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A FIXED POINT THEORY AND SOME OTHER APPLICATIONS OF WEEDS

ZESZYTY NAUKOWE

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Contents

Lech Pasicki

A FIXED POINT THEORY AND SOME OTHER APPLICATIONS OF WEEDS

S u m m a r y. The present paper is devoted to studying of a convex structure called weed. The idea is to replace convex combinations which are natural for linear spaces, by a sequence of mappings for topological spaces (see Def. 1.2.7, p. 18). The convex sets are replaced by overhulls (Def. 1.2.2, p. 16).

We are mainly interested in the fixed point theory, and the results we obtain extend the well known classical theorems even for the case of locally convex spaces. Two final chapters are more complicated as regards the assumptions on weeds under consideration and they are devoted to retractions and continuous selections, respectively.

COMMENTS

The aim of this paper is to present some applications of a convex structure called weed. The idea in brief is to replace the convex combinations $\Sigma \{t_i e_i : i = 1,\dots,n\}$ which are proper for linear spaces, by $Q_n(e,t)$, $e \in E^n$, $t \in P^{n-1}$ (simplex) for topological spaces. The "algebraic" assumptions are very natural (see (1.1), p. 16) and we require the continuity of Q_n only on t (see Def. 1.2.7, p. 18).

A convex structure with such a weak continuity assumption has appeared in print in [3]. Prof. Bielawski and the author of the present paper have defined similar structures at the same time though the idea itself was present in the proofs of [P1, P2, P3].

It should be stressed that numerous mathematicians (e.g. Keimel, Wieczorek) developed an abstract theory of convexity with applications to the fixed point theory and other items where the idea of convex sets in usual sense is present. The difference is that convex hulls cQ (see Def. 1.2.2, p. 16) need not be members of convexity, the definitions are not numerous, and we are interested in the classical consequences.

The main subject here is the fixed point theory and its applications. The proofs are simple and elementary and the results are fairly general (e.g. the spaces are usually not T_1). The situation is especially interesting for weeds being convex subsets of linear topological spaces (with conv in place of cQ). The general theorems lead in particular to natural generalizations of well known facts (locally convex spaces need not be Hausdorff), they are Nos 2.2.4, 2.2.10, 2.2.13. 2.2.19, 2.2.21, 4.2.1, 4.2.3, 4.2.5, 4.3.2, 4.3.5, 4.3.8, 4.3.9, 4.3.10, and probably completely new 3.1.4, 3.1.8. The two final chapters are devoted to some results in the theory of retracts and continuous selections. They are more advanced as regards the structures under consideration.

The first chapter gathers the fundamental facts which will be applied in the remaining parts of this paper. In particular Section 1.2 is devoted to the definition of a weed and simple properties of hulls; 1.4 contains some basic facts concerning KKM mappings and c-compactness which are useful in proving fixed point theorems and other results. As regards 1.3, we develop there an example of a weed defined by S-contractibility which is a generalization of equiconnectedness.

The second chapter contains general theorems which are fundamental for the fixed point results of the chapters to follow. In the first section we extend the classical definition of usc mapping to the set-to-set case being of natural interest in this paper.

The main subject of the third chapter are the fixed point theorems for composition of functions. They are equivalent to fixed point theorems for the Cartesian product of functions - the so-called cross theorems. The two element cross theorems can be identified with coincidence results. Sections 3.2, 3.3 apply to 4.1.

The first section of Chapter 4 is devoted to minimax results, the two other ones are "classical" as they concern linear topological spaces. In section two we consider the problem of nearest points, and the third one consists of a fixed point and coincidence theorems for upper hemicontinuous mappings (Def. 4.3.1, p. 61) including the inward and outward mappings.

The fifth chapter contains specialized theorems. In the first section we extend the results on convex sections and related topics, the subsequent one is devoted to nonempty intersections of families of sets with measures of noncompactness involved. The third section concerns fixed point theorems for maps which are "almost" compact.

In Section 6.1 the original definition of a weed is specialized to enable us to obtain fairly general results on retractions. Some theorems have a stronger form for finite dimensional spaces. The last section of this chapter is devoted to comparing the various concepts of spaces equipped with structure functions.

Chapter 7 contains theorems on continuous selections and extensions for a weed with more restrictive assumptions than in Chapter 6.

The contents of the paper can be described in a more detailed way as follows.

1. Basic definitions and properties

1.1. Terminology. In this paper we are interested mainly in the set-to-set functions. The notations are somewhat specific though they agree in general with [2]. For $F\colon 2^X \to 2^Y$, $H\colon 2^Y \to 2^Z$ $H \circ F$ is defined by $(H \circ F)(A) = H(F(A))$, $A \subset X$; on the other hand we have $F^-(B) = \bigcup\{F^-(y)\colon y \in B\}$, $B \subset Y$. We adopt Kelley's terminology concerning the separation axioms. In place of linear topological space we write linear space whereas the term linear space itself is replaced by linear structure (it appears incidentally). The notation

$\partial_i x$ which is presented in the first part of this section is applied in the crucial Definition 1.2.1. The list of all abbreviations employed in the text is the last point of this section.

1.2. **Weeds.** The two basic definitions are 1.2.1 and 1.2.7. The notion of a convex hull which is proper for linear space is replaced by overhulls and underhulls (Def. 1.2.2). Their properties can be found in Lemmas 1.2.4, 1.2.6. In view of Remark 1.2.5 overhull is a natural extension of the notion of convex hull in classical sense. As regards the examples, the most important one for the proofs of this paper is No 1.2.12. In consequence we note that Def. 1.2.7 extends [P6. Def. 1.6]. Example 1.2.11 is interesting as in particular we state that every equiconnected space (see Def. 6.3.1) can be treated as a weed.

1.3. **Weeds and S-contractible spaces.** This section is devoted to the investigation of the problem when a mapping $S: X \times I \times X \longrightarrow X$ defines a structure of a weed on X. The most important result is Lemma 1.3.4 (see Def. 1.3.1).

1.4. **KKM mappings.** Definition 1.4.2 of KKM mapping extends the respective idea of Dugundji and Granas [13] and of Bielawski [3]. The main results of this section are Theorems 1.4.7 and 1.4.9. They are related to the notion of c-compactness (Def. 1.4.6) which generalizes Lassonde's Def.3 [36, p. 154].

2. General fixed point theorems

2.1. **Upper semicontinuity.** The notion of the upper semicontinuous (usc) mappings is extended to increasing (Def. 2.1.1) set-to-set functions. Some well-known properties of usual usc mappings are preserved in the general case. In the final part of this section we present two lemmas (2.1.14, 2.1.15) which are of importance for the upper semicontinuity of $\overline{cQ}\circ F$; it is convenient to think in terms of linear spaces for conv in place of cQ. In particular, if Z is a convex set in a locally convex space, $F: X \longrightarrow 2^Z$ is usc and $\overline{\text{conv}}\, F(x)$ is compact, $x \in X$, then $\overline{\text{conv}}\circ F$ is usc (Corol. 2.1.16).

2.2. **Fixed point theorems.** The theorems presented here seem to be abstract. Yet it suffices to treat X as being a convex set in a linear space, and $\mathring{E}$ as a subset of X, and Q_n to be convex combinations (see (1.2), p. 17), then the results become "classical". The most important theorems here are 2.2.3, 2.2.8 and 2.2.11. As consequences we obtain generalizations of some important results: 2.2.4 (Browder), 2.2.10 (Fan, Glicksberg), 2.2.13 (Lassonde). In the final part of this section we present a theorem (2.2.20) which leads to a natural extension of one of Hukuhara's results (2.2.21).

3. Cross theorems, coincidences

3.1. Cross theorems. In order to prove fixed point theorems for composition we use a Cartesian product of mappings. This enables us to avoid the problem of non-convex values. The most illustrative are classical Theorems 3.1.5 and 3.1.8. Theorem 7.2 enables us to connect these cases (Remark 3.1.6).

3.2. Coincidences, case $T,H: X \to 2^Y$. The two element cross theorems are related to coincidences. Our main aim in this section is to prove more specialized theorems (3.2.2, 3.2.4) which enable us to avoid an approximation technique in proofs of Chapter 4.

3.3. Coincidences, case $h: X \to Y$, $H: X \to 2^Y$. In order to obtain the results on coincidences we apply Theorems 1.4.5, 1.4.7 which give better effects than cross theorems. Our main interest here are theorems 3.3.4 and 3.3.5 as they are applied in Chapter 4. The remaining results can be divided in two parts: for mappings H with open values (Theorems 3.3.1, 3.3.2), for $\overline{cQ \circ H^-}$ being usc (3.3.7 and subsequent theorems).

4. Examples of applications

4.1. Inequalities. This section is devoted to minimax results. The most important are generalizations of some well known theorems 4.1.7 (Sion), 4.1.10 (Fan), 4.1.11 (Fan), 4.1.12 (Allen). All of them extend the respective results of Lassonde.

4.2. Special inequalities for linear spaces. At first we extend a fixed point theorem of Browder (cp. 2.2.5). As regards the other results of this section, we are interested mainly in theorems on nearest points (4.2.5, 4.2.6) which are consequences of Th. 4.2.3.

4.3. Fixed point theorems. This section concerns fixed point and coincidence theorems for uhc mappings (Def. 4.3.1). Theorem 3.2.2 enables us to prove an elegant generalization of Lassonde's theorem (here Th. 4.3.2). Further results concern theorems with boundary conditions. They generalize the well known results of Fan (e.g. 4.3.5), Browder and Halpern for the inward and outward mappings (4.3.10). Our quite clear proofs are based on Th. 4.2.8.

5. Nonempty intersections, measures of noncompactness

5.1. On a theorem of Fan. We extend theorems of Fan and Lassonde on convex sections (Th. 5.1.1) for subsets of the Cartesian product of spaces. In consequence some other generalizations of classical results are obtained 5.1.2 (Fan, Lassonde), 5.1.3 (Nash), 5.1.4 (Fan) and 5.1.5 (Browder).

5.2. On a theorem of Cellina. We extend results of Kuratowski (5.2.2) and Cellina (5.2.3) concerning a nonempty intersection of family of sets, with measures of noncompactness involved.

5.3. Condensing mappings. We define compacting (Def. 5.3.3) and b-condensing mappings (Def. 5.3.6) - in view of Lemma 5.3.7 this last notion seems to be of less interest. These definitions extend some ideas of Daneš (α-generalized concentrative mapping, $\alpha = cQ$) and Hahn ((Ψ)-kondensirend Abbildung). Our main result in this section is Theorem 5.3.5.

6. Retractions

6.1. General theorems. This section begins with Theorem 6.1.1 which can be of general interest for metric spaces. Main definitions are 6.1.2 and 6.1.5. For weeds satisfying these more restrictive conditions we prove Theorems 6.1.3 and 6.1.6 on retractions, and their immediate consequences are Theorems 6.1.4 and 6.1.7 on continuous extensions. It seems that Theorem 6.1.10 is of interest for the general theory of retracts. As a consequence of 6.1.6 and 2.2.20 we obtain a fixed point theorem 6.1.12.

6.2. Finite dimensional case. It appears that we can relax the assumptions on the structure of a weed, if the space under consideration is finite dimensional. The respective analogs of Theorems 6.1.4 and 6.1.7 are Theorems 6.2.3 and 6.2.5. Th. 6.2.7 is a finite dimensional version of Th. 6.1.10. Theorem 6.2.8 provides information about the dependences between some specializations of weeds and AR($\mathcal{M}$) (ANR($\mathcal{M}$)). The final part is devoted to comparing various types of weeds.

6.3. The notions introduced in 6.1, 6.2 and the general definition of weed are compared here with some concepts of Dugundji, Fox (equiconnecting function) and Himmelberg (CS spaces).

