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Applications of weeds

A b s t r a c t . This paper is devoted to the theorems on retractions, fixed points and selections for a convex structure called weed. The most important results in this paper are Theorems: 2.2, 2.3, 2.5, 3.3 (cp. 3.5), 3.14, 4.11, 4.15. They extend some well known classical results for convex structures due to Dugumdji, Himmelberg (retracts) and Michael (selections). Theorem 3.5 modifies a theorem of Hukuhara.

1. BASIC INFORMATION

First let us establish the terminology. 2^X is the family of all subsets of X. For a set A and a family $\mathcal{B}$ of sets $A\cap\mathcal{B}=\{A\cap B:\ B\in\mathcal{B}\}$, $A\times\mathcal{B}=\{A\times B:\ B\in\mathcal{B}\}$. We adopt the definition of mapping $f: X\to Y$ from [7] as being a subset (of a special kind) of the Cartesian product $X\times Y$. By writing $f: X\to Y$, we understand that f is a mapping of X to Y and f,X,Y are nonempty; on the other hand $f(\emptyset)=\emptyset$ if $\emptyset\notin X$. If A is a nonempty subset of X and $f: X\to Y$, then $f|A$ or $f|: A\to Y$ is the restriction of f to A. Is $\mathcal{F}$ is a family of mappings and $\mathcal{A}$ is a family of mappings or sets, then $\mathcal{F}\circ\mathcal{A}=\{f\circ a:\ f\in\mathcal{F},\ a\in\mathcal{A}\}$ or $\mathcal{F}(\mathcal{A})=\{f(A):\ f\in\mathcal{F},\ A\in\mathcal{A}\}$(we disregard the empty families).

For $F: X\to 2^Y$ and any $A\subset X$ $F(A)=\bigcup\{F(x):\ x\in A\}$; if $H: 2^Y\to 2^Z$, then $(H\circ F)(A):=H(F(A))$, in particular $\overline{F}(A)=\overline{F(A)}$.

In this paper, space means topological space, and map means continuous mapping.

The subsets of any space X are treated as being subspaces of X if the topology is mentioned. The terminology concerning the separation axioms is adopted from [9]. For any point or subset A of a space, U_A, V_A, W_A etc. are neighbourhoods of A, i.e. they contain A in their interiors.

If $\{X_s\}_{s\in T}$ is a family of spaces, then their Cartesian product $X = P_{s\in T} X_s$ has the Tychonoff topology; if $x = P_{s\in T} x_s \in X$, then $\delta_i x = P\{x_s : s \in T,\ s \neq i\}$.

Let $I = \langle 0,1 \rangle$ with the natural topology and let $P^{n-1} = \{t \in I^n : \sum t_i = 1\}$.

Definition 1.1

A pair (D,Q) is a weed in a set X if $Q = (Q_n)_{n\in N}$ is a sequence of mappings such that for each $n \in N$ with $n \leq \overline{\overline{D}}$ and any set $\{e_1,\dots,e_n\} \subset D$, $Q_n(e,\cdot): P^{n-1} \to X$, where $e = (e_1,\dots,e_n)$, satisfies the following condition

$$Q_n(e,t) = Q_{n-1}(\delta_i e, \delta_i t) \quad \text{if} \quad t_i = 0, \quad i = 1,\dots,n \tag{1}$$

If Q_n, e are as in Def. 1.1, $A \subset D$ and $1 \leq n \leq \overline{\overline{A}}$, then we write

$$Q_n(e) = Q_n(e, P^{n-1}), \quad Q_n(A) = \bigcup\{Q_n(e) : \{e_1,\dots,e_n\} \subset A\} \tag{2}$$

In view of (1) it is clear that

$$Q_m(A) \subset Q_n(B), \quad A \subset B, \quad m \leq n, \quad m \leq \overline{\overline{A}}, \quad n \leq \overline{\overline{B}} \tag{3}$$

Definition 1.2

If (D,Q) is a weed in a set X, then for any $\emptyset \neq A \subset D$ $cQ\,A = \bigcup_{n\in N} Q_n(A)$, $cQ\,\emptyset = \emptyset$. If $(A \subset cQ\,A)$ $cQ\,A \subset A$, then A is (Q-underhull) Q-overhull; if $cQ\,A = A$, then A is Q-convex.

Lemma 1.3

If (D,Q) is a weed in a set X and $\emptyset \neq \mathcal{A} \subset 2^D$, then

(a) $\bigcup cQ\mathcal{A} \subset cQ \bigcup \mathcal{A}$,
(b) $cQ \bigcap \mathcal{A} \subset \bigcap cQ\mathcal{A}$,
(c) if $cQ \subset A$, $A \in \mathcal{A}$, then $cQ \bigcap \mathcal{A} \subset \bigcap \mathcal{A}$ (intersection of Q-overhulls is a Q-overhull),
(d) if $x \in cQ\{x\}$, $x \in \bigcap \mathcal{A} = B$, then $B \subset cQ\,B$, i.e. B is a Q-underhull, if in addition $\mathcal{A}$ consists of Q-convex sets, then $\bigcap \mathcal{A}$ is Q-convex.

Proof

We have $A \subset \bigcup \mathcal{A}$, $A \in \mathcal{A}$ and therefore $cQ\,A \subset cQ \bigcup \mathcal{A}$, $A \in \mathcal{A}$ (see (3)) which implies (a). Similarly as in (a), from $cQ \bigcap \mathcal{A} \subset cQ\,A$ we obtain (b). Directly from (b) follows (c). As about (d), we have $B \subset \bigcup\{cQ\{x\}: x \in B\} \subset cQ\,B$ (see (a)). Taking (c) into account, we obtain $\bigcap \mathcal{A} \subset cQ \bigcap \mathcal{A} \subset \bigcap \mathcal{A}$.

Remark 1.4

It is convenient to extend Q_n to be defined on $D^n \times P^{n-1}$, $n \in N$ in such a way that (1) holds. Let us assume that (D,) is a well ordered set. For an arbitrary $e=(e_1,\dots,e_n) \in D^n$ let $e'=(e'_1,\dots,e'_m)$ be a strongly increasing sequence of points of D such that $k \in \{1,\dots,n\}$ iff there exists a $j \in \{1,\dots,m\}$ with $e_k=e'_j$. If $e_{k_1}=\dots=e_{k_s}=e'_j$, then $t'_j:=t_{k_1}+\dots+t_{k_s}$. Now let us consider $Q_n(e,t):=Q_m(e',t')$. Crearly, condition (1) holds, and $Q_n(e)$ depends only on e' consisting of the different points. Therefore $Q_n(A)=\bigcup\{Q_n(e): e \in A^n, n \leq \bar{\bar{A}}\}$. What is more $Q_n(A)$ is defined for all $n \in N$ and $cQ\,A=\bigcup\{Q_n(A): n \in N\}$.

