delta$, $A \in \gamma$ and let $(s, t) \in U$ imply $(f(s, \cdot), f(t, \cdot)) \in W(A(A), V)$ ($U \in \mathcal{J}$, $V \in \mathcal{V}$). Then $(f(s, g(x)), f(t, g(x))) \in V$, $g \in A$, $x \in A$ which means $(\hat{f}(s, g), \hat{f}(t, g)) \in W(A, V)$, $g \in A$ and $(\hat{f}(s, \cdot), \hat{f}(t, \cdot)) \in W(A, W(A, V))$. If we consider $A_1, \dots, A_n \in \delta$, $V_1, \dots, V_n \in \mathcal{V}$, then there exists a $U \in \mathcal{J}$ such that $(s, t) \in U$ implies $(f(s, \cdot), f(t, \cdot)) \in W(A_i, V_i)$, $i=1, \dots, n$ and therefore $(\varphi \hat{f}(s), \varphi \hat{f}(t)) = (\hat{f}(s, \cdot), \hat{f}(t, \cdot)) \in W(A, \bigcap \{W(A_i, V_i): i=1, \dots, n\})$. Thus $\varphi \hat{f}$ is uniformly continuous. If φf is uniformly continuous, then $\{f(\cdot, y): y \in B\}$ is a uniformly equicontinuous family of maps for any $B \in \delta$, as $(f(s, y), f(t, y)) \in V$, $y \in B$, $(s, t) \in U$.

Corollary 1.17

Let X be a set and let (J, \mathcal{J}) , (Y, \mathcal{U}) , (Z, \mathcal{V}) be q -spaces. If (2) holds and $\varphi f: J \rightarrow UC_{\delta}^{\mathcal{U}}(Y, Z)$ is uniformly continuous, then $\varphi \hat{f}: J \rightarrow UC_{\gamma}^{\mathcal{U}}(E_{\delta}(X, Y), F_{\delta}(X, Z))$ is uniformly continuous. If in addition $f(t, \cdot): Y \rightarrow Z$ is strongly uniformly continuous on $E[\delta, Y]$ (or uniformly continuous) for each $t \in J$, then $\varphi \hat{f}: J \rightarrow UC_{\gamma}(E_{\delta}(X, Y), F_{\delta}(X, Z))$ is uniformly continuous.

Proof

In view of Lemma 1.15 $\varphi \hat{f}: J \rightarrow UC_{\delta}^{\mathcal{U}}(E_{\delta}(X, Y), F_{\delta}(X, Z))$ and we apply Lemma 1.16.

Corollary 1.18

Let J, X, Y, Z be q -spaces. If (1), (2) hold, $E(X, Y) \subset C^\beta(X, Y)$ and $\varphi f: J \rightarrow UC_6^\beta(Y, Z)$ is uniformly continuous, then $\varphi f: J \rightarrow UC_6^\beta(E_\delta(X, Y), C_\delta^\beta(X, Z))$ is uniformly continuous. If in addition $f(t, \cdot): Y \rightarrow Z$ is strongly uniformly continuous on $E[\delta, Y]$ (or $f(t, \cdot) \in UC(Y, Z)$), $t \in J$ and $E(X, Y) \subset CC(X, Y)$, then $\varphi f: J \rightarrow UC_6(E_\delta(X, Y), C_\delta(X, Z))$ is uniformly continuous. If $E(X, Y) \subset UC^\beta(X, Y)$, then we may write $UC_6^\beta(X, Z)$ in place of $C_\delta^\beta(X, Z)$ (and $UC_6(X, Z)$ in place of $C_\delta(X, Z)$ for $E(X, Y) \subset UC(X, Y)$).

Proof

We apply Proposition 1.75 to Corollary 1.17. The composition of uniform maps is uniform.

Lemma 1.19

Let X be a set and let (Y, \mathcal{U}) be a q -space. If $\mathcal{A} \subset \mathcal{B} \in c$ in $E_\delta(X, Y)$, $A \in \delta$ and $g(A) \in c$ for each $g \in \mathcal{B}$, then $\mathcal{A}(A) \in c$ in Y . If $\mathcal{A} \in pc(c)$ in $E_\delta(X, Y)$; then $\mathcal{A}(A) \in pc(c)$ in Y , for $g(A) \in pc(c)$, $A \in \delta$, $g \in \mathcal{A}$.