7. Selections

Theorem 7.2 extends a little the respective result of Bielawski as paracompact space need not be T_2. Our main result of this chapter is Theorem 7.11. The convex structure under consideration (Def. 7.8) is more general than those of Michael [40] and Curtis [8] (see the comment following Remark 7.12) and a little more general than Bielawski's one [3, Th. (3.5)]. Theorem 7.18 is a finite dimensional version of Th. 7.11. We obtain some results on extensions of selections (7.13, 7.14) and a generalization of Bartle-Graves theorem (7.15).

1. Basic definitions and properties

1.1. TERMINOLOGY

First we present the most specific conventions of this paper. The domain of each mapping is assumed to be nonempty. The term linear space is replaced by the notion of linear structure. The very word "space" means a "topological space". Thus a linear space is a linear topological space. Sets U_x, V_x, W_x are neighbourhoods of x in a fixed space.

In the present paper the composition of functions is of importance. We define it by the equality $(H \circ F)(A) = H(F(A))$. The notation $H(F(A))$ should be fairly general as H, F may be point-to-point, point-to-set and set-to-set. In this connection we should also avoid the symbol F^{-1} which concerns the inverse relation and is inconvenient, e.g. for point-to-set function F. Therefore, we prefer the following concepts. The notation $F: \mathcal{A} \to \mathcal{B}$, where $\mathcal{A}$, $\mathcal{B}$ are nonempty families of sets, means that for each $A \in \mathcal{A}$ $F(A)$ is a member of $\mathcal{B}$. The family of all subsets of a set X is written 2^X. If $F: 2^X \to \mathcal{B}$, then $F(\{x\})$, $x \in X$ are called values of F. The notation $F: X \to \mathcal{B}$ replaces $F: 2^X \to \mathcal{B}$ if for each $A \subset X$ we have $F(A) = \bigcup\{F(\{x\}): x \in A\}$; in particular $F(\emptyset) = \emptyset$. This way of introducing $F: X \to 2^Y$ is, roughly speaking, the same as in [2, p. 22] ($F(\emptyset) = \emptyset$ was not a part of the definition in [2]). If $F: X \to 2^Y$ and $A \subset X$, then we will use the notations $F: A \to 2^Y$, $F|A$ or $F|: A \to 2^Y$. For $F: X \to 2^Y$ the domain of F is the set $D_F = \{x \in X: F(\{x\}) \neq \emptyset\}$ [2, p. 21]; F is a mapping if $\emptyset \neq D_F = X$. In turn, $f: X \to Y$ means that $f: X \to 2^Y$ and for each $x \in D_f$, $f(\{x\})$ is a singleton. For the simiplicity of notations we assume all the sets in this paper to be disjoint with the families of their subsets. Then we write $F(x) = B$, $F(A) = y$ or $F(x) = y$ in place of $F(A) = B$ if $A = \{x\}$ or $B = \{y\}$, respectively. In particular, for $f: X \to Y$ we obtain the classical notation $f(x) = y$ though its sense is different.

For $F: \mathcal{A} \to 2^Y$ and $B \subset Y$ we define $B \cap F: \mathcal{A} \to 2^Y$ by $(B \cap F)(A) =$

$= B \cap F(A)$, $A \subset X$; if $F\colon X \to 2^Y$, $B \subset Y$, then $B\backslash F\colon X \to 2^Y$ is defined by $(B\backslash F)(A) = B\backslash \bigcap\{F(x)\colon x\in A\}$, $A \subset X$. If for each $F \in \mathcal{F}$ we have $F\colon X \to 2^Y$, then $(\bigcap\mathcal{F})(x) = \bigcap\mathcal{F}(x)$, $x\in X$; if $F,G \in \mathcal{F}$, then $F \subset G$ means that $F(x) \subset G(x)$, $x\in X$.

The composition $H\circ F\colon \mathcal{A} \to 2^Z$ for $F\colon \mathcal{A} \to 2^Y$, $H\colon \mathcal{B} \to 2^Z$ is defined by $(H\circ F)(A) = H(F(A))$, $A\in\mathcal{A}$ if the right side has sense for each $A\in\mathcal{A}$. It is evident that the difference between $H\colon Y \to 2^Z$ and $H\colon 2^Y \to 2^Z$ is meaningless if we consider $H(y)$, $y\in Y$, and it is of great importance for the composition $H\circ F$. We will apply the following natural notations $\mathcal{F}\circ\mathcal{A} = \{F\circ A\colon F\in\mathcal{F}, A\in\mathcal{A}\}$, $\mathcal{F}(\mathcal{A}) = \{F(A)\colon F\in\mathcal{F}, A\in\mathcal{A}\}$ if the right side of the respective equality has sense.

Symbol F^- [2, p. 25] is extended to $F\colon 2^X \to 2^Y$ as follows: if $B \subset Y$, then $F^-(B) = \{x\in X\colon B \cap F(x) \neq \emptyset\}$, i.e. $x \in F^-(y)$ if and only if $y \in F(x)$. Clearly, $F^-(\emptyset) = \emptyset$ and $F^-(B) = \bigcup\{F^-(y)\colon y\in B\}$, $B \subset Y$ which means that $F^-\colon Y \to 2^X$. It can be easily verified that for $F\colon X \to 2^Y$, $H\colon Y \to 2^Z$ we have $(F^-)^- = F$ and $(H\circ F)^- = F^-\circ H^-$. If $f\colon X \to Y$ is a mapping in the traditional sense, then f^- may be identified with f^{-1} ($f^{-1}[B] = f^-(B)$). In other cases it is more convenient to use f^-. For example, if $f\colon \mathbb{C} \to 2^{\mathbb{C}}$, $f(z) = z^{1/2}$, $z\in\mathbb{C}$, then $f^-(z) = z^2$ while $f^{-1}(z)$ has no sense.

In this paper space means a topological space; subsets of a space will be considered subspaces if topology is needed. The terminology concerning general topology is adopted from Kelley's book [31], e.g. compact space need not be T_2 (some special notations are the same as in [15]). If $A \subset X$ or $A\in X$, then U_A, V_A, W_A are neighbourhoods of A, i.e. they contain A in their interiors (X must be a space). For $F\colon \mathcal{A} \to 2^Y$, where Y is a space, $\overline{F}\colon \mathcal{A} \to 2^Y$ is defined by $\overline{F}(A) = \overline{F(A)}$, $A\in\mathcal{A}$. If X,Y are spaces, then $C(X,Y)$ is the set of all maps (i.e. continuous mappings) $h\colon X \to Y$.

As regards uniform spaces, we assume the uniformities to consist of symmetric entourages. Each member U of a uniformity in X may be treated as a mapping $U\colon X \to 2^X$, where $U(x) = \{y\in X\colon (x,y) \in U\}$. Composition of n-times U is written nU.

The Cartesian product $X = P\{X_s\colon s\in S\}$ of a family of spaces ($S \neq \emptyset$) is a space equipped with the Tychonoff topology; if $x\in X$, then $\partial_i x \in \partial_i X = P\{X_s\colon s\in S\backslash\{i\}\}$ is defined by $p_s(\partial_i x) = p_s(x)$, $s \in S\backslash\{i\}$ (p_s is the projection into the s-th coordinate set). For any $A \subset X$ we write $\partial_i A = \{\partial_i x\colon x\in A\}$; on the other hand, $z = \langle y_i, \partial_i x\rangle$ ($z\in X$) means that $p_i(z) = y_i$, $p_s(z) = p_s(x)$, $s \neq i$, $i,s \in S$.

Let R denote the real numbers (with the natural topology if necessary). We use the notations $\overline{R} = [-\infty,\infty]$, $I = [0,1]$ and $P^{n-1} =$

$= \{t \in I^n: \sum t_i = 1\}$, $n \in N = 1, 2, \ldots$

As regards the special abbreviations, they are listed below, with reference to the respective definitions

KKM - G: $E \to 2^Y$, Def. 1.4.2, p. 23,

usc - F: $2^X \to 2^Y$ upper semicontinuous, Def. 2.1.3, p. 26,

lsc - F: $2^X \to 2^Y$ lower semicontinuous, Def. 2.2.15, p. 36,

uhc - F: $X \to 2^Y$ upper hemicontinuous, Def. 4.3.1, p. 61,

scs - f: $X \to \overline{R}$ upper semicontinuous, Def. 4.1.5, p. 52,

sci - f: $X \to \overline{R}$ lower semicontinuous, Def. 4.1.5, p. 52.

The notion of c-compactness appears in Def. 1.4.6, p. 24. For compact functions see Def. 2.1.11, p. 28. We will often apply quasiconvex (Def. 4.1.1, p. 51) and convex mappings (Def. 4.1.3, p. 51).

1.2. WEEDS

In the proofs of many classical fixed point theorems the convex combinations play the crucial role. We develop this idea by replacing $tx = \sum \{t_i x_i: i = 1, \ldots, n\}$ by $Q_n(x,t)$, $n \in N$, where Q_n are mappings and x belong to an abstract set E.

1.2.1. <u>Definition</u> (cp. Def. 1.2.7). *A pair* (E,Q) *is a weed in a set* X, *if* E *is a (nonempty) set,* $Q = (Q_n)_{n \in N}$ *is a sequence of mappings* $Q_n: E^n \times P^{n-1} \to X$ *satisfying*

(1.1) *if* $t_i = 0$, *then* $Q_n(e,t) = Q_{n-1}(\partial_i e, \partial_i t)$, $e \in E^n$, $i = 1, \ldots, n$

for each $n \in N$.

As it is seen, each of Q_n directs the "growth" of E (which is not necessarily a subset of X) in X.

1.2.2. <u>Definition</u>. *If* (E,Q) *is a weed in a set* X, *then* $cQ\,A = \bigcup\{Q_n(e,t): e \in (A \cap E)^n, t \in P^{n-1}, n \in N\}$; A *is an underhull if* $A \subset cQ\,A$ *and* A *is an overhull, if* $cQ\,A \subset A$. *A set* A *is convex whenever* $A = cQ\,A$.

To avoid ambiguities we will sometimes use the terms *Q-underhull, Q-overhull, Q-convex set*.

If A is a fixed set, then for Q, X as in 1.2.2 we have $cQ: 2^A \to 2^X$ $(cQ(C) = cQ\,C,\ C \subset A)$.

1.2.3. __Example.__ *Let* $(M,+)$ *be a semigroup such that* $tx \in M$, $t\in I$, $x\in M$ *and* $0x = \Theta$, $x\in M$ (Θ *is the neutral element in* M). *Then for any nonempty set* $E \subset M$ *the formula*

$$(1.2)\quad Q_n(e,t) = \Sigma\,\{t_i e_i\colon i=1, \ldots, n\}, \quad e \in E^n, \quad t \in P^{n-1}, \quad n \in N$$

defines a sequence Q *as to satisfy the assumptions of* Def. 1.2.1 *for* $\mathrm{conv}\, E \subset X \subset M$. *In particular, if* X *is a set in a linear structure (over a field containing* I*) and* $\mathrm{conv}\, E \subset X$, *then* (E,Q) *is a weed in the set* X. *What is more we have* $cQ\, A = \mathrm{conv}(A\cap E)$.