Definition 1.5

A pair (D,Q) is a local k-weed in X if X is a space, (1) holds for all $n \leq k$ (see 1.4) and

$$\text{for each } z \in D \text{ there exists a } U_z \text{ such that } Q_k(x,\ ): P^{k-1} \to X \text{ is continuous, } x \in (D \cap U_z)^k \tag{4}$$

If in addition $U_z=D$ in (4), then (D,Q) is a k-weed in X. For $D=X$ we say that (X,Q) is a local k-weed.

Definition 1.6

A pair (D,Q) is a (local) weed in X, if Q is as in Def. 1.1, Remark 1.4 and (D,Q) is a (local) k-weed for all $k \in N$.

Definition 1.7

We write $X \in LWkl(D)$ if there exists a Q such that (D,Q) is a local k-weed in X and the following condition is satisfied

$$\text{for each } z \in D \text{ and any } V_z \text{ there exists an } U_z \text{ such that } Q_k(D \cap U_z) \subset V_z \tag{5}$$

If (D,Q) is a local weed in X and for each $k \in N$ (5) holds, then $X \in LWl(D)$. For (D,Q) being a k-weed in X or a weed in X, we obtain $X \in Wkl(D)$ or $X \in Wl(D)$ respectively. If $D=X$, then we write $X \in (L)Wkl$ or $X \in (L)Wl$.

Proposition 1.8

If X is a T_1-space and $X \in LWk1(D)$ or $X \in LW1(D)$, then $A \subset Q_1(A)$, $A \subset cQ\,A$ respectively, for each $A \subset D$.

Proof

It is enough to consider the case $A \neq \emptyset$. If $z \in A$, then for any V_z we have $Q_1(z) \subset V_z$ and therefore $Q_1(z) \subset \{z\}$. On the other hand $Q_1(z)$ is nonempty and hence $Q_1(z) = \{z\}$. Thus $cQ\{z\} = Q_1(z) = \{z\}$ and we apply Lemma 1.3.

Now it is clear that condition (1.1) of [3]: , i.e. $Q_1(x,1)=x$, $x \in X$ is usually superfluous.

Definition 1.9

We write $X \in LW2(D)$ if (D,Q) is a local weed in X, U_z in (4) does not depend on k and

$$\text{for each } z \in D \text{ and any } V_z \text{ there exists a } U_z \text{ such that } cQ(D \cap U_z) \subset V_z \tag{6}$$

holds. If (D,Q) is a weed in X ($U_z = D$ in (4) for all $k \in N$) and (6) holds, then $X \in W2(D)$. For $D=X$ we write ($X \in LW2$) $X \in W2$.

The above definition extends the notion of the local simplicial convexity on X [3, Def. /1.9/].

It is obvious that $X \in (L)W2(D)$ implies $X \in (L)W1(D)$.

Remark 1.10

If ($X \in LW2$) $X \in W2$ and f is an r-map, then ($f(X) \in LW2$) $f(X) \in W2$. If $g: Y \to X$ is a right inverse, then the respective Q' is defined as follows $Q'_k(u,t) = f(Q_k((g(u_1),\dots,g(u_k)),t))$, $u \in (f(X))^k$, $t \in P^{k-1}$, $k \in N$ (see Rem. 1.4). This conclusion enables us to generate many spaces $X \in W2$ with the help of retractions, as any convex set in a locally convex topological vector space is of type I [16] and therefore it belongs to the class W2.

2. RETRACTIONS

R. Bielawski has proved that if X is metrizable, then $X \in AR(\mathcal{M})$ iff $X \in W2$ [3, Corol. (2.2)]. We will present here some more precise results and we will prove that $X \in ANR(\mathcal{M})$ iff X is metrizable and $X \in LW2$.

Let A,B be nonempty sets in a metric space (X,d) and let $r>0$. Then dia $A = \sup\{d(x,y): x,y\in A\}$, $d(A,B) = \inf\{d(x,y): x\in A, y\in B\}$, $B(A,r) = \{x\in X: d(A,x) < r\}$ and $\delta(A,B) = \max\{\sup\{d(x,B): x\in A\}, \sup\{d(A,y): y\in B\}\}$ (Hausdorff distance).

The lemma below will be helpful in the proofs of this section.

Lemma 2.1

Let A be a closed set in a metric space (X,d) and let $p: X\setminus A \to (0,\infty)$ be lower semi-continuous. Then for any nonempty set $B\subset A$ there exists a locally finite open cover $\mathcal{U}=\{U_s\}_{s\in T}$ of $X\setminus A$ and a set $\{y_s: s\in T\}\subset B$ such that if $x\in St(U_s,\mathcal{U})$, then $d(x,y_s)< p(x)+d(x,B)$.

P r o o f

For continuous p this result was stated in [20]. Therefore if suffices to prove that there exists a map $q: X\setminus A \to (0,p(x))$, $x\in X\setminus A$. The sets $P(x)=(0,p(x))$ are convex and for any $y\in(0,\infty)$ $P^{-}(y)=\{x: y<p(x)\}$ is open, p being lower semi-continuous. Thus $P: X\setminus A \to 2^{(0,\infty)}$ admits a continuous selection q [4, Th. 1, p. 285].

Theorem 2.2

If a is a closed set in a metric space (X,d) and $A\in W2(\mathrm{Fr}\,A)$, then A is retract of X.

P r o o f

Let $p(x) = d(x,\mathrm{Fr}\,A)$, $x\in X\setminus A$ and let $U_s;y_s$, $s\in T$ be as in Lemma 2.1 for $B=\mathrm{Fr}\,A$ and T being well ordered. For each $x\in X\setminus A$ there exists a W_x such that $T_x=\{s\in T: U_s\cap W_x\neq\emptyset\}$ is finite. We may assume that $T_x=\{s_1,\ldots,s_m\}$, $s_1\prec\ldots\prec s_m$ $(m=m(x))$. Then for $b_s(x)=d(x,X\setminus U_s)$, $s\in T$ and

$$c(x)=(b_{s_1}(x),\ldots,b_{s_m}(x))/\sum_{i=1}^{m} b_{s_i}(x),\quad x\in X\setminus A \tag{1}$$

it is seen that $c(x)\in P^{m-1}$ and $Q_m(y,c(x))$ for $y=(y_{s_1},\ldots,y_{s_m})$ is well defined (see condition (I.1), Remark 1.4). From the continuity of c on $X\setminus A$ it follows that $r: X\setminus A\to A$

$$r(x)=\begin{cases} x, & x\in A,\\ Q_m(y,c(x)), & x\in X\setminus A\end{cases} \tag{2}$$

is continuous on $X\setminus A$ (Def. 1.6). Clearly, $r(x)\in cQ[B(x,d(x,\mathrm{Fr}\,A)+p(x))\cap \mathrm{Fr}\,A]$ which guarantees the continuity of r on Fr A (see (I.6)).