Proof

Let $\{V_{j, g} := V^Y(y) : y \in \mathcal{B}(A)\}$ be a cover of $\mathcal{B}(A)$ for $V^Y \in \mathcal{U}$. We have $g(A) \in C \cup \{V_{j, g} := V_{z_{j, g}} : z_{j, g} \in g(A), j \in T_g\}$ for a finite set T_g and any $g \in \mathcal{B}$. In view of [6, Corol. 4.21 p. 55] there exists a $U^g \in \mathcal{U}$ with $U^g(g(A)) \subset C \cup \{V_{j, g} : j \in T_g\}$. The compactness of \mathcal{B} implies $\mathcal{B} \subset \{W(A, U^{g_k}) : k \in K\}$ for a finite set K . Thus $\mathcal{B}(x) \subset \{U^{g_k}(x) : k \in K\} \subset \{V_{j, g_k} : j \in T_{g_k}, k \in K\}$, $x \in A$, i.e. $\mathcal{B}(A)$ is compact and $\mathcal{A}(A) \subset \mathcal{B}(A)$ is relatively compact. If $\mathcal{A} \in c$, we take $\mathcal{B} = \mathcal{A}$. If $\mathcal{A} \in pc$ and $g(A) \in pc$ for each $g \in \mathcal{A}$, then it is enough to consider $\mathcal{B} = \mathcal{A}$, $V^Y = V$, $y \in \mathcal{A}(A)$. Then for any $U \in \mathcal{V}$ such that $U^2 \subset V$ we have $\mathcal{A} \subset \{W(A, U)(g_k) : k \in K\}$ for a finite set K , i.e. $\mathcal{A}(x) \subset C \cup \{U(g_k(x)) : k \in K\}$, $x \in A$. We have $g_k(A) \subset \{U(z_{j, g_k}) : j \in T_{g_k}\}$ ($z_{j, g_k} \in g_k(A)$), T_{g_k} - finite and $U(g_k(A)) \subset C \cup \{U^2(z_{j, g_k}) : j \in T_{g_k}\} \subset C \cup \{V(z_{j, g_k}) : j \in T_{g_k}\}$. Hence $\mathcal{A}(A) = \bigcup \{\mathcal{A}(x) : x \in A\} \subset C \cup \{V(z_{j, g_k}) : j \in T_{g_k}, k \in K\}$, i.e. $\mathcal{A}(A) \in pc$.

Remark 1.20

If (Y, \mathcal{U}) is a uniform space and $A \subset Y$, then for any $U \in \mathcal{U}$ we have $\bar{A} \subset U(A)$ is equivalent to $U(x) \cap A = \emptyset$ (U is symmetric); if A is compact, then $\bar{A} \subset U(B)$ for a finite set $B \subset A$ (see [4, Th. 33 p. 199]) and \bar{A} is compact [4, B(b) p. 161], i.e. $\bar{A} \in pc$.

Corollary 1.21

Let X, Y be q -spaces. If $\delta \subset \beta \cap c$ in X and $E(X, Y) \subset C^\beta(X, Y)$, then for any $\mathcal{A} \in c$ ($\mathcal{A} \in rc$) in $E_\delta(X, Y)$, $\mathcal{A}(A) \in c$ ($\mathcal{A}(A) \in rc$) in Y . If $E(X, Y) \subset UC^\beta(X, Y)$, $\delta \subset \beta \cap pc$ in X , $\mathcal{A} \in pc$ in $E_\delta(X, Y)$, then $\mathcal{A}(A) \in pc$ in Y (i.e. we have $\mathcal{A}[\delta] \subset \delta$ for $\delta \subset \beta \cap c$, $\mathcal{A} = c$ (rc), $\delta = c$ (rc) or for $\delta \subset \beta \cap pc$, $\mathcal{A} = pc$, $\delta = pc$).

Proof

We follow the first part of the proof of Corollary 1.11 and then we apply Lemma 1.19. If $g \in E(X, Y)$ is uniformly continuous on $A \in \delta \subset pc$, then $g(A) \in pc$ in Y .