In order to shorten the notations we adopt the following (Q_n is as in Def. 1.2.1)

$$(1.3)\quad Q_n(e) = Q_n(e,P^{n-1}), \quad e \in E^n, \quad Q_n(e)=\emptyset,\ e\notin E^n,$$

$$(1.4)\quad Q_n(A) = \bigcup\{Q_n(e)\colon e \in (A[illegible])^n\}.$$

From (1.1) we obtain the following two properties

$$(1.5)\quad Q_m(A) \subset Q_n(B) \subset cQ\, B, \quad A \subset B, \quad m \le n,$$

$$(1.6)\quad cQ\, A \subset cQ\, B, \quad A \subset B.$$

1.2.4. __Lemma.__ *Let* (E,Q) *be a weed in a set* X *and let* $\mathcal{A}$ *be a nonempty family of sets. Then the following inclusions are satisfied*

$$(1.7)\quad \bigcup\{cQ\, \mathcal{A}\} \subset cQ\, \bigcup\mathcal{A},$$

$$(1.8)\quad cQ\, \bigcap\mathcal{A} \subset \bigcap\{cQ\, \mathcal{A}\}.$$

In particular, the sum of underhulls is an underhull, the intersection of overhulls is an overhull. If the set $\bigcap\mathcal{A}$ *is an underhull and* $\mathcal{A}$ *consists of overhulls, then* $\bigcap\mathcal{A}$ *is convex.*

Proof. From $A \subset \bigcup\mathcal{A}$ and (1.6) we obtain $cQ\, A \subset cQ\, \bigcup\mathcal{A}$ and (1.7). On the other hand, (1.6) implies that $cQ\, \bigcap\mathcal{A} \subset cQ\, A$, $A\in\mathcal{A}$ and (1.8) is established. If $\mathcal{A}$ consists of underhulls, then $\bigcup\mathcal{A} \subset \bigcup\{cQ\, \mathcal{A}\}$ and (1.7) lead to $\bigcup\mathcal{A} \subset cQ\, \bigcup\mathcal{A}$, i.e. $\bigcup\mathcal{A}$ is an underhull. Similarly, if $\mathcal{A}$ is a family of overhulls, then we obtain $\bigcap\{cQ\ \mathcal{A}\} \subset \bigcap\mathcal{A}$ which, considering (1.8) implies that $cQ\{\bigcap\mathcal{A}\} \subset \bigcap\mathcal{A}$ and consequently, $\bigcap\mathcal{A}$ is an overhull. The last part of our lemma follows from the preceding inclusion. □

1.2.5. <u>Remark</u>. *If* $x \in cQ\,\{x\}$, $x \in E$ *(e.g. if* $x \in Q_1(x)$, $x \in E$), *then each set* $A \subset E$ *is an underhull; in such a case overhulls in* E *are identical with the convex sets.*

The above remark explains why the intuitions which are good for linear structures are not suitable for weeds. As regards the general case, it is more natural to consider overhulls. For example, let $Q_n(e,t) = x_o \in X$, $e \in E^n$, $t \in P^{n-1}$, $n \in N$ while $x_o \in E$. Then every set containing x_o is an overhull.

It should be noted that in view of Lemma 1.2.4 (the intersection property), if X is an overhull, then the family of all overhulls in X is a convexity in the sense of [44]. We however prefer to avoid this terminology as being too close to the intuitions proper for linear structures. It must be emphasized that the equality $cQ\ A = cQ(cQ\ A)$ usually does not hold.

As a supplement of 1.2.4 we prove the following

1.2.6. <u>Lemma</u>. *Let* (E,Q) *be a weed in a set* X. *If a family* $\mathcal{A}$ *of underhulls is directed by* $\supset$ *(inclusion), then* $B = \bigcup\mathcal{A}$ *is an underhull.*

Proof. It suffices to consider $\mathcal{A} \subset 2^X$ (see Def. 1.2.2). If e is an element of B^n, then $e \in A^n$ for an $A \in \mathcal{A}$ [31, p. 65]. Hence, $Q_n(e) \subset Q_n(A) \subset cQ\ A \subset A \subset B$ and $cQ\ B \subset B$, e, n being arbitrary. □

Now we define a weed in a space.

1.2.7. <u>Definition</u>. *A pair* (E,Q) *is a weed in* X, *if* X *is a space,* (E,Q) *is a weed in the set* X *and for each* $e \in E^n$, $n \in N$ $Q_n(e,\cdot)\colon P^{n-1} \to X$ *is continuous.* (X,Q) *is a weed if* (X,Q) *is a weed in* X.

1.2.8. <u>Example</u>. *Let* X *be a set in a linear structure over a field containing* I *(or* $X \subset M$ *for* $(M,+)$ as in Ex. 1.2.3). *Let* conv E *be a nonempty subset of* X *which is equipped with a topology for which* $Q_n(e,\cdot)\colon P^{n-1} \ni t \mapsto \sum t_i e_i \in X$ *is continuous,* $e \in E^n$, $n \in N$.

Then (E,Q) *is a weed in* X. *In particular, each convex space* [36, Def. 2 p. 153], *with the sequence of convex combinations, is a weed.*

1.2.9. <u>Example</u>. *If* (E,Q) *is a weed in* X, Y *is a space, then* $FC(X,Y)$ *is the family of mappings* $h: X \to Y$ *which are finitely continuous, i.e. such that for each* $e \in E^n$, $n \in N$ $h: Q_n(e) \to Y$ *is continuous. Then for any* $h \in FC(X,Y)$ $(E, h \circ Q)$ *is a weed in* Y, *where* $(h \circ Q)_n(e,t) = h(Q_n(e,t))$, $e \in E^n$, $t \in P^{n-1}$, $n \in N$. *Clearly, we have* $c(h \circ Q)A = h(cQ\ A)$.

1.2.10. <u>Example</u>. *Assume that* $f \in FC(X,Y)$ *and let* $g: Y \to X$ *be the right inverse of* f *(*g *need not be continuous),* $E \subset Y$ *and* $(g(E),Q)$ *a weed in* X. *Then* (E,Q^*) *is a weed in* Y *for* Q^* *defined as follows*

$$(1.9) \quad Q^*_n(e,t) = f(Q_n(g(e),t)), \quad g(e) = (g(e_1), \ldots, g(e_n)),$$
$$e \in E^n, \quad t \in P^{n-1}, \quad n \in N.$$

What is more $cQ^*A = f(cQ\ g(A))$. *Thus, in particular, if* f *is an r-map on* X *to* Y, *then* (E,Q^*) *is a weed in* Y. *If additionally* (X,Q) *is a weed, then* (Y,Q^*) *is a weed.*

1.2.11. <u>Example</u>. *Let* $\{S_x: x \in X\}$ *be a family of homotopies, where* S_x *joins* ~~a map g: X→X~~ identity *with the constant map equal to* g(x) (g: X → X) $x \in X$. *Then in view of* Lemma 1.3.4 X *can be equipped with a* Q *such that* (X,Q) *is a weed (see* (1.13)*)* if each compact set K⊂X is regular.

1.2.12. <u>Example</u>. *Let* (E,Q) *be a weed in* X. *We will define* Q'_n *depending on* $\{e_1, \ldots, e_n\}$ *(the order in* $(e_1, \ldots, e_n)$ *becomes meaningless) and such that* (E,Q') *is a weed in* X. *We may assume that* E *is linearly ordered. Let us consider* $Q_k(e,t)$ *for* $e \in E^k$, $t \in P^{k-1}$ *such that* $e_i \neq e_j$, $i \neq j$, $i,j = 1, \ldots, k$ and $e_1 < \ldots < e_k$, $k \leq \text{card } E$, $k \in N$. *For* $e = (e_1, \ldots, e_n) \in E^n$ *let* $e' = (e'_1, \ldots, e'_k)$ *be an increasing sequence such that we have* $i \in \{1, \ldots, n\}$ *iff there exists a* $j \in \{1, \ldots, k\}$ *for which* $e_i = e'_j$. *Then for* $t'_j = \Sigma\{t_i: e_i = e'_j\}$, $j = 1, \ldots, k$ *we define* Q' *as follows*

$$(1.10) \quad Q'_n(e,t) = Q_k(e',t'), \quad e \in E^n, \quad t \in P^{n-1}.$$

1.2.13. Remark. *The above construction shows that* Def. 1.2.7 *extends* [P6, Def. 1.6]. *Clearly we have* $Q'_n(e) \subset Q_n(e)$, $n \in N$ *and* $cQ'A \subset cQ\,A$ *for* Q' *defined as above. What is more* $cQ'A$ *is compact for each finite set* A *as being equal to* $Q_n(e)$ *for an* $n \in N$.

1.3. WEEDS AND S-CONTRACTIBLE SPACES

Weed (as in Def. 1.2.7) is a kind of a convex structure. The comparison with some other concepts introduced by Fox [24], Dugundji [12], Himmelberg [28] and Curtis [7] will be shown in Chapter 6. As regards more complicated definitions of Michael [40] and Curtis [8], they will be presented in Chapter 7. Now we point out that the assumption of continuity for weeds is less restrictive than in the above mentioned structures. It accords with the concept of Bielawski [3, p. 158] we but do not require either $Q_n(\{x\}) = \{x\}$ or $E = X$. This section is devoted to S-contractible spaces (see Ex. 1.2.11, 6.3).

If $f\colon T \times X \longrightarrow Y$ is a mapping, then it defines an *exponential mapping* $\varphi f\colon T \longrightarrow F(X,Y)$ on T into the set $F(X,Y)$ of all mappings on X to Y; it is defined by the equality $(\varphi f(t))(x) = f(t,x)$, $t \in T$, $x \in X$. If X,Y are spaces, then $F(X,Y)$ can be equipped with the compact open topology - a subbase for this topology consists of sets $M(K,U) = \{g \in F(X,Y)\colon g(K) \subset U\}$, where $K \subset X$ is compact and $U \subset Y$ open. This space is written $F_c(X,Y)$. The family of all mappings on X to Y which are continuous on compacta is $C^c(X,Y)$. Hence $C^c_c(X,Y) \subset F_c(X,Y)$ consists of mappings which are continuous on compacta and the topology of this space is compact open.

1.3.1. Definition. *A space* X *is S-contractible if for each* $x \in X$ *mapping* $\varphi \cdot S_x\colon I \longrightarrow C^c_c(X,X)$ *is continuous and the following is satisfied*

$$(1.11)\quad S_x(0,y) = y,\quad S_x(1,y) = g(x),\quad x,y \in X.$$

The above definition differs from [P1, Def. 1] where we assumed $g = id_X$, X a T_2-space and S_x a homotopy (i.e. $\varphi S_x\colon I \longrightarrow C_c(X,Y)$ is continuous) joining the identity with the constant map equal to x.

If X is an S-contractible space, then

$$(1.12)\quad S^n(e,t) = S_{e_1}(t_1, S_{e_2}(t_2, \ldots, S_{e_{n-1}}(t_{n-1}, e)\ldots)$$
$$e = (e_1,\ldots,e_n) \in X^n,\quad t = (t_1,\ldots,t_{n-1}) \in I^{n-1}$$

defines a mapping $S^n: X^n \times I^{n-1} \to X$, $n \in N$. We will show that Q_n: $X^n \times P^{n-1} \to X$ defined by

(1.13) $Q_n(e,t) = S^n(e, \partial_n t/\max\{t_1, \ldots, t_n\})$, $e \in E^n$, $t \in$ $\in P^{n-1}$, $n \in N$

satisfy condition (1.1) of Def. 1.2.1. Indeed, if $t_i = 0$ for an $i < n$, then in view of (1.11) S_{e_i} in (1.12) can be omitted; if $i =$ $= n$, then $u_j = t_j/\max\{t_1, \ldots, t_n\} = 1$ for a $j < n$, which means that $S_{e_j}(u_j, y) = g(e_j)$, $y \in X$. In both cases we have $Q_n(e,t) =$ $= Q_{n-1}(\partial_i e, \partial_i t)$ if $t_i = 0$.

1.3.2. Remark. *The above reasoning suggests that* Q_1, Q_2 *as in* Def. 1.2.1 *suffice to define* $S_x(t,y) = Q_2((x,y),(t,1-t))$ *and then* Q' *(as in* (1.13)*)* *satisfies* (1.1). *Unfortunately the assumption of continuity (see* Def. 1.2.7*) may fail to be satisfied for such a* Q'.