From Remark 1.10 and Th. 2.2 follows [3, Corol. (2.2)].

Theorem 2.3

If A is a closed set in a metric space (X,d) and $A \in LW2(\mathrm{Fr}\ A)$ for Q, then A is retract of its neighbourhood $W \cup A$ for W defined as follows

$$W = \{x \in X\colon \text{there exists a } q>0 \text{ such that for } A(x,q)= B(x,d(x,\mathrm{Fr}\ A)+q)\cap \mathrm{Fr}\ A, \text{ all } m\in N \text{ and each } y\in (A(x,q))^m \ \ Q_m(y,\cdot)\colon P^{m-1}\to A \text{ is continuous}\} \tag{3}$$

Proof

One can state that $A\cup W$ is a neighbourhood of A (cp. the proof of [16, Th. 3]). Let us consider

$$q(x)=\sup\{q\colon Q_m(y,\cdot)\colon P^{m-1}\to A \text{ is continuous for each } y\in(A(x,q))^m,\ m\in N\},\quad x\in W\setminus A \tag{4}$$

If $(x_n)_{n\in N}$ is any sequence of points converget in $W\setminus A$, say to x, then $B(x_n,d(x_n,\mathrm{Fr}\ A)+q(x)-\varepsilon)\subset B(x,d(x,\mathrm{Fr}\ A)+q(x))$ for n big enough and therefore $\underline{\lim}_{n\to\infty}\, q(x_n)\geq q(x)$ which proves the lower semi-continuity of q. Thus we may apply Lemma 2.1 to $p(x)=\min\{d(x,\mathrm{Fr}\ A),q(x)\}$, $x\in W\setminus A$ and $X=W$. The continuity of r (defined by (2)) on $W\setminus A$ follows from (4) and (3). We have $r(x)\in cQ[B(x,2d(x,\mathrm{Fr}\ A))\cap \mathrm{Fr}\ A]$, and therefore r is continuous on Fr A.

Corollary 2.4

$X\in ANR(\mathcal{M})$ iff X is metrizable and $X\in LW2$.

Theorem 2.5

Let A be a closed subset of a metrizable space X. If $f\colon A\to Y$ is continuous and $Y\in LW2(f(\mathrm{Fr}\ A))$ or $Y\in W2(f(\mathrm{Fr}\ A))$, then f can be extended to a map $h\colon U\to Y$, where U is a neighbourhood of A in X or U=X respectively, in such a way that $h(U)\subset f(A)\cup cQ\ f(\mathrm{Fr}\ A)$.

Proof

It suffices to consider $z=(f(y_{s_1}),\dots,f(y_{s_m}))$ and $h\colon U\to Y$ defined as follows

$$h(x)=\begin{cases} f(x), & x\in A,\\ Q_m(z,c(x)), & x\in W\setminus A\end{cases}$$

for y,c as in the proof of Theorem 2.3 or 2.2 respectively.

As a corollary we obtain [3, Th. (2.1)] and [6, Th. (10.4) p. 92].

[20] contains some results for the finite dimensional spaces of the classes LWkl, Wkl. These theorems stay valid for the definitions from the present paper if we assume the respective dimensions to be $\leq k-1$. For example let us prove the following.

Theorem 2.6

Let A be a closed subset of a metrizable space X. If $f: A \to Y$ is continuous, $\dim (X \setminus A) \leq k-1$ and $Y \in LWkl(f(Fr\ A))$ or $Y \in Wkl(f(Fr\ A))$, then f can be extended to a map $h: U \to Y$, where U is a neighbourhood of A in X or $U=X$ respectively, in such a way, that $h(U) \subset f(A) \cup Q_k(f(Fr\ A))$.

Proof

In the proof of Th. 2.5 m does not exceed k as $\dim(U \setminus A) \leq k-1$ (if $Y \in LWkl(f(Fr\ A))$ then the respective neighbourhood of A exists).

Let us conclude this section with a fixed point theorem.

Theorem 2.7

(cp. [18, Th. 1])

Let (X,d) be a metric space and let $A \in W2$ be a closed subset of X. If $f: A \to X$ is a compact mapping satisfying

$$\begin{array}{l} \text{there exists a lower sem. continuous mapping } p: C=f(A)\setminus A \to (0,\infty) \\ \text{such that } 0 < d(f^{-1}(z), cQ[B(z,d(z,A)+p(z)) \cap A], \ z \in C, \end{array} \quad (5)$$

then f has a fixed point.

Proof

In view of Lemma 2.1 (see the proof of Th. 2.2) there exists a retraction $r: C \setminus A \to A$ such that for any $z \in C \setminus A$ $r(z) \in cQ[B(z,d(z,A)+p(z)) \cap A]$. Theorem 3.3 implies $r \circ f: A \to A$ has a fixed point, say $x \in A$. If $z=f(x) \notin A$, then $x \neq r(z)$. Thus $f(x) \in A$ and $x=(r \circ f)(x) = f(x)$.

The above theorem generalizes [16, Th. 4].

3. FIXED POINT THEOREMS

Definition 3.1 (cp. [1, pp. 114, 115])

Let X,Y be spaces and let $F: X \to 2^Y$. Then F is usc if for each open set $V \subset Y$ $F^+(V)=\{x\in X: F(x) \subset B\}$ is open in X.

Definition 3.2

A mapping $F: X \to 2^Y$ is (relatively) compact if it is usc and ($\overline{F(X)}$ is relatively compact) $\overline{F(X)}$ is compact.

Theorem 3.3

If $f: X \to X$ is relatively T_2-compact and $X\in W2(f(X))$, then f has a fixed point.