Theorem 1.22

Let J, X, Y, Z be q -spaces and let $\delta, \beta \subset 2^X$, $\delta \subset \beta \cap c$ in X , $\mathcal{A} \in c$ ($\mathcal{A} \in rc$) in $E_\delta(X, Y)$, $\mathcal{B}, \sigma \subset 2^Y$, $c \in \sigma$ ($rc \subset \sigma$) in Y . If $\varphi f: J \rightarrow UC_\sigma^{\mathcal{A}}(Y, Z)$ is uniformly continuous, (1) holds and $E(X, Y) \subset C^\beta(X, Y)$, then $\varphi \hat{f}: J \rightarrow UC_\sigma^{\mathcal{A}}(E_\delta(X, Y), C_\sigma^\beta(X, Z))$ is uniformly continuous. If in addition $f(t, \cdot): Y \rightarrow Z$ is strongly uniformly continuous on $E[\delta, Y]$ (or $f(t, \cdot) \in UC(Y, Z)$) for each $t \in J$ and $E(X, Y) \subset C(X, Y)$, then $\varphi \hat{f}: J \rightarrow UC_\sigma^{\mathcal{A}}(E_\delta(X, Y), C_\sigma^\beta(X, Z))$ is uniformly continuous; if $E(X, Y) \subset UC^\beta(X, Y)$, then we may write $UC_\sigma^\beta(X, Z)$ in place of $C_\sigma^\beta(X, Z)$ (and $UC_\sigma(X, Z)$ for $E(X, Y) \subset UC(X, Y)$) $f(t, \cdot): Y \rightarrow Z$ being strongly uniformly continuous on $E[\delta, Y]$ and we may write pc in place of c . In all above cases $\{\hat{f}(\cdot, g): g \in \mathcal{A}\}$ is a uniformly equicontinuous family of maps for each $\mathcal{A} \in \mathcal{A}$.

Proof

We apply Corollaries 1.18, 1.21 and Lemma 1.16.

The theorem below is formally similar to the last part of Theorem 1.13; we have given up the midstep between 1.22 and the theorem to follow.

Theorem 1.23

Let J, X, Y, Z be q -spaces. If $\varphi f: J \rightarrow UC_c^c(Y, Z)$ is uniformly continuous and $E(X, Y) \subset C^c(X, Y)$, then $\varphi \hat{f}: J \rightarrow UC_c^c(E_c(X, Y), C_c^c(X, Z))$ is uniformly continuous. We may omit the upper index "c". In place of c one may insert pc for $E(X, Y) \subset UC^{pc}(X, Y)$ (or $UC(X, Y)$).

Proof

We have $E[c, Y] \subset c$ ($E[pc, Y] \subset pc$), i.e. (1) holds and we apply Theorem 1.22.

Remark 1.24

In view of Remark 1.3, Theorem 1.23 is a specialized version of Theorem 1.13 (if Q is a q -space and $\varphi f: J \rightarrow (Q, \tau_Q)$ is continuous, the $\varphi f: J \rightarrow (Q, \tau_c)$ is continuous).

Now let us consider the cases when f is continuous on the Cartesian product.

Lemma 1.25

Let X be a set and let J, Y, Z be topological spaces. Assume $f: J \times Y \rightarrow Z$ and $\delta \subset 2^X$ are such that for each $t \in J$, $g \in E(X, Y)$, $A \in \delta$ and any open $V \subset Z$ for which $(t, g(A)) \subset f^{-1}(V)$ there exist the open neighbourhoods G of t and U of $g(A)$ such that $G \times U \subset f^{-1}(V)$. Then $\hat{f}: J \times E_\delta(X, Y) \rightarrow F_\delta(X, Z)$ is continuous and $\{f(\cdot, g) : g \in E(X, Y)\}$ is an evenly continuous family of maps.

Proof

We have $(s, h) \in \hat{f}^{-1}(M(A, V))$ iff $(s, h(A)) \subset f^{-1}(V)$ and therefore $(t, g(A)) \subset G \times U \subset f^{-1}(V)$ implies $(t, g) \in G \times M(A, U) \subset \hat{f}^{-1}(M(A, V))$, i.e. \hat{f} is continuous and the condition of the even continuity holds. It is seen that if $E(X, Y)$ in our lemma contains all constants, then f is continuous.

Lemma 1.26

Let X be a set and let J, Y, Z be topological spaces. Assume $f: J \times Y \rightarrow Z$ is continuous and for each $g \in E(X, Y)$, $A \in \delta$ the set $g(A)$ is compact. Then $\hat{f}: J \times E_\delta(X, Y) \rightarrow F_\delta(X, Z)$ is continuous.