Let us present the requirements which guarantee Q as in (1.13) to satisfy the continuity assumptions of Def. 1.2.7.

The family of compact subsets of a space under consideration is denoted by c. A mapping $f: T \times X \to Y$ is *continuous on* $T \times c$ if for each $K \in c \subset 2^X$, $f: T \times K \to Y$ is continuous.

1.3.3. Lemma. *Let* T, X, Y *be spaces. If* $f: T \times X \to Y$ *is continuous on* $T \times c$, *then* $\varphi f: T \to C^c_c(X,Y)$ *is continuous; if* φf: $T \to C^c_c(X,Y)$ *is continuous and compacta in* X *are regular (e.g. if* X *is regular or* T_2*), then* f *is continuous on* $T \times c$.

Proof. Suppose that $\varphi f(t) \in M(K,V)$, where $t \in T$, $K \in c \subset 2^X$ and $V \subset Y$ is open. If f is continuous on $T \times K$, then $f(t,K) \subset V$ means that for each $x \in K$ there exist neighbourhoods $F^x \subset T$ of t and $G_x \subset K$ of x, respecectively, such that $f(F^x \times G_x) \subset V$ [31, Th. 12 p. 142]. Let $G_i = G_{x_i}$, $i = 1, \ldots, n$ cover K. Then for $F_t = \bigcap F^i$ we obtain $f(F_t \times K) \subset V$, i.e. $\varphi f(F_t) \subset M(K,V)$. A similar inclusion holds for any element B of a base of the compact open topology in $C^c(X,Y)$. Hence, $\varphi f: T \to C^c_c(X,Y)$ is continuous. Conversely, if $\varphi f: T \to C^c_c(X,Y)$ is continuous and $f(t,x) \in V$ for an open set $V \subset Y$, then for each compact set $K \subset X$ there exists a neighbourhood G_x such that $f(t, G_x \cap K) \subset V$ ($f(t,\cdot)$ is continuous on compacta). It follows from the regularity of K that

$G_x \cap K$ may be considered closed [31, p. 113] and therefore compact. In view of the continuity of φf there exists an F_t such that $f(F_t \times [G_x \cap K]) \subset V$ which means that f is continuous on $T \times K$. □

1.3.4. Lemma. *If* X *is an S-contractible space, and each compact set* $K \subset X$ *is regular, then* (X,Q) *is a weed for* Q *as in* (1.13).

Proof. From the second part of Lemma 1.3.3, by induction we obtain the compactness of $Q_n(e)$, $e \in E^n$ (see (1.13), (1.3)), as a continuous image of any compact set is compact. Assume $Q_n(e,t) \in U$ for an open set $U \subset X$. In view of the continuity of S_{x_1} on $I \times c$ there exist neighbourhoods $F_1 \subset I$, $U_1 \subset Q_n(e)$ of $t_1/\max\{t_1,\dots,t_n\}$, $S^{n-1}((e_2,\dots,e_n),(t_2,\dots,t_{n-1}))$, respectively, such that $S_{x_1}(F_1 \times U_1) \subset U$. In turn, we find neighbourhoods $F_2 \subset I$, $U_2 \subset Q_n(e)$ of $t_2/\max\{t_1,\dots,t_n\}$, $S^{n-2}((e_3,\dots,e_n),(t_3,\dots,t_{n-1}))$ such that $S_{x_2}(F_2 \times U_2) \subset U_1$, etc. By induction we define a neighbourhood $F = F_1 \times F_2 \times \dots \times F_{n-1}$, of $\partial_n t/\max\{t_1,\dots,t_n\}$ for which $S^n(e,F) \subset U$. Clearly, F defines a neighbourhood of $t \in P^{n-1}$ and, consequently, the assumption of continuity (as in Def. 1.2.7.) holds. □

In view of the above (see Remark 1.3.2) it seems that continuity assumptions (as in Def. 1.3.1) are the weakest ones which, under the reasonable hypothesis on X, lead from (1.11) to the requirements of Def. 1.2.7.

We recall that X is a *k-space* provided that any set A in X is open if and only if $A \cap K$ is open in each compact set $K \subset X$.

1.3.5. Remark. *If* X *is a k-space, then* $C^c(X,X) = C(X,X)$ *and mappings* S_x *as in* Def. 1.3.1 *become homotopies. It is known* [15, Th. 3.3.20 p. 201] *that any* T_2*-space* X *satisfying the first axiom of countability (e.g. metrizable) is a k-space; what is more* $I \times X$ *is then a k-space* [15, Th. 3.3.27 p. 203]. *Now, in view of* Lemma 1.3.3, *continuity of* $\varphi S_x\colon I \to C_c^c(X,X)$ *implies then continuity of* $S_x\colon I \times X \to X$. *On the other hand, if* S_x *is continuous on* $I \times c$, *then* $\varphi S_x\colon I \to C_c^c(X,X)$ *is a map (first part of* 1.3.3*); in particular, if* S_x *is continuous, then* $\varphi S_x\colon I \to C_c(X,X)$ *is continuous too.*

1.4. KKM MAPPINGS

The proofs of the fixed point theorems presented in this paper are in fact based on the following theorem of Knaster-Kuratowski-Mazurkiewicz [33, p. 134]. We present it in a shorter form. Theorems 1.4.7, 1.4.9 are useful tools in proving many results of this paper.

1.4.1. <u>Theorem</u>. *Let* $M_1, \ldots, M_n$ *be closed subsets of* P^{n-1}. *If for each* $t \in P^{n-1}$ *we have* $t \in \bigcup\{M_i\colon\ t_i \neq 0\}$, *then* $\bigcap\{M_i\colon i = 1, \ldots, n\}$ *is nonempty.*

A simple proof of the above can be found, e.g. in [14, p. 102].

The subsequent definition extends the idea of KKM mappings of Dugundji and Granas [13].

1.4.2. <u>Definition</u>. *Let* (E,Q) *be a weed in* X, *then* $G\colon E \to 2^X$ *is KKM (for* Q*) if the following is satisfied (see* (1.10))

$$(1.14)\quad Q'_n(e) \subset \bigcup\{G(e_i)\colon i = 1, \ldots, n\}, \quad e \in E^n, \quad n \in N.$$

It follows from Def 1.2.1 that G is a mapping. It is worth noting that if $G\colon E \to 2^X$ is KKM for a Q and $h \in FC(X,Y)$ (see 1.2.9), then $h \circ G\colon E \to 2^Y$ is KKM for $h \circ Q$.

A similar definition of Bielawski [2, Def. (4.1)] is much more restrictive as in place of $Q'_n(e)$ Bielawski uses the least convex set containing $\{e_1, \ldots, e_n\}$.

The following is a natural generalization of [13, Th. 1.2].

1.4.3. <u>Theorem</u>. *Let* (E,Q) *be a weed in* X *and* $G\colon E \to 2^X$ *KKM such that* $G(e_i) \cap Q'_n(e)$ *is closed in* $Q'_n(e)$, $e \in E^n$, $n \in N$. *Then the family* $\{G(x)\colon x \in E\}$ *has the finite intersection property.*

Proof. Let $\{e_1, \ldots, e_n\} \subset E$ be arbitrary. We may assume (see Ex. 1.2.12) that $e_1 < \ldots < e_n$. In view of Def. 1.2.7 $g = Q_n(e,\cdot)\colon P^{n-1} \to X$ is a map and therefore, $M_i = g^-(G(e_i)) = g^-(G(e_i) \cap Q_n(e))$ is a closed set. What is more, $g(t) = Q_n(e,t) = Q'_n(e,t) \in \bigcup\{G(e_i)\colon t_i \neq 0\}$ (see (1.10), (1.1)) which means that $t \in g^-(\bigcup\{G(e_i)\colon t_i \neq 0\}) = \bigcup\{M_i\colon t_i \neq 0\}$. Now in view of Th. 1.4.1 $\bigcap g^-(G(e_i)) \neq \emptyset$, so in consequence we obtain $\bigcap G(e_i) \neq \emptyset$. □

The definition to follow is natural for X being not a k-space.

1.4.4. <u>Definition</u> [36, Def. 4 p. 154]. *A subset* A *of a space* X *is compactly closed (open) if* $A \cap C$ *is closed (open) in each compact set* $C \subset X$.

1.4.5. Theorem. *Let* (E,Q) *be a weed in* X *and let* $L: E \to 2^Y$, $h: X \to Y$ *satisfy the following*

(i) *all values of* L *are compactly closed,*

(ii) $G = h^{-} \circ L$ *is KKM.*

If X *is compact and* $h \in C(X,Y)$ or if Y is compact and $h \in FC(X,Y)$, then $\bigcap\{L(x): x \in E\} \neq \emptyset$.

Proof. According to Ex. 1.2.9, if $h \in FC(X,Y)$ and (E,Q) is a weed in X, then $(E,h \circ Q)$ is a weed in Y. If $G: E \to 2^X$ is KKM then $h \circ G = L$ (see (ii)) is KKM for each $h \in FC(cQ\ E,Y)$ and compactness of Y together with condition (i) imply the assumptions of Th. 1.4.3 for $(E,h \circ Q)$, L in place of (E,Q), G, respectively. If X is compact and $h \in C(X,Y)$, then $h(X)$ is compact and it can be identified with Y as $h \circ G = L$. In both cases the finite intersection property of $\{L(x): x \in E\}$ means that $\bigcap L(x) \neq \emptyset$. □

On the analogy of [36, Def. 3 p. 154] we present the following.

1.4.6. Definition. *Let* (E,Q) *be a weed in* X. *A nonempty set* $K \subset E$ *is c-compact, if for each finite set* $Z \subset E$ *there exists a compact set* $K_Z \subset X$ *such that* $cQ'(K \cup Z) \subset K_Z$ *(see (1.10)), i.e.* $cQ'(K \cup Z)$ *is relatively compact.*

1.4.7. Theorem. *Let* (E,Q) *be a weed in* X *and let* $L: E \to 2^Y$ *and an* $h \in C(X,Y)$ *satisfy the following conditions*

(i) *all values of* L *are compactly closed,*

(ii) $G = h^{-} \circ L$ *is KKM,*

(iii) $B = \bigcap\{L(x): x \in K\} \subset Y$ *is compact for a c-compact set* $K \subset E$.

Then $B \cap \bigcap\{L(x): x \in E\}$ *is* nonempty.

Proof. Let Z, K_Z be as in Def. 1.4.6. Clearly, $(K \cup Z, Q')$ is a-weed in K_Z and hence $K_Z \cap (h^{-} \circ L)$ is KKM on $K \cup Z$. In view of Th. 1.4.5, when applied to $E = K \cup Z$, $X = K_Z$ and $h: K_Z \to Y$ set $\bigcap\{L(x): x \in K \cup Z\} = \bigcap\{B \cap L(x): x \in Z\}$ is nonempty. Thus, $\{B \cap L(x): x \in E\}$ is a family of closed (in B) compact sets which has the finite intersection property. □

Theorem 1.4.7 extends [36, Th. I p. 154] which, in turn, is more general than [23, Th. 1 p. 151] (see [36, 2^o p. 154]).

1.4.8. <u>Remark</u>. *In view of* Rem. 1.2.13 *each finite set* $Z \subset E$ *is c-compact. Hence* Th. 1.4.7 *contains* [18, Lemma 1]. *If* Y *is a compact space, then* (iii) *is clearly satisfied. The same is when* X *is compact as then we consider* $Y = h(X)$.

The above comment informs us that Th. 1.4.7 becomes interesting if neither X nor Y is compact (see Th. 1.4.5) and if K is not finite (this case follows from Th. 1.4.3).