Proof

Suppose f has no fixed point. First we will show that there exists an open cover $\mathcal{W}=\{W_x\}_{x\in Z}$ of a compact set Z containing D=f(X) which satisfies

$$f^{-1}(W_x)\cap cQ(D\cap W_x) = \emptyset,\ x\in Z \tag{1}$$

(Q as in Def. 1.9). As $f(x)\neq x$, $x\in Z$ and Z is a T_2-space, there exist the open neighbourhoods $P = P_x$, $U=U_{f(x)}$ in X such that $P\cap U\cap Z = \emptyset$. Let us consider $V = P\cap f^{-1}(U)$. In view of (I.6) there exists an open neighbourhood W_x in Z for which $cQ(D\cap W_x)\subset V$. Thus $f^{-1}(W_x)\cap cQ(D\cap W_x) \in f^{-1}(V)\cap V\subset f^{-1}(P)\cap f^{-1}(U)= f^{-1}(P\cap U)=\emptyset$. The open cover $\mathcal{W}$ has an open locally finite star refinement $\mathcal{U}$ (Z is paracompact). We may assume $\mathcal{U}=\{U_i: i=1,\dots,m\}$ Z being compact. Let $y_i\in D\cap U_i$ (if $D\cap U_i \neq \emptyset$) and $A_i = X\setminus f^{-1}(U_i)=\bar{A}_i$, $i=1,\dots,m$. We define the mapping $h: P^{m-1}\to X$ as follows (for simplicity we assume $D\cap U_i\neq\emptyset$ for all $i\leq m$)

$$h(t)=Q_m(y,t),\ y=(y_1,\dots,y_m) \tag{2}$$

In view of Definition 1.6 h is continuous. Let $P=\{t\in P^{m-1}: t_i=0 \text{ for } i\neq i_j,\ j=1,\dots,n\}$ be a face of P^{m-1}. We are going to show that $h(P)\subset \bigcup\{A_{i_j}: j=1,\dots,n\}$. If $\bigcap U_{i_j}\neq\emptyset$, then $\bigcup U_{i_j}\subset St(U_{i_1},\mathcal{U})\subset W$ for a $W\in\mathcal{W}$. Therefore (1) implies $h(P)\subset cQ(D\cap W)\subset X\setminus f^{-1}(W)\subset X\setminus f^{-1}(\bigcap U_{i_j}) = \bigcup X\setminus f^{-1}(U_{i_j})= \bigcup A_{i_j}$.
If $\bigcap U_{i_j}=\emptyset$, then from the last two equalities we obtain $\bigcup A_{i_j}=X$. Thus the closed sets $H_i=h^{-1}(A_i)$, $i=1,\dots,m$ satisfy the assumptions of the Knaster-Kuratowski-Mazurkiewicz theorem [10, p. 134] and $\emptyset\neq\bigcap A_i=X\setminus\bigcup f^{-1}(U_i)= X\setminus X=\emptyset$. This contradiction proves that f has a fixed point.

From the above we obtain

Corollary 3.4 (cp.[15, Th. 1])

Any compact self map on a T_2-space $X \in W2(f(X))$ has a fixed point.

Obviously every locally convex topological vector space is of type I (see [16]) and therefore 3.4 is a generalization of the Tychonoff fixed point theorem. Let us quote this theorem with a shorter version of the proof of Theorem 3.3.

Theorem 3.5

Let X be a convex subset of a locally convex topological vector space. If $f: X \to X$ is continuous and $\overline{f(X)}$ is T_2-compact, then f has a fixed point.

P r o o f

Suppose f has no fixed point. Clearly, there exists an open, convex neighbourhood W of zero in X, such that

$$f^{-1}(x+W) \cap (x+W) = \emptyset, \quad x \in Z = \overline{f(X)} \qquad (1')$$

Let U be open, convex and such that $2U \subset W$ and $Z \subset \bigcup\{y_i+U:\ i=1,\ldots,m\}$. Let us consider $A_i = X \setminus f^{-1}(y_i+U) = \overline{A}_i$, $i=1,\ldots,m$. We define a map $h: P^{m-1} \to X$ as follows

$$h(t) = \sum_{i=1}^{m} t_i y_i \qquad (2')$$

For $P = \{t \in P^{m-1}:\ t_i = 0 \text{ for } i \neq i_j,\ j=1,\ldots,n\}$ we will show that $h(P) \subset \bigcup_{j=1}^{n} A_{i_j}$. If $\bigcap\{y_i+U\} \neq \emptyset$, then $\bigcup\{y_{i_j}+U\} \subset y_{i_1}+W$. Therefore (1′) implies $h(P) \subset (y_{i_1}+W) \subset X \setminus f^{-1}(y_{i_1}+W) \subset X \setminus f^{-1}(\bigcap\{y_{i_j}+U\}) = \bigcup A_{i_j}$. If $\bigcap\{y_{i_j}+U\} = \emptyset$, then the last equality implies $\bigcup A_{i_j} = X$. In view of the Knaster-Kuratowski-Mazurkiewicz theorem [10, p. 134] we have $\emptyset \neq \bigcap h^{-1}(A_i)$ and therefore $\emptyset = \bigcap A_i = X \setminus \bigcup f^{-1}(y_i+U) = X \setminus X = \emptyset$. This contradiction proves that f has a fixed point.

Theorem 3.6

If $f: X \to X$ is relatively compact with $f(X) \subset Z$, $\dim Z < \infty$ for a compact set $Z \subset X$ and $X \in W1(f(X))$, then f has a fixed point.

Proof

Assume that $\dim Z \leq k-1$. By applying (I.5) in place of (I.6) we conclude that there exists an open cover $\mathcal{W}$ of Z such that

$$f^{-1}(W_x)\cap Q_k(W_x\cap D) = \emptyset, \quad x\in Z \tag{3}$$

If $\bigcap U_{i_j} \neq \emptyset$ (see the proof of Theorem 3.3), then $n \leq k$ as we may assume $\mathcal{U}$ to be of order $\leq k-1$. Thus (3) implies $h(P) \subset Q_k(D\cap W)$. Now we can follow the remaining part of the proof of Theorem 3.3.

The corollary below is a generalization of [17, Th. 1.1].

Corollary 3.7

If f is a compact operator on a T_2-space $X\in Wl(f(X))$ and $\dim \overline{f(x)} < \infty$, then f has a fixed point.

Below we present the proper version of the theorem from [17].

Theorem 3.8

Let A be a compact set of type I in a metric space (M,d). Then $E\circ G := \bigcap_{r>0} \overline{co}S(P_r\circ G)\colon A\to C(A)$ has a fixed point if $G\colon A\to C(M)$ is compact and continuous.