Proof

Assume that $(t, g(A)) \subset f^{-1}(V)$. In view of the Wallace theorem [4, Th. 12 p. 142] there exist the open neighbourhoods G, U of $t, g(A)$ respectively for which $G \times U \subset f^{-1}(V)$ and we can apply Lemma 1.25.

Theorem 1.27

Let J, X, Y, Z be topological spaces and let $\delta, \beta \subset 2^X$, $\delta \subset \beta \cap c$ in X . If $f: J \times Y \rightarrow Z$ is continuous and $E(X, Y) \subset C^\beta(X, Y)$, then $\hat{f}: J \times E_\delta(X, Y) \rightarrow C_\delta^\beta(X, Z)$ is continuous and $\{\hat{f}(\cdot, g) : g \in E(X, Y)\}$ is an evenly continuous family of maps. In particular $\hat{f}: J \times E(X, Y) \rightarrow C^c(X, Z)$ for $E(X, Y) \subset C^c(X, Y)$. If $E(X, Y) \subset C(X, Y)$, then $\hat{f}: J \times E(X, Y) \rightarrow C(X, Z)$.

Proof

See Lemma 1.26, Proposition 1.5.

Lemma 1.28

Let X, J be a set and a topological space respectively and let $(Y, \mathcal{U}), (Z, \mathcal{V})$ be q -spaces. Assume $f: J \times Y \rightarrow Z$, $\delta \subset 2^X$ are such that for each $t \in J$, $g \in E(X, Y)$, $A \in \delta$, $V \in \mathcal{V}$ there exist a $U \in \mathcal{U}$ and a neighbourhood G of

t for which $x \in A$, $h \in E(X, Y)$ and $(g(x), h(x)) \in U$ imply $(f(t, g(x)), f(s, h(x))) \in V$, $s \in G$. Then $\hat{f}: J \times E_\delta(X, Y) \rightarrow F_\delta(X, Z)$ is continuous and $\hat{f}(t, \cdot): E_\delta(X, Y) \rightarrow F_\delta(X, Z)$ is uniformly continuous for any $t \in J$. If $\{f(\cdot, y): y \in g(A)\}$ is equicontinuous for every $g \in E(X, Y)$, $A \in \delta$, then $\{\hat{f}(\cdot, g): g \in E(X, Y)\}$ is equicontinuous.

Proof

If $(g, h) \in W(A, U)$, i.e. $(g(x), h(x)) \in U$, $x \in A$, then $(f(t, g(x)), f(s, h(x))) \in V$, $x \in A$, i.e. $(\hat{f}(t, g), \hat{f}(s, h)) \in W(A, V)$. The uniform continuity follows from Lemma 1.15. We have $\hat{f}(G, g) \in W(A, V)(\hat{f}(t, g))$ iff $f(G, g(x)) \subset V(f(t, g(x)))$, $x \in A$ and it is seen that $\{f(\cdot, g): g \in E(X, Y)\}$ is equicontinuous.

Lemma 1.29

Let X, J be a set and a topological space respectively and let (Y, \mathcal{U}) , (Z, \mathcal{V}) be q -spaces. If $f: J \times Y \rightarrow Z$ is continuous, $E[\delta, Y] \subset \mathcal{C}$ in Y and $(f(t, g(A)), \mathcal{V}|f(t, g(A)))$ is uniformizable for each $t \in J$, $g \in E(X, Y)$, $A \in \delta$, then $\hat{f}: J \times E_\delta(X, Y) \rightarrow F_\delta(X, Z)$ is continuous and $\hat{f}(t, \cdot)$ is uniformly continuous for any $t \in J$. What's more $\{\hat{f}(\cdot, g): g \in E(X, Y)\}$ is equicontinuous.