Below we present a stronger form of [38, Th. III p. 156].

Every uniform space and consequently each linear space is regular. Therefore, the relative compactness of conv $(K \cup Z)$ means the compactness of $\overline{\text{conv}(K \cup Z)}$ [31, B (b) p. 161] which can be considered equal to K_Z (Def. 1.4.6).

1.4.9. <u>Theorem</u>. *Let* E *be a nonempty subset of a convex set* X *in a linear space (or* E, X *as in* 1.2.8 *and* X *regular). If* $L\colon E \to 2^Y$ *and an* $h \in C(X,Y)$ *satisfy the following*

(i) *all values of* L *are compactly closed,*

(ii) $G = h^{-}\circ L$ *is KKM,*

(iii) $C = \{y \in X\colon$ *if* $x \in E \cap \text{conv}(K \cup \{y\})$, *then* $y \in (h^{-}\circ L)(x)\}$ *is relatively compact for a c-compact set* $K \subset E$,

then $h(\overline{C}) \cap \bigcap\{L(x)\colon x \in E\} \neq \emptyset$.

Proof. First we will show that $\{C \cap (h^{-}\circ L)(x)\colon x \in E\}$ has the finite intersection property. If $z \in X$ and $Z = \{z_1, \ldots, z_n\} \subset E$, then for $K_1 = \overline{\text{conv}(K \cup Z \cup \{z\})}$ which is compact $K_1 \cap (h^{-}\circ L)\colon E \cap K_1 \to 2^{K_1}$ is KKM, K_1 being convex. The values of $K_1 \cap (h^{-}\circ L)$ are compact closed, and in view of Th. 1.4.3 there exists a $y \in \bigcap\{K_1 \cap (h^{-}\circ L)(x)\colon x \in E \cap K_1\} \subset \bigcap\{(h^{-}\circ L)(x)\colon x \in E \cap K_1\}$. From the fact that $y \in K_1$ follows the inclusion $\text{conv}(K \cup Z \cup \{y\}) \subset \overline{\text{conv}(K \cup Z \cup \{z\})}$, and hence $y \in \bigcap\{(h^{-}\circ L)(x)\colon x \in E \cap K_1\} \subset \bigcap\{(h^{-}\circ L)(x)\colon x \in E \cap \text{conv}(K \cup Z \cup \{y\})\} \subset \bigcap\{(h^{-}\circ L)(x)\colon x \in Z \cup [E \cap \text{conv}(K \cup \{y\})]\}$. Thus we obtain $y \in C \cap \bigcap\{(h^{-}\circ L)(x)\colon x \in Z\}$. It follows from the compactness of $\overline{C}$ that $\overline{C} \cap G(x) = \overline{C} \cap (h^{-}\circ L)(x) = \overline{C} \cap h^{-}(h(\overline{C}) \cap L(x))$ is closed, $x \in E$ (see (i)). This fact and the finite intersection property mean that $\overline{C} \cap h^{-}(\bigcap\{L(x)\colon x \in E\}) \neq \emptyset$ and, consequntly $h(\overline{C}) \cap \bigcap\{L(x)\colon x \in E\} \neq \emptyset$. □

Let us compare sets B (Th. 1.4.7) and C (Th. 1.4.9) for $Y = X$, $h = id_X$. We have $C = \{y \in X\colon$ if $x \in E \cap \text{conv}(K \cup \{y\})$, then $y \in L(x)\} \subset B = \{y \in X\colon$ if $x \in K$, then $y \in L(x)\}$. Thus, for linear spaces, 1.4.9 is more effective than 1.4.7.

2. General fixed point theorems

2.1. UPPER SEMICONTINUITY

Set-to-set functions are of natural interest in this paper and there is the need of extending the classical definition of upper semicontinuity to suit this case. The initial part of this section is devoted to the properties of usc mappings and in the final one we obtain some dependencies between the upper semicontinuity of F and of $\overline{F}$, $\overline{cQ}\circ F$. In consequence, we obtain an important property of usc functions in linear spaces (Corol. 2.1.16).

2.1.1. Definition. *We say that* $F: 2^X \to 2^Y$ *is increasing (weakly) if for each* $A,B \subset X$ *from* $A \subset B$ *follows* $F(A) \subset F(B)$.

As it is seen, each $F: X \to 2^Y$ is increasing; for topological spaces operations of closure and interior are increasing. It follows from (1.6) that if (E,Q) is a weed in X, then for any set $Z \subset Q$: $2^Z \to 2^X$ is increasing.

2.1.2. Lemma. *If* $F: 2^X \to 2^Y$, $H: 2^Y \to 2^Z$ *are increasing, then* $H\circ F$ *is increasing.*

Proof. From $A \subset B \subset X$ follows $F(A) \subset F(B)$ which, in turn, implies $(H\circ F)(A) = H(F(A)) \subset H(F(B)) = (H\circ F)(B)$. □

2.1.3. Definition. *An increasing* $F: 2^X \to 2^Y$ *is usc (upper semicontinuous), if* X, Y *are spaces and for each* $x\in X$ *and any neighbourhood* W *of* $F(x)$ *there exists a neighbourhood* U *of* x *such that* $F(U) \subset W$.

Clearly, for $F: X \to 2^Y$ the inclusion $F(U) \subset W$ is equivalent to $F(z) \subset W$, $z\in U$. Hence in such a case 2.1.3 agrees with the classical definition [2, p. 114]. If $h \in C(X,Y)$ holds, then h is usc.

Let us present some properties.

2.1.4. Lemma. *If* $F: 2^X \to 2^Y$, $H: Y \to 2^Z$ *are usc, then* $H \circ F$ *is usc. If* $h \in C(X,Y)$ *and* $H: 2^Y \to 2^Z$ *is usc, then* $H \circ h$ *is usc.*

Proof. In view of Lemma 2.1.2, it suffices to consider the second requirement of Def. 2.1.3. If we have $H(F(x)) \subset W$ for an open set $W \subset Z$, then each element y of $F(x)$ has a neighbourhood V_y such that $H(V_y) \subset W$ holds. For $V = \bigcup\{V_y: y \in F(x)\}$ there exists a neighbourhood U of x such that $F(U) \subset V$. Hence we obtain $H(F(U)) \subset H(V) = \bigcup\{H(V_y): y \in F(x)\} \subset W$. If $F = h \in C(X,Y)$, then we have $V = V_y$ for $y = f(x)$. □

2.1.5. Lemma. *If* $F: 2^X \to 2^Y$ *is usc and* C *is a closed subset of* Y, *then* $F^-(C)$ *is closed.*

Proof. Assume $x \notin F^-(C)$. Then we have $F(x) \cap C \neq \emptyset$ and $F(x) \subset \subset Y \setminus C$. The set $Y \setminus C$ is open and, therefore, $F(U) \subset Y/C$ for a $U = U_x$, F being usc. From the fact that $F(z) \subset F(U)$, $z \in U$ follows $U \cap F^-(C) = \emptyset$ which means that $F^-(C)$ is closed. □

For $F: 2^X \to 2^Y$ let us write $G_F = \{(x,y) \in X \times Y: y \in F(x)\}$.

2.1.6. Lemma. *Let* $F: 2^X \to 2^Y$ *be usc with closed values. If* Y *is a regular space or if* Y *is a* T_2*-space and* $F(x)$ *is compact,* $x \in X$, *then* G_F *is a closed subset of* $X \times Y$.

Proof. If $(x,y) \notin G_F$, then there exist disjoint neighbourhoods $V = V_y$, $W = W_{F(x)}$. For a $U = U_x$ we have $F(U) \subset W$, F being usc. The fact that F is increasing implies that $F(z) \subset W$, $z \in U$ and hence $(U \times V) \cap G_F = \emptyset$ holds. □

2.1.7. Lemma. *Under the hypothesis of Lemma* 2.1.6 *the set* $\mathrm{Fix}\, F$ $(= \{x \in X: x \in F(x)\})$ *is closed.*

Proof. If $x \notin \mathrm{Fix}\, F$, then $(x,x) \notin G_F$ and for a $U = U_x$ we have $(U \times U) \cap G_F = \emptyset$ (see 2.1.6). Hence, $F(z) \cap U = \emptyset$, $z \in U$ which implies that $U \subset X \setminus (\mathrm{Fix}\, F)$. □

2.1.8. Lemma [36, Lemma 1 p. 157]. *If for* $F: X \to 2^Y$ $F(X)$ *is relatively compact and* G_F *is closed, then* F *is usc with closed compact values.*

Proof. For the sake of simplicity let us assume that Y is compact. Suppose $W = W_{F(x)}$ to be such that for each $U = U_x$ we have $F(U) \not\subset W$. Hence, there exists a net $((x_s,y_s))_{s\in S}$ for which $x_s \to x$, $y_s \in F(x_s)$, $y_s \notin W$, $s\in S$. It follows from the compactness of Y that (y_s) has a convergent subnet [31, Th.2 p.136]. The respective subnet of $((x_s,y_s))$ is convergent to an element $(x,y) \in G_F$, this last set being closed. Hence, $(x_s,y_s) \in U \times W$ for an $s\in S$, and a contradiction results. Thus, for each $W_{F(x)}$ there exists a $U_x \subset W_{F(x)}$, which means that F is usc. In particular, by considering $((x,y_s))$ we assert that $F(x)$ is closed. □

2.1.9. <u>Lemma</u> [36, Lemma 1 p. 157]. *If* $F: X\to 2^Y$ *is usc with compact values, then for each compact set* $A \subset X$, $F(A)$ *is compact.*

Proof. Let $\mathcal{W}$ be an open cover of $F(A)$. For each $x\in A$ we can find a finite cover $\mathcal{W}_x$ of $F(x)$ by members of $\mathcal{W}$. For a U_x we have $F(U_x) \subset W_{F(x)} = \bigcup \mathcal{W}_x$. The cover $\{U_x\colon x\in A\}$ has a finite subcover $\{U_{x_1},\dots,U_{x_n}\}$. Hence $\mathcal{W}_{x_1} \cup \dots \cup \mathcal{W}_{x_n}$ is a finite cover of $F(A)$. □

2.1.10. <u>Corollary</u>. *If* $F: 2^X\to 2^Y$, $H: Y\to 2^Z$ *are usc with compact values, then* $H\circ F$ *is usc with compact values (see 2.1.4).*

2.1.11. <u>Definition</u>. *Let* $F: 2^X \to 2^Y$ *be usc. Then* F *is relatively compact provided that* $F(X)$ *is relatively compact (i.e. contained in a compact subset of* Y*);* F *is compact if* $\overline{F(X)}$ *is a compact set.*

2.1.12. <u>Remark</u>. *In each regular space* Y *the closure of any compact set is compact* [31, B (b) p. 161]. *In each* T_2*-space* Y *every compact set is closed. Therefore, in these cases compactness and relative compactness of* $F: 2^X\to 2^Y$ *coincide.*

For the sake of further applications we are interested in the dependency between the upper semicontinuity of $F: X\to 2^E$ and of $\overline{cQ}\circ F$ when E is a uniformizable space (see, e.g. Th. 2.2.12). We recall that a space is uniformizable if and only if it is completely regular [31, Corol. 17 p. 188]. For the sake of simplicity we will just require $(E,\mathcal{V})$ to be a uniform space.

2.1.13. Lemma. *Let* X *be a space,* $(E,\mathcal{V})$ *a uniform space,* $F: 2^X \to 2^E$ *increasing and such that all values of* $\overline{F}$ *are compact. Under these assumptions* $\overline{F}$ *is usc if and only if the following is satisfied*

(2.1) *for every* $x \in X$, $V \in \mathcal{V}$ *there exists a neighbourhood* U *of* x *such that* $F(U) \subset V(F(x))$.