As a corollary from the above we obtain:

Theorem 3.9

Let $E=\overline{conv}\, E$ be a compact set in a normed space $(X,\|\cdot\|)$. If a map $F\colon E\to (K(X),\delta)$ ($K(X)$ is the family of all closed, convex and nonempty sets in X) satisfies

$$\text{for each } x\in E \text{ if } F(x)\cap E = \emptyset, \text{ then } d(E,F(x)) < d(x,F(x)) \tag{4}$$

then $H_F\colon E\ni x\mapsto B(F(x),d(E,F(x))\cap E\in 2^E$ has a fixed point.

In the sequel we are going to give some conditions which guarantee the existence of a fixed point though the mapping under consideration is not compact.

The lemma below will not be appllied here, it seems but to be interesting enough to be worth of being proved.

Lemma 3.10

If X is a space and for $F\colon X\to 2^X\setminus\{\emptyset\}$ there exists a nonempty compact set $B\subset X$ such that $F(B)\subset B$, then there exists a nonempty set $E\subset X$ with $\overline{F(E)}=E$.

Proof

Let $\mathcal{A}=\{A=\bar{A}\in 2^X: F(A)\subset A\}$. Obviously $\mathcal{A}\neq\emptyset$ as $X\in\mathcal{A}$. Let $\mathcal{G}\subset\mathcal{A}$ be a maximal chain with respect to "$\subset$" consisting of the elements G of $\mathcal{G}$ for which $G\cap B\neq\emptyset$. Then $E=\bar{E}=\bigcap\mathcal{G}$ is nonempty as $\bigcap\{\mathcal{G}\cap B\}\neq\emptyset$ (see [7, Th. 3.1.1 pp. 166,177]). We have $F(E)=F(\bigcap\mathcal{G})\subset\bigcap F(\mathcal{G})\subset\bigcap\overline{F(\mathcal{G})}\subset\bigcap\mathcal{G}=E$. Therefore $F(\overline{F(E)})\subset F(E)\subset\overline{F(E)}$ and $\overline{F(E)}\in\mathcal{A}$. In addition $\emptyset\neq F(E\cap B)\subset F(E)\cap F(B)\subset E\cap B$ which means $F(E)\in\mathcal{G}$ and $\overline{F(E)}=E$ $\mathcal{G}$ being maximal.

Lemma 3.11

If (Y,Q) is a weed in the set X and for $F: X\to 2^Y\setminus\{\emptyset\}$ there exists a nonempty compact set B in space X such that $F(B)\subset B$, then there exists a nonempty set $E\subset X$ for which $\overline{cQ}\,F(E)=E$ holds.

Proof

Let $\mathcal{A}=\{\bar{A}=A\in 2^X: \overline{cQ}\,F(A)\subset A\}$. Obviously $\overline{cQ}\,F(X)\in\mathcal{A}$. Let $E,\mathcal{G}$ be defined as in the proof of Lemma 3.10 (for the "new" $\mathcal{A}$). We can see that $\overline{cQ}\,F(E)=$ $=\overline{cQ}\,F(\bigcap\mathcal{G})\subset\overline{cQ}\bigcap F(\mathcal{G})\subset\bigcap\mathcal{G}=E$ (cp. Lemma 1.3, Prop. 1.8). Thus $cQ\,F(\overline{cQ}\,F(E))$ $\subset cQ\,F(E)$ and $\overline{cQ}\,F(E)\in\mathcal{A}$. In addition $\emptyset\neq F(E\cap B)\subset F(E)\cap F(B)\subset E\cap B$ implies $B\cap\overline{cQ}\,F(E)\neq\emptyset$. Thus $\overline{cQ}\,F(E)$ and in view of the maximality of $\mathcal{G}$ we have $\overline{cQ}\,F(E)=E$.

Definition 3.12 (cp. [17, Def. 3.4 p. 786])

Let (Y,Q) be a weed in the set X. A mapping $F: X\to 2^Y\setminus\{\emptyset\}$ is generally condensing if X is a space and the following conditions are satisfied

for each $E\subset X$, if $F(E)\subset E$ and $\mathrm{card}(E\setminus\overline{F(E)})\leq 1$ then $\overline{F(E)}$ is compact (5)

for each $\emptyset\neq E\subset X$, if $E=\overline{cQ}\,F(E)$, then $F|E$ is relatively compact (6)

Lemma 3.13

If (Y,Q) is a weed in the set X and $F: X\to 2^Y\setminus\{\emptyset\}$ is generally condensing for a space X such that $\{\overline{x_0}\}=\{x_0\}$ for a $x_0\in X$, then there exists a set $E=\overline{cQ}\,F(E)$ such that $F|E$ is relatively compact.

Proof

We will show that the assumptions of Lemma 3.11 are satisfied. Let $\mathcal{G}=\{G=\bar{G}\subset X: x_0\in G$ and $F(G)\subset G\}$. Clearly, $X\in\mathcal{G}\neq\emptyset$. For $B=\bigcap\mathcal{G}$ we have $x_0\in B\neq\emptyset$ and $F(B)=F(\bigcap\mathcal{G})\subset\bigcap F(\mathcal{G})\subset\bigcap\mathcal{G}=B$. Hence $F(\overline{F(B)}\cup\{x_0\})=F(\overline{F(B)})\cup$ $\cup F(x_0)\subset\overline{F(B)}\cup\{x_0\}$ and therefore $B=\overline{F(B)}\cup\{x_0\}$. In view of (5) B is compact. Now we apply Lemma 3.11 and (6).

Theorem 3.14

Let f be a generally condensing self map on a T_1-space $X \in W2(f(X))$ ($X \in W1(f(X))$ and $\dim Z < \infty$ for a closed set $Z \subset X$ containing $f(X)$). Then f has a fixed point.

Proof

In view of Lemma 3.13 there exists a set $E = \overline{cQ}\, f(E)$ such that $f|E$ is relatively compact. We have $E \in W2(f(E))$ ($E \in W1(f(E))$ as $f(E) \subset f(X)$. Thus in view of Theorem 3.3 (3.6) f has a fixed point (see [7, Th. 7.1.8 p. 474]).