Proof

For any $y \in g(A)$ there exists a neighbourhood G_y of t and $U^y \in \mathcal{U}$ such that for $U_y := U^y(y)$ we have $f(G_y, U_y) \subset V(f(t, y))$. Then $g(A) \subset \bigcup \{U_i := U_{y_i} : i = 1, \dots, n\} =: V_2$ $g(A)$ being compact. There exists a $U \in \mathcal{U}$ with $U(g(A)) \subset V_2$ [6, Th. 4.20 p. 54]. For $G := \bigcap \{G_{y_i} : i = 1, \dots, n\}$ and any $y \in g(A)$ there exists an i such that $f(G, U(y)) \subset f(G, U_{y_i}) \subset V(f(t, y_i))$, and $f(t, y) \in V(f(t, y_i))$ implies $f(t, y_i) \in V(f(t, y))$ as we may assume that $V|f(t, g(A))$ is symmetric. Therefore $f(G, U(y)) \subset V^2(f(t, y))$, $y \in g(A)$, i.e. $\hat{f}(G, W(A, U)(g)) \subset W(A, V^2)(\hat{f}(t, g))$ which implies the continuity of \hat{f} and the uniform continuity of $\hat{f}(t, \cdot)$. From the above it follows that $\{f(\cdot, y): y \in g(A)\}$ is equicontinuous and (Lemma 1.28) $\{\hat{f}(\cdot, g): g \in E(X, Y)\}$ is equicontinuous.

Theorem 1.30

Let X, J be topological spaces, (Y, \mathcal{U}) , (Z, \mathcal{V}) q -spaces, and let $\delta, \beta \subset 2^X$, $\delta \subset \beta \subset \mathcal{C}$ in X . If $f: J \times Y \rightarrow Z$ is continuous, $E(X, Y) \subset \mathcal{C}^\beta(X, Y)$ and $(f(t, g(A)), \mathcal{V}|f(t, g(A)))$ is uniformizable for each $t \in J$, $g \in E(X, Y)$, $A \in \delta$, then $\hat{f}: J \times E_\delta(X, Y) \rightarrow \mathcal{C}_\beta^\beta(X, Z)$ is continuous and $\hat{f}(t, \cdot)$ is uniformly continuous for any $t \in J$; $\{\hat{f}(\cdot, g): g \in E(X, Y)\}$ is equicontinuous. If in addition $E(X, Y) \subset \mathcal{C}(X, Y)$, then $\hat{f}: J \times E(X, Y) \rightarrow \mathcal{C}(X, Z)$.

Proof

We apply Proposition 1.5 and Lemma 1.29.

Lemma 1.31

Let X be a set and let (J, \mathcal{T}) , (Y, \mathcal{U}) , (Z, \mathcal{V}) be q -spaces. If $f: J \times Y \rightarrow Z$ is strongly uniformly continuous on $J \times E[\delta, Y]$ (or uniformly continuous on $J \times Y$), then $\hat{f}: J \times E_{\delta}(X, Y) \rightarrow F_{\delta}(X, Z)$ is uniformly continuous.

Proof

Let $(s, t) \in T$, $(y, z) \in U$, $y, z \in A$ imply $(f(s, y), f(t, z)) \in V$. By taking $y = g(x)$, $z = h(x)$ we obtain $((s, g), (t, h)) \in T \times W(A, U)$ implies $(\hat{f}(s, g), \hat{f}(t, h)) \in W(A, V)$, i.e. \hat{f} is uniformly continuous.

Theorem 1.32

Let J, X, Y, Z be q -spaces. If $E(X, Y) \subset C^{\beta}(X, Y)$, $\delta \in \beta$ and $f: J \times Y \rightarrow Z$ is strongly uniformly continuous on $J \times E[\delta, Y]$ (or uniformly continuous on $J \times Y$), then $\hat{f}: J \times E_{\delta}(X, Y) \rightarrow C_{\delta}^{\beta}(X, Z)$ is uniformly continuous. If in addition $E(X, Y) \subset UC^{\beta}(X, Y)$, then $\hat{f}: J \times E(X, Y) \rightarrow UC^{\beta}(X, Z)$. For $\gamma \in \beta$ we may omit β .

Proof

We apply Lemma 1.31, Proposition 1.5 and Lemma 1.15.

Remark 1.33

If J is a (q -space) topological space, Y a set and (Z, \mathcal{V}) is a q -space, then the (uniform) equicontinuity of $\{f(\cdot, y): y \in Y\}$ implies the (uniform) continuity of $\varphi: J \rightarrow F_{\delta}(Y, Z)$ ($F_{\delta}(Y, Z)$ is a q -space) for any $\delta \in \mathcal{V}$; it is seen that $(\varphi(t), \varphi(s)) \in W(B, V)$ is equivalent to $(f(t, y), f(s, y)) \in V$, $y \in B$.

2. COMPACTNESS

The problem of when $\tau_{\mathcal{U}|C} = \tau_{\mathcal{U}}|C$, for q -spaces, is of particular importance. Some results were obtained in [5]. Alas, they were based on [6, Corol. p. 56] which is false as it is shown in the example below.