In particular, if F *is usc with relatively compact values, then* $\overline{F}$ *is usc.*

Proof. Assume (2.1) is satisfied. For each $V \in \mathcal{V}$, $A \subset E$ we have $\overline{A} \subset V(A)$ [31, Th. 7 p. 179] and hence $F(U) \subset V(F(x))$ implies that $\overline{F}(U) \subset 2V(F(x)) \subset 2V(\overline{F}(x))$. In view of Lemma 2.1.2 $\overline{F}$ is increasing. Every neighbourhood of the compact set $\overline{F}(x)$ contains a uniform neighbourhood [31, Th. 33 p. 189] and, therefore, the above inclusion means that $\overline{F}$ is usc. Conversely, if $\overline{F}(U) \subset V(\overline{F}(x))$ holds, then $F(U) \subset \overline{F}(U) \subset V(\overline{F}(x)) \subset 2V(F(x))$ and consequently (2.1) is satisfied. □

2.1.14. Lemma. *Let* X *be a space,* $(E,\mathcal{V})$, $(Z,\mathcal{W})$ *uniform spaces and let an increasing* $F: 2^X \to 2^E$ *satisfy* (2.1) *(e.g.* F *is usc). If* (E,Q) *is a weed in* Z, *all values of* $\overline{cQ \circ F}$ *are compact and the following is satisfied*

(2.2) *for every* $W \in \mathcal{W}$ *there exists a* $V \in \mathcal{V}$ *such that for each* $A \in \{F(x): x \in X\}$ *we have* $cQ\, V(A) \subset W(cQ\, A)$,

then $\overline{cQ \circ F}$ *is usc.*

Proof. In view of Lemma 2.1.2 and condition (1.6) $cQ \circ F$ is increasing. Let U,V,W be as in (2.1), (2.2). We have $cQ\, F(U) \subset cQ\, V(F(x)) \subset W(cQ\, F(x))$, i.e. (2.1) for $cQ \circ F$ in place of F. Now we apply Lemma 2.1.13. □

2.1.15. Lemma. *Let* $(E,\mathcal{V})$, $(Z,\mathcal{W})$ *be uniform spaces,* (E,Q) *a weed in* Z *and* $F: 2^X \to 2^E$. *If all values of* F *are overhulls, then from* (2.2) *follows*

(2.3) *for each* $W \in \mathcal{W}$ *there exists a* $V \in \mathcal{V}$ *such that for every* $A \in \{F(x): x \in X\}$ *we have* $cQ\, V(A) \subset W(A)$.

If (2.3) holds and all values of F *are underhulls, then (2.2) is satisfied. In particular, if all values of* F *are convex, then (2.2) and (2.3) are equivalent. By (2.3)* $\overline{F}(x)$ *is an overhull,* $x \in X$.

Proof. From (2.2) and Def. 1.2.2 we obtain $cQ\ V(A) \subset W(cQ\ A) \subset \subset W(A)$, i.e. (2.3). If (2.3) holds and $A \subset cQ\ A$, then $cQ\ V(A) \subset \subset W(A) \subset W(cQ\ A)$ implies (2.2). In view of $\overline{A} = \bigcap\{W(A): W \in \mathcal{W}\}$ [31, Th. 7 p, 179] (2.3) means that $cQ\ \overline{A} \subset \overline{A}$, i.e. $\overline{A}$ is an overhull. □

It is worth noting that Lemmas 1.14, 1.15 remain valid for (2.2), (2.3) modified as follows: for each $W \in \mathcal{W}$, $A \in \{F(x): x \in X\}$ there exists a $V \in \mathcal{V}$, etc. Nevertheless the "uniform" versions presented above are of greater interest in this paper.

If M is a locally convex space (every neighbourhood W of zero contains a convex neighbourhood V of zero), then for each $A \subset M$ we obtain (see Ex. 1.2.3.) $cQ(A+V) = conv[(A+V) \cap E] \subset [V + conv(A \cap E)] \cap \cap X \subset (W + cQ\ A) \cap X$.

2.1.16. Corollary. *If* Z *is a convex set in a locally convex space, then every mapping* $F: X \to 2^Z$ *satisfies (2.2) for* $cQ = conv$ *and it satisfies (2.3) whenever all values of* F *are convex. Hence if* F *satisfies (2.1) (e.g.* F *is usc), then* $\overline{conv \circ F}$ *(closure in* Z*) is usc whenever its values are compact.*

2.2. FIXED POINT THEOREMS

This section contains fairly general theorems on fixed points. Though being abstract, they can be easily applied to linear spaces (see, e.g. Th. 2.2.4). As a consequence of the main theorem of this section (2.2.11) we obtain a very clear Th. 2.2.13. Some other classical results are Theorems 2.2.10, 2.2.17, 2.2.19, 2.2.21. Two initial lemmas play the crucial role in this paper.

2.2.1. Lemma. *Let* (E,Q) *be a weed in* X *and* $F: X \to 2^E$. *If* $G = X \setminus F^-: E \to 2^X$ *is not KKM, then there exists a set* $Z = \{z_1, \ldots, z_m\} \subset E$ *and a point* $x \in Q_m(z) \subset Q_m(F(x)) \subset cQ\ F(x)$ $(z = (z_1, \ldots, z_m))$.

Proof. Since G is not KKM (see Def. 1.4.2), then for a set $Z = \{z_1, \ldots, z_m\} \subset E$ inclusion $Q_m'(z) = Q_m(z) \subset \bigcup\{X \setminus F^-(z_i): i = 1, \ldots, m\} = X \setminus \bigcap\{F^-(z_i): i = 1, \ldots, m\}$ does not hold. Hence there exists a point $x \in Q_m(z)$ such that $x \in F^-(z_i)$, $i = 1, \ldots, m$ which means $z_i \in F(x)$, $i = 1, \ldots, m$. This last implies

$Q_m(z) \subset Q'_m(F(x)) \subset Q_m(F(x))$ (Remark 1.2.13) and thus our lemma is established (see (1.5)). □

The subsequent lemma is a specialization of 2.2.1.

2.2.2. <u>Lemma</u>. *Let* (E,Q) *be a weed in* X, $F: X \to 2^E$ *and let* $Q_n(e) \cap F^-(y)$ *be open in* $Q_n(e)$ *for each* $y \in E$, *while* $E = \{e_1, \ldots, e_n\}$ *and* $F^-(E) = X$. *Then there exists a set* $Z = \{z_1, \ldots, z_m\} \subset E$ *and a point* $x \in X$ *such that* $x \in Q_m(z) \subset Q_m(F(x))$; *in particular,* $x \in Q_n(e) \cap Q_n(F(x)) \subset Q_n(e) \cap cQ\, F(x) \subset cQ\, F(x)$ *for* $e = (e_1, \ldots, e_n)$.

Proof. Let us consider $G = X \backslash F^-$. Clearly, the values of $Q_n(e) \cap G$ are closed and $\bigcap \{G(y): y \in E\} = X \backslash F^-(E) = \emptyset$. In view of Th. 1.4.3 G cannot be KKM. Now it suffices to recall Lemma 2.2.1 and to take into account the inclusion $Q_m(z) \subset Q_n(e)$. □

Directly from 2.2.2 follows

2.2.3. <u>Theorem</u>. *Let* X *be a compact space,* (E,Q) *a weed in* X *and* $F: X \to 2^E$ *a mapping such that all values of* F^- *are open. Then* $cQ \circ F$ *has a fixed point. If, in addition, all values of* F *are overhulls, then* F *has a fixed point.*

Proof. There exists a finite set $E_1 \subset F(X)$ such that $F^-(E_1) = X$, X being compact and $D_F = X$. Now Lemma 2.2.2 applies to (E_1,Q) in place of (E,Q) and $E_1 \cap F$ in place of F as we have $F^-(y) = (E_1 \cap F)^-(y)$, $y \in E_1$. □

If in the above all values of F are overhulls, then from $cQ\, F(x) \subset X$ it follows that $cQ\, F(x) \subset X \cap F(x)$, $x \in X$ and one may assume $F: X \to 2^X$.

As a very particular consequence of 2.2.3 we obtain the following generalization of a Browder's theorem [5, Th. 1 p. 285]

2.2.4. <u>Theorem</u>. *Let* X *be a compact convex set in a linear space,* $F: X \to 2^X$ *a mapping and let all values of* F^- *be open. Then there exists a point* $x \in \mathrm{conv}\, F(x)$. *If, in addition, all values of* F *are convex, then* F *has a fixed point.*

The theorem to follow is a consequence of Th. 1.4.7.

2.2.5. <u>Theorem</u>. *Let* (E,Q) *be a weed in* X *and* $F: X \to 2^E$ *a mapping satisfying*

(i) *all values of* F^{-} *are compactly open*

(ii) $X\setminus F^{-}(K)$ *is compact for a c-compact set* $K \subset E$.

Then $cQ\circ F$ *has a fixed point. If in addition all values of* F *are overhulls, then* F *has a fixed point.*

Proof. Let us adopt $X = Y$, $h = id_X$ and $L = X\setminus F^{-}$ in Theorem 1.4.7. It is seen that all values of $L\colon E \to 2^X$ are compactly closed and $\bigcap\{L(x)\colon x \in E\} = X\setminus F^{-}(E) = \emptyset$. In view of Th. 1.4.7 L is not KKM and Lemma 2.2.1 applies. □

Now we are going to prove some results for usc mappings.

2.2.6. Lemma. *Let* $(X,\mathcal{U})$ *be a uniform precompact space,* (E,Q) *a weed in* X *and* $F\colon X \to 2^E$ *a mapping. Then for each* $U \in \mathcal{U}$ *there exists a finite set* $E_1 = \{e_1, \ldots, e_n\} \subset F(X)$ *and a point* $x \in Q_n(e) \cap Q_n((F\circ U)(x)) \subset cQ\, F(U(x))$.

Proof. We may require U to be open and symmetric [31, Th. 6 p. 179]. There exists a finite cover $\{U(x_i)\colon i = 1, \ldots, n\}$ of X as X is precompact. Then for $e_i \in F(x_i)$, $i = 1, \ldots, n$ and $E_1 = \{e_1, \ldots, e_n\}$ we have $X = \bigcup U(x_i) = (U\circ F^{-})(E_1) = (U^{-}\circ F^{-})(E_1) = (F\circ U)^{-}(E_1)$ ($U = U^{-}$ – symmetry). Clearly, all values of $(F\circ U)^{-} = U\circ F^{-}$ are open and we may apply Lemma 2.2.2 to E_1 and $E_1 \cap (F\circ U)$ in place of E and F, respectively. □

A simpler version of the above lemma is the following

2.2.7. Corollary. *If* X *is a precompact convex set in a linear space* Z, *and* $F\colon X \to 2^X$ *is a mapping, then for each neighbourhood* U *of zero in* Z *there is an* $x \in X$ *such that* $x \in \operatorname{conv}[F(x+U) \cap X]$.

2.2.8. Theorem. *Let* $(X,\mathcal{U})$ *be a compact uniform space,* (E,Q) *a weed in* X *and let* $F\colon X \to 2^E$, $T\colon X \to 2^X$ *be mappings satisfying*

(2.4) *for each* $x \in X$, $W \in \mathcal{U}$ *there exists a* $U \in \mathcal{U}$ *such that* $cQ(F\circ U)(x) \subset (W\circ T)(x)$.