Definition 3.15

A mapping $b: 2^X \to \langle 0, \infty \rangle$ is a measure of noncompactness on a space X if

$$\text{for each } E \subset X \quad b(E) = 0 \text{ iff } E \text{ is relatively compact} \tag{7}$$

$$\text{for each } C, E \subset X \quad \text{if } C \subset E, \text{ then } b(C) \leqslant b(E) \tag{8}$$

Definition 3.16 (cp. [8, p. 7])

Let (Y,Q) be a weed in a set X. A mapping $F: X \to 2^Y \setminus \{\emptyset\}$ is generally b-condensing if b is a measure of noncompactness on the space X and

$$\text{for each } \emptyset \neq E \subset X \quad \text{if } F(E) \subset E \text{ and } b(E) \leqslant b(\overline{cQ}\, F(E))$$
$$\text{then } F|E \text{ is compact} \tag{9}$$

Lemma 3.17

Let (Y,Q) be a weed in the set X and $y \in cQ\,\{y\}$, $y \in Y$. If b is a measure of noncompactness on X and

$$b(E \cup \{x\}) = b(E), \; E \subset X, \; x \in X, \tag{10}$$

then every generally b-condensing mapping is generally condensing.

Proof

Let $F: X \to 2^Y$ be generally b-condensing. If $F(E) \subset E \neq \emptyset$ and $\operatorname{card}(E \setminus \overline{F(E)}) \leqslant 1$, then (8), (10) imply $b(E) \leqslant b(\overline{F(E)} \cup \{x\}) = b(\overline{F(E)}) \leqslant$ $\leqslant b(\overline{cQ}\, F(E))$ (see Lemma 1.3 (d)). Therefore (9) can be applied and we obtain the compactness of $\overline{F(E)}$, i.e. (5) is satisfied. If $\emptyset \neq E = \overline{cQ}\, F(E)$, then $b(E) \leqslant b(\overline{cQ}\, F(E))$ (see ((8)) and (6) follows from (9).

It is seen that condition (7) of the classical definiton can be disregarded here.

4. SELECTIONS

Definition 4.1 [1, pp. 114, 115]

Let X,Y be spaces and let $F\colon X \to 2^Y$. Then F is lsc if for each open set $V \subset Y$ $F^-(V) = \{x \in X\colon F(x) \cap V \neq \emptyset\}$ is open in X. We say that F is continuous if it lsc and usc (see Def. 3.1). Selection means continuous selection in this paper. First let us recall some properties of the lsc mappings.

Proposition 4.2 [11, Ex. 1.3' p. 362]

If $F\colon X \to 2^Y$ is lsc, $C = \bar{C} \subset X$ and $g\colon C \to Y$ is a selection for $F|C$, then mapping $G\colon X \to 2^Y$

$$G(x) = \begin{cases} g(x), & x \in C, \\ F(x), & x \in X \setminus C \end{cases} \tag{1}$$

is lsc.

Proposition 4.3 (cp. [11, Prop. 2.3 p. 366])

A mapping $F\colon X \to 2^Y$ is lsc iff $\bar{F}$ is lsc.

The two facts below are probably known.

Lemma 4.4

Let X be a space and (Y,d) a metric space. If $F\colon X \to 2^Y \setminus \{\emptyset\}$ satisfies

$$\text{for each } p > 0,\ x \in X \text{ there exists a } U_x \text{ such that for any } z \in U_x \quad F(x) \subset B(F(z),p) \tag{2}$$

then F is lsc. If the values of F are compact and F is lsc then (2) holds.

Proof

Let $V \subset Y$ be open. If $x \in F^-(V)$, then there exists a point $y \in F(x) \cap V$. Thus for a $p > 0$ $B(y,p) \subset V$. It follows that for all $z \in X$ such that $F(x) \subset B(F(z),p)$ we have $F(z) \cap V \neq \emptyset$ and (2) implies $F^-(V)$ is a neighbourhood of x. The second part of 4.4 follows from [12, Lemma 11.3 p. 578].

Let us recall that δ is the Hausdorff metric (section 2) in the family $T(Y)$ of all closed nonempty subsets of Y.

Proposition 4.5 (cp. [1, Th. 1 p. 133])

If (Y,d) is a metric space and $F\colon X \to (T(Y),\delta)$ is a map, then mapping F is lsc.

P r o o f

The set $\{z\in X: F(x)\subset B(F(z),p)\}$ contains the neighbourhood $F^{-1}(B_\delta(F(x),p))$ of x and therefore (2) holds.

R. Bielawski has proved a theorem on selections [3, Th. /3.5/]. The terminology connected with weeds is more precise and we will formulate a generalization of this theorem for weeds.

Definition 4.6

Let (X,d) be a metric space. Then $X\in W4(\mathcal{A})$ if $\mathcal{A}\subset 2^X\setminus\{\emptyset\}$, and $X\in W2(\bigcup\mathcal{A})$ for a Q such that

$$\text{for each } q>0 \text{ there exists a } p>0 \text{ such that for any } A\in\mathcal{A}\ cQ(B(A,p)\cap\bigcup\mathcal{A})\subset B(cQ\ A,q) \qquad (3)$$

If $\mathcal{A}=2^X\setminus\{\emptyset\}$, then we write $X\in W4$.

One can define $LW4(\mathcal{A})$ with $LW2(\bigcup\mathcal{A})$ in place of $W2(\bigcup\mathcal{A})$.

Lemma 4.7

Let X be a space and (Y,d) a metric space $Y\in W4(\{F(x): x\in X\})$. If $F: X\to 2^Y\setminus\{\emptyset\}$ satisfies (2), then $cQ\circ F$ is lsc.

P r o o f

For any $q>0$ there exists a $p>0$ such that $cQ[F(X)\cap B(F(z),p)]\subset$ $\subset B(cQ\ F(z),q)$, $z\in X$ which means (2) holds for $cQ\circ F$ and in view of Lemma 4.4 $cQ\circ F$ is lsc.

Corollary 4.8

Let X be a space and let (Y,d) be a metric space $Y\in W4(\{F(x): x\in X\})$. If the values of $F: X\to 2^Y$ are compact and F is lsc, then $\overline{cQ}\circ F$ is lsc.

P r o o f

See 4.4, 4.7, 4.3.

We will say that a partition of unity $\mathcal{G}$ has a property which is suitable for covers if $\mathcal{U}_\mathcal{G}=\{g^{-1}((0,1\rangle): g\in\mathcal{G}\}$ has the respective property.

The following (known) fact will be helpful.

Theorem 4.9

Every open cover of any paracompact space (see [9, p. 156]) has a star refinement being a locally finite partition of unity.

P r o o f

Any open cover $\mathcal{W}$ of a paracompact space X has a closed locally finite refinement [9, Lemma 29 p. 157] X being normal [9, Corol. 32 p. 159]. Thus $\mathcal{W}$ has an open star refinement $\mathcal{V}$ [7, Lemmas 5.1.13, 5.1.15 p. 377]. The cover $\mathcal{V}$ has an open locally finite refinement and there exists a locally finite partition of unity $\mathcal{G}$ which refines $\mathcal{V}$ [9, W p. 171]. Thus $\mathcal{G}$ is a locally finite partition of unity being a star refinement of $\mathcal{W}$.