Example

Let $X=I$ with the natural topology and let \mathcal{U}, \mathcal{P} be the metric uniformity and the Pervin q -uniformity, respectively. The mapping $\text{id}: (X, \mathcal{U}) \rightarrow (X, \mathcal{P})$ is obviously continuous. For any $\alpha > 0$ we have $\mathcal{U} \ni \{(x, y) \in I: |x-y| < \alpha\} \not\subset \{x < 1/2, 1/2 < x \in I\} \in \mathcal{P}$, i.e. id is not uniformly continuous. The uniform space (X, \mathcal{U}) is compact, (X, \mathcal{P}) is a q -space, $f = \text{id}$ is continuous but not uniformly continuous.

Below we list the necessary modifications of the results of Morales.

Theorem 2.1. (cp. [5, Th. (1.1) p. 7])

Let $f \in C(X \times Y, Z)$, where $Z = (Z, \mathcal{V})$ is a q -space. If X is compact, regular and for each $y \in Y$ $(f(X, y), \mathcal{V}|f(X, y))$ is uniformizable (cp. Def. 1.1), then the family $\{f(x, \cdot) : x \in X\}$ is equicontinuous.

Proof, (cp. [5])

Let \mathcal{U} be the uniformity on X and let $y_0 \in Y, V \in \mathcal{V}$. Since $f(\cdot, y_0)$ is uniformly continuous [4, p. 198] there exists a $U \in \mathcal{U}$ such that $(x_1, x_2) \in U$ implies $f(x_1, y_0), f(x_2, y_0) \in V$. For each $x \in X$ there exists a $U^x \in \mathcal{U}$ and a neighbourhood W_x of y_0 such that $U^x \subset U$ and $f(U^x \times W_x) \subset V(f(x, y_0))$ for $U_x := U^x(x)$. If $\bigcup \{U_i := U_{x_i} : i=1, \dots, n\} = X$, then $W = \bigcap \{W_{x_i} : i=1, \dots, n\}$ is a neighbourhood of y_0 . Let $(x, y) \in X \times W$. For some $i, x \in U_i$ and therefore $f(x, y) \in V(f(x_i, y_0))$. Hence $f(x, y) \in V(f(x_i, y_0)) \subset V^2(f(x, y_0))$, $x \in X, y \in W$ as we may assume that $\bigvee_n [f(X, y)]^2$ is symmetric.

To prove the results below it is enough to follow the respective proofs from [5] and to apply the above theorem.

Lemma 2.2 (see [5, Lemma (1.3) p. 7])

If $E(X, Y)$ is an evenly continuous family of mappings from a topological space X to a q -space (Y, \mathcal{U}) such that

(3) $(\overline{E(x)}, \mathcal{U}|E(x))$ is uniformizable for all $x \in X$

and $\overline{E(x)}$ is compact for all $x \in X$, then $E(X, Y)$ is equicontinuous.

Theorem 2.3 (cp. [5, Th. (1.5) p. 8])

Let $f \in C(Y \times Y, Z)$, where $Z = (Z, \mathcal{V})$ is a q -space. If X is compact, Z is regular and $(f(X, y), \mathcal{V}|f(X, y))$ is uniformizable for any $y \in Y$, then the family $\{f(x, \cdot) : x \in X\}$ is equicontinuous.

Remark

In view of [4, Corol. 30 p. 198] the topological space $(f(X, y), \tau_{\mathcal{V}|f(X, y)})$ is uniformizable but it does not mean that the q -space $(f(X, y), \mathcal{V}|f(X, y))$ is uniformizable (see Example).

Let (Y, \mathcal{U}) be a fixed q -space and let $\tau_q = \tau_{\mathcal{U}|c}, \tau_c = \tau_{\mathcal{U}|c}$.

Theorem 2.4 (see [5, Th. (2.4) p. 8])

Let $E \in C^c(X, Y)$. If X is Hausdorff or regular or if Y is regular, then τ_q is the smallest of the topologies on E which are jointly continuous on compacta if

(4) for any $A \in c, f \in E$ $(f(A), \mathcal{U}|f(A))$ is uniformizable

In [5, (2.5), (2.6), (2.7), (3.2)] we additionally assume that (4) holds.

Corollary 2.5 (see [5, (2.4), (2.5) p. 9])

If X is Hausdorff or regular, or if Y is regular, then $\tau_c = \tau_q$ on $E \subset C^c(X, Y)$ if (4) holds.