Then $\overline{T}$ *has a fixed point.*

Proof. Suppose $\overline{T}$ has no fixed point. Then for each $x \in X$ there exist $U^x, W^x \in \mathcal{U}$ such that $U^x(x) \cap (W^x\circ T)(x) = \emptyset$. Assume $2U^x(x) \cap cQ(F\circ 2U^x)(x) = \emptyset$, $x \in X$. There exists a finite set $Z \subset X$ such that

$X = \bigcup\{U^z(z): z\in Z\}$. Let $U = \bigcap\{U^z: z\in Z\}$. For each $x\in X$ there exists a $z\in Z$ such that $x \in U^z(z)$ and hence $U(x) \cap cQ(F\circ U)(x) \subset \subset (U\circ U^z)(z) \cap cQ[(F\circ U)(U^z(z))] \subset 2U^z(z) \cap cQ(F\circ 2U^z)(z) = \emptyset$. Thus $cQ\circ F\circ U$ has no fixed point which contradicts Lemma 2.2.6. □

In consequence we obtain

2.2.9. Theorem. *Let $(X,\mathcal{U})$ be a uniform space, (E,Q) a weed in X and $F: X\to 2^E$ a mapping such that $\overline{cQ}\circ F$ is compact. Then $\overline{cQ}\circ F$ has a fixed point. If in addition all values of F are closed overhulls, then F has a fixed point.*

Proof. The set $\overline{cQ}\, F(X)$ is compact (Def. 2.1.11) and we have $\overline{cQ}\circ F: \overline{cQ}\, F(X) \to 2^{\overline{cQ}\, F(X)}$. Thus Theorem 2.2.8 applies to $X_1 = \overline{cQ}\, F(X)$, $E = F(X_1)$ and $T = cQ\circ F$. □

Corollary 2.1.16 and Th. 2.2.9 imply the following generalization of a theorem of Ky Fan [16, Th. 1 p. 122] and Glicksberg [25, Theorem p. 171] (we do not require X to be T_2).

2.2.10. Theorem. *Let X be a convex set in a locally convex space. If $F: X\to 2^X$ is an usc mapping with closed convex values and $\overline{\text{conv}}\, F(X)$ is compact, then F has a fixed point.*

The subsequent theorem is more interesting from the point of view of locally convex spaces (for convex $F(x)$ $V(F(x))$ may be considered convex in (2.5))

2.2.11. Theorem. *Let $(X,\mathcal{U})$, $(E,\mathcal{V})$ be uniform spaces, (E,Q) a weed in X and $F: X\to 2^E$, $T: X\to 2^X$ mappings. If F satisfies (2.1) (e.g. if F is usc), $F(X)$ is precompact $C = \overline{T}(X)$ is a compact set and*

(2.5) *for each $W\in\mathcal{V}$ there exists a $V\in\mathcal{V}$ such that for every $x\in X$ we have $cQ(V\circ F)(x) \subset (W\circ T)(x)$,*

holds, then $\overline{T}$ has a fixed point.

Proof. Let $U\in\mathcal{U}$, $V\in\mathcal{V}$ be symmetric, U open and let $E_1 \subset F(X)$ be a finite set for which $F(X) \subset V(E_1)$ holds. We have $(V\circ F\circ U)^- = U\circ(F^-\circ V)$, i.e. all values of $(V\circ F\circ U)^-$ are open. What is more, $X = (F^-\circ V)(E_1) = (U\circ F^-\circ V)(E_1)$. Thus $E_1,V\circ F\circ U$ satisfies the assumptions of Lemma 2.2.2 and consequently $A(U) = \{x\in X: x \in cQ(V\circ F\circ U)(x)\} \neq \emptyset$.

Clearly, inclusion $U_1 \subset U_2$, $U_1, U_2 \in \mathcal{U}$ implies $A(U_1) \subset A(U_2)$ and for $U_1, \ldots, U_n \in \mathcal{U}$ there exists a $U \in \mathcal{U}$ such that $U \subset U_1, \ldots, U_n$. Thus $\{A(U): U \in \mathcal{U}\}$ has the finite intersection property. What is more E_1 does not depend on U and therefore the family $\{A(U) \cap Q_n(e): U \in \mathcal{U}\}$ has the finite intersection property (see 2.2.2). In turn, $Q_n(e)$ is compact as being a continuous image of P^{n-1} and therefore there exists an $x \in A = \bigcap\{\overline{A(U)} \cap Q_n(e): U \in \mathcal{U}\}$. If $x \in A$ then we have $U(x) \cap U(A(U)) \neq \emptyset$, $U \in \mathcal{U}$, as $\overline{A(U)} \subset U(A(U))$ [31, Th. 7 p. 179]. Now it is seen that $\emptyset \neq U(x) \cap cQ(V \circ F \circ U)(U(x)) = U(x) \cap cQ(V \circ F \circ 2U)(x)$ and (2.1) implies $x \in U(cQ(V \circ F)(x))$, $U \in \mathcal{U}$, i.e. $x \in \overline{cQ}(V \circ F)(x)$. Now from (2.5) we find out that $B(V) = \{x \in X: x \in \overline{cQ}(V \circ F)(x)\} \subset 2W(C)$ for "small" V. We have $B(V) \cap W(C) \neq \emptyset$, $V \in \mathcal{V}$, $W \in \mathcal{U}$ as $V_1 \subset V_2$, $V_1, V_2 \in \mathcal{V}$ imply $B(V_1) \subset V(V_2)$. Therefore, $C \cap W(B(V)) \neq \emptyset$, $V \in \mathcal{V}$, $W \in \mathcal{U}$. The family $\{C \cap W(B(V))\}$ has the finite intersection property and in view of the compactness of C there exists an $x \in C \cap \bigcap\{\overline{W(B(V))}: V \in \mathcal{V}, W \in \mathcal{U}\} \subset C \cap \bigcap\{2W(B(V)): V \in \mathcal{V}, W \in \mathcal{U}\}$. Hence $\emptyset \neq U(x) \cap \overline{cQ}(V \circ F \circ U)(x) \subset W(x) \cap \overline{cQ}(2V \circ F)(x) \subset W(x) \cap W(T(x))$ for each $W \in \mathcal{U}$ and the respective $U \in \mathcal{U}$, $V \in \mathcal{V}$ (see (2.1), (2.5)). This means that $x \in \overline{T}(x)$. □

Theorem 2.2.11 can be specialized as follows

2.2.12. Theorem. *Let* $(X,\mathcal{U})$, $(E,\mathcal{V})$ *be uniform spaces* (E,Q) *a weed in* X *and* $F: X \to 2^E$ *a mapping such that* $F(X)$ *is precompact. Assume that* F *satisfies* (2.1) *(e.g.* F *is usc) and* $C = \bigcup\{cQ\, F(x): x \in X\}$ *is compact. If* (2.2) *is satisfied for* $(Z,\mathcal{W}) = (X,\mathcal{U})$, *then* $\overline{cQ} \circ F$ *is usc and has a fixed point. If in addition* $\overline{cQ}\, F(x) \subset \overline{F}(x)$, $x \in X$ *(e.g. all values of* F *are overhulls), then* $\overline{F}$ *has a fixed point. If* F *or* $\overline{F}$ *is compact and* (2.3) *holds for* $(Z,\mathcal{W}) = (X,\mathcal{U})$, *then* F *satisfies* (2.1), C *is compact and* $\overline{F}$ *has a fixed point.*

Proof. By replacing T by $cQ \circ F$ in 2.2.11 we obtain the first part of 2.2.12 (the fact that $\overline{cQ} \circ F$ is usc follows from Lemma 2.1.14). If F is compact and (2.3) is satisfied, then we obtain (2.1) (see Lemma 2.1.13), and $C \subset \overline{F}(X)$ as all values of $\overline{F}$ are overhulls (Lemma 2.1.15). In view of Def. 2.1.11 C is compact. Now we may apply 2.2.11 to $T = F$. □

In place of $F(X)$ being precompact, in 2.2.11, 2.2.12 it suffices to assume that there exists a precompact set $D \subset F(X)$ such that $F^{-}(\overline{D}) = X$. In the respective proofs one considers $E \subset D$.

From the last part of Th. 2.2.12 we derive a more general version of [36, Corol. 1.18 p. 173] (we do not require X to be T_2)

2.2.13. __Theorem__. *Let X, C be convex sets in a locally convex space and $F: X \to 2^{X+C}$ a compact mapping with closed convex values. If one of the following conditions is satisfied*

(a) $C = \{\Theta\}$,

(b) X *is compact and* C *closed*,

(c) X *is closed and* C *compact*,

then there exists a point $x \in F(x)-C$, *i.e. such that* $(x+C) \cap F(x) \neq \emptyset$.

Proof. The case (a) is a particular version of Th. 2.2.12 (see Corol. 2.1.16). Let us consider $H(x) = X \cap (F(x)-C)$, $x \in X$. Clearly, $H: X \to 2^X$ is a mapping with convex values which are closed [32, 13.1 (iv) p. 111]. Therefore, for (b) H satisfies hypothesis (a). Let us consider case (c). In view of [32, 13.1 (iii)] $F(x)-C$ is compact as being closed ($F(x)$, C are compact convex). What is more, $F(X)-C$ is compact [32, 5.2 (iv) p. 35]. The closure of a compact set in a regular space is compact [31, B (b) p. 161] and, therefore, for (c) H satisfies (a). If $F(U_x) \subset V + F(x)$, then $F(U_x) - C \subset V + F(x) - C$ holds, which means that H is usc as it suffices to consider uniform neighbourhoods of $H(x)$ [31, Th. 33 p. 199]. □

The above proof is much simpler than Lassonde's especially if we consider only locally convex spaces in Th. 2.2.11 and Lemma 2.2.2.

In particular, if $F: X \to (X+C)$ is a map and X is a T_2-space, then 2.2.13 (a) becomes a theorem of Hukuhara [29, Th. 2]. In the final part of this section we present a simple direct proof of an extension of this theorem.

The theorem to follow seems to be of less interest than 2.2.11.

2.2.14. __Theorem__ (cp. Th. 2.2.8). *Let $(X,\mathcal{U})$ be a uniform space, (E,Q) a weed in X and $F: X \to 2^E$, $T: X \to 2^X$ mappings. If $C = \overline{T}(X)$ is compact and for each $U \in \mathcal{U}$ there exists a finite set $E_1 \subset F(X)$ such that $X = (U \circ F^-)(E_1)$ and the following is satisfied*

(2.6) *for each $W \in \mathcal{U}$ there exists a $U \in \mathcal{U}$ such that for each $x \in X$ we have $cQ(F \circ U)(x) \subset (W \circ T)(x)$,*

then $\overline{T}$ has a fixed point.

Proof. Similarly as in the proof of Lemma 2.2.6 we assert that $E_1 \cap (F \circ U)$ satisfies the hypothesis of Lemma 2.2.2 which means that $A(U) = \{x \in X : x \in cQ\ F(U(x))\}$ is nonempty, $U \in \mathcal{U}$. For each $W \in \mathcal{U}$ there exists a $U \in \mathcal{U}$ such that $A(U) \subset W(C)$ (see (2.6)) and hence $C \cap W(A(U)) \neq \emptyset$, $U, W \in \mathcal{U}$. Then for an $x \in C \cap \bigcap\{\overline{A(U)}: U \in \mathcal{U}\}$ and U "small" we have $\emptyset \neq U(x) \cap cQ\ F(2U(x)) \subset W(x) \cap W(T(x))$. Hence $x \in \overline{T}(x)$ holds. □

Now we present some elementary results for lower semicontinuous mappings. First let us extend the classical definition.

2.2.15. <u>Definition</u> (cp. [2, Th. 1 p. 115]). *Let* X, Y *be spaces. Then* $F: 2^X \to 2^Y$ *is lsc if* $F^-(V)$ *is open for each open* $V \subset Y$.