If Y is a uniform space and (D,Q) is a weed in Y, then C(D) means a family of all nonempty, complete and Q-convex subsets of D.

Theorem 4.10 (cp. [3, Th. /3.5/])

Let X be a paracompact space and (Y,d) a metric space $Y\in W4(\{F(x)\colon x\in X\}\cup\{\{y\}\colon y\in F(X)\})$ for $F\colon X\to C(D)$ being lsc. Then F admits a selection.

P r o o f (cp. [11] proof of Th. 3.2'', [19] proof of Th. 1)

From the fact that $\mathcal{V}=\{F^{-}(B(y,p_1))\}_{y\in F(X)}$ is an open cover of X it follows that there exists a locally finite partition of unity $\mathcal{G}$ being a barycentric refinemnt of $\mathcal{V}$. Let $\mathcal{U}_{\mathcal{G}}=(U_s)_{s\in T}$ for a well ordered set T. For each $x\in X$ there exists a W_x such that $T_x=\{s\in T\colon U_s\cap W_x\neq\emptyset\}$ is finite, say $T_x=\{s_1,\dots,s_m\}$, $s_1\prec\dots\prec s_m$ $(m=m(x))$. Let $y_s\in F(x)$ for an $x\in U_s$, $s\in T$, $g(x)=(g_{s_1}(x),\dots,g_{s_m}(x))$, $x\in X$. Then

$$f_1(x)=Q_m(y,g(x)),\quad x\in X \tag{4}$$

defines a map $f_1\colon X\to Y$ (see (I.1), Def. 1.6) and $f_1(x)\in cQ[F(X)\cap B(F(x),p_1)]\subset B(cQ\,F(x),q_1)=B(F(x),q_1)$ $F(x)$ being Q-convex. Let $F_1(x)=F(x)\cap B(f_1(x),q_1)$. We have $F_1(x)\neq\emptyset$, $x\in X$ and $F_1^{-}(V)=F^{-}(V)\cap f_1^{-1}(B(V,q_1))$ is open if $V\subset Y$ is open which means F_1 is lsc. Now we repeat the above reasoning for $p_2=p_2(q_2)$ and F_1 in order to define $f_2(x)\in cQ[F(X)\cap B(F_1(x),p_2)]\subset B(cQ\,F_1(x),q_2)\subset B(F(x),q_2)$, $x\in X$ (see (I.3)). Afterwards we define $F_2(x)=F(x)\cap B(f_2(x),q_2)$, $x\in X$. By induction we obtain

$$f_n(x)\in cQ[F(X)\cap B(F_{n-1}(x),p_n)]\subset B(cQ\,F_{n-1}(x),q_n))\subset \subset B(F(x),q_n),\quad x\in X \tag{5}$$

and

$$F_n(x)=F(x)\cap B(f_n(x),q_n),\quad x\in X \tag{6}$$

From (6) follow $\operatorname{dia} F_n(x)\leq 2q_n$, $d(f_n(x),F_n(x))<q_n$. In view of (5) we get

$$d(f_n(x),f_{n+1}(x)) \leq d(f_n(x),F_n(x)+\text{dia } F_n(x)+d(F_n(x),f_{n+1}(x)) \leq$$

$$\leq q_n+2q_n+\text{dia cQ } F_n(x)+q_{n+1} \leq 3q_n+\sup\{\text{dia } [cQ\ B(y,2q_n)\cap F(X)]:$$

$$y\in F(X)\} + q_{n+1} = 3g_n+r_n+q_{n+1}$$

From (3) it follows that we may write $q_n = q_n(r_n)$, $n\in N$ and $\sum(q_n+r_n) < <\infty$ as $q_n(r_n) \to 0$ if $r_n \to 0$. Thus we have $d(f_m(x),f_n(x)) \leq d(f_m(x), f_{m+1}(x))+\ldots+d(f_{n-1}(x),f_n(x)) \leq \varepsilon$, $x\in X$ if $n_o \leq m \leq n$. Hence $(f_n)_{n\in N}$ is a uniformly convergent sequence of maps, say to f and $f(x)\in F(x)$, $x \in X$ (see (5)).

It is interesting to compare the weeds as in Def. 4.6 with the respective conditions of the convex structure of Michael [14, Def. 1.1 p. 558]. Let us assume that $M_n = D^n$ and let $Q_n(x,t)=k_n(x,t)$ (see Remark 1.4), then (b) is identic with (I.1). The continuity assumption (d) is identic with that of Def. 1.6. From (e) it follows that $Q_n(D\cap B(A,p))\subset B(Q_n(A),q)$, $A\in\mathcal{A}$, $D=\cup\mathcal{A}$, $n\in N$ and therefore $cQ[D\cap B(A,p)] \subset B(cQ\ A,q)$, i.e. (3). It seems to be difficult to derive (e) from (3). Condition (c) can be disregarded. In consequence (see condition 3) of [5]) Theorem 4.10 is more general than the Curtis result [5, Th. 2.2].

Theorem 4.11 (cp. [14, Th. 1.5 p. 559])

Let X be paracompact and (Y,d) a metric space. Let $F: X\to C(D)$ be lsc, $Y\in W4(\{F(x): x\in X\} \cup \{\{y\}: y\in F(X)\})$ and let g be a selection for $F|C$ where $C=\bar{C} \subset X$. Then g can be extended to a selection f for F.

P r o o f

Let us consider G as in (1). It is clear that $G(X)\subset F(X)$, $G: X\to C(D)$ (see Prop. 1.8) and G is lsc (Prop. 4.2). Thus in view of Theorem 4.10 G admits a selection f and obviously $f|C=g$.

Remark 4.12

If F(X) is compact, then it is enough to assume $Y\in W4(\{F(x): x\in X\})$ in 4.10, 4.11 as then $\sup\{\text{dia } [cQB(y,2q_n)\cap F(X)]: y\in F(X)\}$ tends to zero as $q_n \to 0$ (the standard proof of this fact based on (I.6) is left to the reader). It is possible to assume that (I.6) holds uniformly on F(X) to simplify, in the same way, the type of the weed (cp.[3, Corol. /3.6/]).

Theorem 4.13 (cp. [13, Th. 8.1 p. 388])

Let X be paracompact, (Y,d) a metric space $Y\in W4(\{F(x): x\in X\}\cup\{\{y\}: y\in F(X)\})$, $F: X \to (C(D),\delta)$ being a map. Then if $C = \bar{C} \subset X$ and g is a selection for $F|C$, g can be extended to a selection for F.