The above result explains why using the same symbol $E_g(X, Y)$ in two meanings is reasonable.

In [5, (3.5)] we additionally assume (3).

For a topological space (Y, τ) and any set $D \in Y$, D^* is the R -saturated of D , i.e. $D^* = \bigcup \{ \bigcap \{ G \in \tau : x \in G \} : x \in D \}$ (cp. [5, p. 11]).

Theorem 2.6 (cp. [5, Th. (5.1) p. 12])

Let (8) hold for $E \subset (C(X, Y), \tau_q)$. If Y is regular, then E is compact iff

- (5) E^* is closed in $C(X, Y)$,
- (6) $\overline{E(x)}$ is compact for all $x \in X$,
- (7) E is equicontinuous.

If X is a k -space or Y is Hausdorff, then (5), (6), (7) are necessary for the compactness of E .

Theorem 2.7 (cp. [5, Corol. (5.2) p. 12])

Let (3), (4) hold for $E \subset (C^c(X, Y), \tau_q)$. If Y is regular, then E is compact iff (5), (6) and the following are satisfied

- (8) E is equicontinuous on compacta.

If Y is Hausdorff, then (5), (6), (8) are necessary for the compactness of E .

Theorem 2.8 (cp. [5, Corol. (5.3) p. 12])

Let (4) hold for $E \subset (C(X, Y), \tau_q)$. If Y is regular, then (5), (6), (7) are sufficient for the compactness of E . If X is a k_3 -space, Y is regular and in addition (3) holds, then (5), (6), (7) are necessary for the compactness of E .

Remark 2.9

In the sufficiency part of the above three theorems we may take $\delta \subset c$ in (Y, \mathcal{U}) , $E \subset C^\delta(X, Y)$ and E_g in place of $E_c \subset (C^c(X, Y), \tau_q)$.

The three lemmas below generalize the Bourbaki criterion of precompactness [2, Th. 2 p. 199].

Lemma 2.10

Let (X, \mathcal{U}) , (Y, \mathcal{V}) be q -spaces and let $(\mathcal{A}(A), \mathcal{V}|\mathcal{A}(A))$, $A \in \delta$ be uniformizable. If $\mathcal{A} \subset C^\delta(X, Y)$ and \mathcal{A} is precompact in $F_\delta(X, Y)$, then

- (9) for every $x \in U_\delta$, $\mathcal{A}(x)$ is precompact,
 (10) \mathcal{A} is equicontinuous on the sets of δ .

Proof

The condition $\mathcal{A} \subset \bigcup \{W(A, V)(g_i) : g_i \in \mathcal{A}, i \in T_{\mathcal{A}}\}$ for a finite set $T_{\mathcal{A}}$ implies $\mathcal{A}(x) \subset \bigcup \{V(g_i(x)) : i \in T_{\mathcal{A}}\}$, $x \in A$, i.e. $\mathcal{A}(x)$ is precompact. Let $V^3 \subset V_1$, $V, V_1 \subset \mathcal{V}$. For any $g \in \mathcal{A}$ there exists an index i with $g(x) \in V(g_i(x))$, $x \in A$. The continuity of $g_i|A$ implies the existence of $U_i \in \mathcal{U}$ with $g_i(U_i(x) \cap A) \subset V(g_i(x))$, $i \in T_{\mathcal{A}}$. For $U = \bigcap \{V_i : i \in T_{\mathcal{A}}\}$ we have $g_i(U(x) \cap A) \subset V(g_i(x))$, $i \in T_{\mathcal{A}}$. Therefore $g(U(x) \cap A) \subset V(g_i(U(x) \cap A)) \subset V^2(g_i(x))$. As we may assume that $V|\mathcal{A}(A)$ is symmetric, $g(x) \in V(g_i(x))$ implies $g_i(x) \in V(g(x))$. Hence for any $V_1 \in \mathcal{V}$, and any $x \in U_\delta$ there exists a $U \in \mathcal{U}$ such that $g(U(x) \cap A) \subset V^3(g(x)) \subset V_1(g(x))$ for all $g \in \mathcal{A}$.