P r o o f

The mapping $F: X \to C(D)$ is lsc (Corollary 4.8) and 4.13 follows from Theorem 4.11.

Theorem 4.14 (cp. [2, Corol. 7.5 p. 92])

Let C be a closed subset of a paracompact space X, (Y,d) a complete metric space $Y \in W4(\{\{y\}: y \in Y\})$. Then any map $g: C \to Y$ can be extended to a map $f: X \to Y$.

P r o o f

In view of Proposition 4.2 $G: X \to C(Y)$ defined by (1) for $F(x)=Y$, $x \in X \setminus C$ is lsc and we apply Theorem 4.11.

Let us modify the Bartle-Graves theorem.

Theorem 4.15 (cp. [11, p. 364])

Let $f: Y \to X$ be an open mapping of a metric space (Y,d) $Y \in W4$ in a paracompact space X. If $f^{-1}(x)=cQ\ f^{-1}(x) \in C(Y)$, $x \in X$, then there exists a map $g: X \to Y$ such that $f \circ g = id_X$, i.e. $g(x) \in f^{-1}(x)$, $x \in X$.

P r o o f

In view of Theorem 4.10 it is enough to prove that $F=f^{-1}$ is lsc. If $V \subset Y$ is open, then $F^{-}(V)=\{x \in X: f^{-1}(x) \cap V \neq \emptyset\}=f(V)$ is open.

Now let us consider the finite dimensional case.

Definition 4.16

Let (X,d) be a metric space. Then $X \in Wk3(\mathcal{A})$ $(X \in W3(\mathcal{A}))$ if $\mathcal{A} \subset 2^X \setminus \{\emptyset\}$, and $X \in Wk1(\cup \mathcal{A})$ $(X \in W1(\cup \mathcal{A}))$ for a Q such that

for each $q > 0$ there exists a $p > 0$ such that for any $A \in \mathcal{A}$ $Q_k(B(A,p) \cap \cup \mathcal{A}) \subset B(Q_k\ A,q)$. (7)

If $\mathcal{A} = 2^X \setminus \{\emptyset\}$, then we write $X \in Wk3$ $(X \in W3)$.

One can define $LWk3(\mathcal{A})$, $LW3(\mathcal{A})$ with $LWk1(\cup \mathcal{A})$ or $LW1(\cup \mathcal{A})$ respectively in place of $Wk1(\cup \mathcal{A})$, $W1(\cup \mathcal{A})$ (see Def. 1.7).

It is convenient to extend the definition of the covering dimension [7, p. 472] as follows.

Definition 4.17

Let X be a paracompact space [9, p. 156]. Then $\dim X \leq k$ if for each open cover $\mathcal{W}$ of X there exists a partition of unity of order $\leq k$ which refines $\mathcal{W}$.

The above definition is proper as for any paracompact T_1- space (normal and T_2 [7, Th. 5.1.5 p. 373]) from dim $X \leq k$ follows the existence of the respective partition of unity [7, Th. 7.2.4 p. 484], [9, W p. 171].

If Y is a uniform space and (D,Q) is a k-weed in Y, then $C_k(D)$ means the family of all nonempty, complete subsets A of D for which $Q_k(A) = A$.

Now we can state the "finite-dimensional" version of Theorem 4.10.

Theorem 4.18

Let X be a paracompact space, dim $X \leq k-1$ and (Y,d) a metric space $Y \in Wk3(\{F(x) : x \in X\} \cup \{\{y\} : y \in F(X)\})$ for $F : X \to C_k(D)$ being lsc. Then F admits a selection.

Proof

We follow the proof of Theorem 4.10 by writing Q_k, C_k in place of cQ,C respectively for $\mathcal{U}_\xi$ being of order $\leq k-1$.

To obtain the "finite-dimensional" versions of 4.7, 4.8, 4.10 - 4.15 we write Q_k in place of cQ we additionally assume dim $X \leq k-1$.

It is possible to state a general version of the Browder theorem [4, Th. 1.3 p. 285].

Theorem 4.19 (cp. [3, Prop. /3.9/])

Let X be a paracompact space [9, p. 156] (and dim $X \leq k-1$). If $F : X \to 2^Y \setminus \{\emptyset\}$ is such that for each $x \in X$ (F(x),Q) is a weed (k-weed) in Y, $cQ\,F(x) \subset F(x)$ $(Q_k(F(x)) \subset F(x))$, and $F^-(y)$ is open for each $y \in Y$, then F admits a selection.

Proof

It is enough to consider $\mathcal{V} = \{F^-(y)\}_{y \in F(X)}$ in the proof of Theorem 4.10 (4.18) and then f_1 defined by (4) is a selection for F.

We conclude this section with two fixed point theorems.

Theorem 4.20

Let X be a (finite-dimensional) paracompact space $X \in W2(F(X))$ $(X \in W1(F(X)))$ for $F : X \to 2^X \setminus \{\emptyset\}$. If the values of F are Q-convex, $\overline{F(X)}$ is compact and $F^-(y)$ is open for each $y \in X$, then F has a fixed point.

Proof

Every paracompact space is normal [9, Corol. 32 p. 159] and in view of Theorem 4.19 F has a selection f. From Theorem 3.3 (3.6) it follows that f has a fixed point as $f(X) \subset F(X)$.

Theorem 4.21

If (X,d) is a (finite-dimensional) metric space $X \in W4(\{F(x): x \in X\} \cup \{\{x\}: x \in F(X)\})$ ($X \in W3(\ldots)$), $F: X \to C(D)$ is lsc and $F(X)$ is compact, then F has a fixed point.

P r o o f

In view of Theorem 4.10 (4.18) F admits a selection f. From Theorem 3.3 (3.6) it follows that f has a fixed point as $f(X) \subset F(X)$.

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Streszczenie

Zastosowania chwastów

Praca zawiera twierdzenia o retrakcjach, punktach stałych i ciągłych selekcjach dla pewnego rodzaju struktury wypukłej zwanej chwastem. Twierdzenia 2.2, 2.3, 2.5 wzmacniają klasyczne dla teorii retraktów wyniki Dungundji i Himmelberga. Znane twierdzenie Hukuhary uzyskuje się bezpośrednio z Tw. 3.5, które jest wnioskiem z 3.3. Jeśli chodzi o selekcje, to najważniejsze Tw. 4.11 uogólnia odpowiedni rezultat z pracy Michaela [14].