Lemma 2.11

Let (X, \mathcal{U}) , (Y, \mathcal{V}) be q -spaces and let $(\mathcal{A}(A), \mathcal{V}|\mathcal{A}(A))$, $A \in \delta$ be uniformizable. If $\mathcal{A} \subset C^\delta(X, Y)$, $\delta \subset c$ and (9), (10) hold, then \mathcal{A} is precompact in $F_\delta(X, Y)$.

Proof

For any $V_1 \in \mathcal{V}$ we can find a $V \in \mathcal{V}$ such that $V^3 \subset V_1$. For any $x \in A \in \delta$ there exists a $U_x \in \mathcal{U}$ such that $f(U_x \cap A) \subset V(g(x))$ for $U_x := U_x^x(x)$, $g \in \mathcal{A}$, \mathcal{A} being equicontinuous on δ . We have $A \subset \bigcup \{U_i := U_{x_i} : i=1, \dots, n\}$ and hence for any $x \in A$ there exists an i with $g(x) \in g(U_i \cap A) \subset V(g(x_i))$. The precompactness of $\mathcal{A}(\{x_1, \dots, x_n\})$ implies $g(x_i) \in \bigcup \{V(g_j(x_i)) : j=1, \dots, k, i=1, \dots, n\}$ and we obtain $g(x) \in V(g(x_i)) \cap \mathcal{A}(A) \subset \bigcup \{V^2(g_j(x_i)) \cap \mathcal{A}(A) : j=1, \dots, k\} \subset \bigcup \{V^3(g_j(x)) : j=1, \dots, k\}$ if $V|\mathcal{A}(A)$ is symmetric which can be assumed. Thus $\mathcal{A} \subset \bigcup \{W(A, V_1)(g_j) : j=1, \dots, k\}$ which implies the precompactness of \mathcal{A} .

Lemma 2.12

Let (X, \mathcal{U}) , (Y, \mathcal{V}) be q -spaces. Assume that $\mathcal{A} \subset C^\delta(X, Y)$, $\delta \subset pc$ in X , (9) and

- (11) \mathcal{A} is uniformly continuous on δ

hold. If $(\mathcal{A}(A), \mathcal{V}|\mathcal{A}(A))$ or $(A, \mathcal{U}|A)$ is uniformizable for all $A \in \delta$, then \mathcal{A} is precompact in $F_\delta(X, Y)$.

Proof

The uniform continuity of \mathcal{A} on A implies $U^x=U$, $x \in A$ in the proof of Lemma 2.11. If $(\mathcal{A}(A), \mathcal{V}|\mathcal{A}(A))$ is uniformizable, then we follow the proof of Lemma 2.11. If $(A, \mathcal{U}|A)$ is uniformizable, then $x \in U(x_1) \cap A$ implies $x_1 \in U(x) \cap A$ $U_n A \times A$ being symmetric. Hence (11) gives $g(x_1) \in g(U(x) \cap A) \subset V(g(x))$, $g \in \mathcal{A}$ and $V^2(g_j(x_1)) \subset V^3(g_j(x))$, $j=1, \dots, k$, $x \in A$ (cp. proof of Lemma 2.11).

References

- [1] Bourbaki N.: Topologie générale (Hermann Paris, 1971).
- [2] Burbaki N.: Obscaia topologia (Nauka Moskwa, 1975).
- [3] Engelking R.: General topology (PWN. Warszawa, 1977).
- [4] Kelley J.L.: General topology (Springer Verlag N.Y.-Heidelberg-Berlin, 1975).
- [5] Morales P.: Non-Hausdorff Ascoli theory, Dissert. Math., 119 (1974).
- [6] Murdeshwar M.G., Naimpally S.A.: Quasi-uniform topological spaces (Noordhoff, 1966).
- [7] Pervin W.J.: Quasi-uniformization of topological spaces, Math. Ann., 147 (1962), 316-317.

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Streszczenie

O przestrzeniach funkcyjnych

Praca dotyczy zależności między ciągłością $f: J \times Y \rightarrow Z$, $\varphi f: J \rightarrow E(X, Z)$, gdzie $E(X, Y)$ jest rodziną odwzorowań z X w Y oraz $\hat{f}: J \times E(X, Y) \rightarrow E(X, Y)$ i $\varphi \hat{f}: J \rightarrow E(E(X, Y), E(X, Z))$, z odpowiednimi topologiami w przestrzeniach funkcji. Najważniejsze tu są Twierdzenia 1.13, 1.23, 1.27, 1.32.

Część druga poświęcona jest głównie zwartości zbiorów odwzorowań.