2.2.16. <u>Lemma</u>. *Let* (E,Q) *be a weed in* X *and* $F: X \to 2^E$ *an lsc mapping. If* $(E,\mathcal{V})$ *is a precompact uniform space, then for each* $V \in \mathcal{V}$ *and* $E = \{e_1,\ldots,e_n\} \subset F(X)$ *such that* $F(X) \subset V(E)$ *there exists an* $x \in Q_n(e) \cap Q_n(V(F(x))) \subset cQ\ V(F(x))$.

Proof. Let V be open and symmetric [31, Th. 6 p. 179]. From the fact that F is lsc it follows that for each $y \in F(X)$ the set $(V \circ F)^-(y) = (F^- \circ V)(y)$ is open. There exists a finite set $E_1 \subset F(X)$ such that $F(X) \subset V(E_1)$ and, therefore, $(V \circ F)^-(E_1) = F^-(V(E_1)) = F^-(F(X)) = X$ holds, F being a mapping. Now we apply Lemma 2.2.2 to $E_1 \cap (V \circ F)$. □

Lemma 2.2.16 is a generalization of a Lassonde's theorem [14, (13.16) p. 104], as from the above follows

2.2.17. <u>Theorem</u>. *Let* X *be a convex set in a linear space* M *and* $F: X \to 2^X$ *an lsc mapping such that* $F(X)$ *is precompact. Then for each neighbourhood* V *of zero in* M *there exists a point* $x \in \mathrm{conv}\,(V + F(x))$; *if in addition* $V, F(x)$ *are convex, then* $x \in -V+F(x)$ *and* $(x+V) \cap F(x) \neq \emptyset$ *hold.*

From Lemma 2.2.2 one can obtain

2.2.18. <u>Lemma</u>. *Let* X *be a precompact set in a uniform space* $(E,\mathcal{V})$ *and* $H: X \to 2^E$ *an lsc mapping such that for an open* $V \in \mathcal{V}$ *we have* $H(X) \subset V(X)$ *and* $(X \cap (V \circ H)(X), Q)$ *is a weed in* X. *Then there exists an* $x \in cQ(F \circ H)(x)$.

Proof. From $H(X) \subset V(X)$ we obtain $(V \circ H)(x) \cap X \neq \emptyset$, $x \in X$. Now

$F = E_1 \cap (V \circ H)$, for a finite set $E_1 \subset X$, satisfies the assumptions of Lemma 2.2.2. Thus, there exists a point $x \in cQ\ F(x) \subset cQ(V \circ H)(x)$. □

In particular, from 2.2.18 we obtain the following more general version of [14, (13.15) p. 104]

2.2.19. Theorem. *Let* X *be a convex precompact set in a linear space* M *and* $F\colon X \longrightarrow 2^M$ *an lsc mapping such that for an open neighbourhood* V *of zero in* M *we have* $F(X) \subset X+V$. *Then there exists an* $x \in \operatorname{conv}(V+F(x))$. *If in addition* V, F(x) *are convex, then there exists an* $x \in -V+F(x)$, *i.e. such that* $(x+V) \cap F(x) \neq \emptyset$.

Now we present a fixed point theorem which generalizes a Hukuhara's result [29, Th. 2 p. 2]

2.2.20. Theorem. *Let* $f\colon X \longrightarrow X$ *be a* T_2*-compact map while* (f(X),Q) *is a weed in* X. *If the following is satisfied (cp. (6.3))*

(2.7) *for each* $y \in f(X)$ *and each* $W = W_y$ *there exists a* $V = V_y$ *such that* $cQ\ V \subset W$,

then f *has a fixed point.*

Proof. Let Z be a T_2-compact subspace of X such that $f(X) \subset Z$. Suppose f has no fixed point. Then by (2.7) there exists an open cover $\mathcal{W} = \{W_y\colon y \in Z\}$ of Z such that the following is satisfied

(*) $f^-(W) \cap cQ\ W = \emptyset$, $W \in \mathcal{W}$.

There exists an open cover $\mathcal{U} = \{U_1, \ldots, U_n\}$ being a barycentric refinement of $\mathcal{W}$ [15, Th. 5.1.12 p. 377]. Then for $e_i \in U_i$ we adopt $G(e_i) = X \setminus f^-(U_i)$. Clearly, $G(e_i)$ is closed for each $i = 1,\ldots,n$ and $\bigcap G(e_i) = X \setminus f^-(U_i) = \emptyset$. Hence in view of Th. 1.4.3 G is not KKM. We have $G(e_i) = X \setminus F^-(e_i)$ for $F(x) = \{e_i\colon f(x) \in U_i\}$ and by Lemma 2.2.1 there exists a point x such that $x \in cQ\ F(x) \subset cQ\ St(f(x),\mathcal{U}) \subset cQ\ W$ for a $W \in \mathcal{W}$. From the fact that $cQ\ \{y\} = \{y\}$, $y \in f(X)$ we obtain $x \in f^-(W)$ and, consequently, $x \in f^-(W) \cap cQ\ W$ which contradicts (*). □

Let us quote an extended version of Hukuhara's theorem which follows from 2.2.20

2.2.21. Theorem. *Let* X *be a convex set in a locally convex space and* $f\colon X \longrightarrow X$ *a* T_2*-compact map. Then* f *has a fixed point.*

In connection with Theorems 2.2.20, 2.2.12 it is interesting to compare conditions (2.7) and (2.3).

2.2.22. Lemma. *Let* $(X,\mathcal{U})$ *be a uniform space and* (E,Q) *a* T_1*-compact weed in* X, $E \subset X$. *Then the following conditions are equivalent*

(a) *for each* $W \in \mathcal{U}$, $x \in E$ *there exists a* $V \in \mathcal{U}$ *such that* $cQ\, V(x) \subset W(x)$,

(b) *for each* $W \in \mathcal{U}$ *there exists a* $V \in \mathcal{U}$ *such that for each* $x \in E$ $cQ\, V(x) \subset W(x)$ *holds.*

Proof. There exists a finite set $Z \subset E$ such that $E \subset \bigcup V^i(z_i)$ and $cQ\, 2V^i(z_i) \subset W(z_i)$. For $V = \bigcap V^i$ and every $x \in E$ there exists a point z_i such that $x \in V^i(z_i)$, i.e. $z_i \in V^i(x)$. Hence we obtain $V(x) \subset 2V^i(z_i)$ and $cQ\, V(x) \subset W(z_i)$. Condition (a) implies $\emptyset \neq cQ\,\{x\} \subset \{x\}$, $x \in E$ and, therefore, $x \in cQ\, V(x) \subset W(z_i)$. Now we have $cQ\, V(x) \subset W(z_i) \subset 2W(x)$, $x \in E$, i.e. from (a) follows (b). □

2.2.23. Remark. *From the proofs of* Theorems 2.2.8, 2.2.11 *and* 2.2.14 *it follows that we may assume* Q *in* (2.4), (2.5), (2.6) *to depend on* W, *i.e. there exists a family* $\{Q^W\colon W \in \mathcal{U}\}$ *such that for each* $W \in \mathcal{U}$ $(F(X),Q^W)$ *is a weed in* X *and the respective inclusion holds for* Q^W *in place of* Q. *Similarly,* Q *in* (2.7) *may depend on* $\mathcal{W} = \{W_y\colon y \in f(X)\}$.

3. Cross theorems, coincidences

3.1. CROSS THEOREMS

This section is devoted to the fixed point theorems for the composition of mappings. The main resluts are 3.1.3, 3.1.5. Their versions for linear spaces, Theorems 3.1.4, 3.1.8 and Remark 3.1.6 seem to be of importance.

It appears that composition of two mappings with convex values may fail to have a convex value. This is the case, e.g. for $H \circ F: R^2 \to 2^{R^2}$, where $F(x) = [-1,1] \subset R$, $x \in R^2$, $H(0) = \{0\} \times R$ and $H(x) = (x,0)$ for $x \neq 0$, $x \in R$. To avoid these difficulties we use the Cartesian product of mappings.

For $F_i: X_i \to 2^{E_{i+1}}$, $i = 1, \ldots, m-1$, $F_m: X_m \to 2^{E_1}$ we define the cross function $F: X \to 2^E$ as follows

$$F(x) = (F_m \times F_1 \times \ldots \times F_{m-1})(x_1, \ldots, x_m) = F_m(x_m) \times F_1(x_1) \times \ldots \times F_{m-1}(x_{m-1}) \subset E = P\{E_i: i = 1, \ldots, m\}, \quad x = (x_1, \ldots, x_m) \in X = P\{X_i: i = 1, \ldots, m\}. \tag{3.1}$$

As it is seen $F_m \circ F_{m-1} \circ \ldots \circ F_1$ has a fixed point in X_1 if and only if there exist $x_i \in X_i$, $i = 1, \ldots, m$ such that $x_1 \in F_m(x_m)$, $x_m \in F_{m-1}(x_{m-1})$, $\ldots$, $x_2 \in F_1(x_1)$, i.e. if and only if F as in (3.1) has a fixed point in X.

3.1.1. <u>Remark</u>. *We have proved some fixed point theorems for weeds and therefore it is convenient to define the respective structure for the Cartesian product. If* (E_s, Q^s) *is a weed in* X_s, $s \in S$, *then for* $E = P\{E_s: s \in S\}$, $X = P\{X_s: s \in S\}$ *sequence* Q *is defined as follows: if* $e = (e_1, \ldots, e_n) \in E^n$, *then we write* $e_s = (p_s(e_1), \ldots, p_s(e_n))$ (p_s *- projection into* E_s) *and* $Q_n = P\{Q_n^s: s \in S\}$, *where* $Q_n(e,t) = P\{Q_n^s(e_s,t): s \in S\}$, $t \in P^{n-1}$. *Clearly,* $Q_n: E^n \times P^{n-1} \to X$ *fulfil all requirements of* Def. 1.2.7 *(the Tychonoff topology in* X*), i.e.*

(E,Q) is a weed in X. What is more, for $A = P\{A_s: s\in S\} \subset X$ we have $Q_n(A) = P\{Q_n^s(A_s): s\in S\}$ and $cQ\,A = P\{cQ^s A_s: s\in S\}$. The last two equalities are very useful for cross mappings.

Let us present an assumption which will appear here in each of general *cross theorems* (i.e. fixed point theorems for cross mappings)

3.1.2. *Mapping $F: X \to 2^E$ is defined by (3.1), (E_i, Q^i) is a weed in X_i, $i = 1, \dots, m$ and $Q_n = Q_n^1 \times \dots \times Q_n^m$, $n\in N$.*

Theorem 1.4.5 and Lemma 2.2.1 enable us to prove the following

3.1.3. <u>Theorem</u>. *If 3.1.2 holds, all values of $F_1^-, \dots, F_m^-$ are compactly open and for at least one index i we have*

(3.2) $X_i \setminus F_i^-(K)$ *is compact for a c-compact set* $K \subset E_j$ $(j = 1$ if $m = 1)$

and $cQ^j F_i(x_i) \subset T_i(x_i)$ (for the respective j), $x_i \in X_i$, $i = 1, \dots, m$, then there exists a point $x \in X$ such that $(x_1, \dots, x_m) \in T_m(x_m) \times T_1(x_1) \times \dots \times T_{m-1}(x_{m-1})$. If in addition $E = X$, all values of F_i are Q^j-overhulls (the respective j), $i = 1, \dots, m$, then $F_m \circ \dots \circ F_1$ has a fixed point.

Proof. Assume that $K \subset E_1$ is c-compact (Def. 1.4.6). Then the set $X_m \setminus F_m^-(K)$ is compact and all values of F_m^- are compactly open (Def. 1.4.4). In consequence there exists a finite set $Z_1 \subset E_1$ such that for $E_1' = K\cup Z$ we have $X_m = F_m^-(E_1')$. The compactness of $K_Z \subset X_1$ implies that there exists a finite set $E_2' \subset E_2$ for which $X_1' = K_Z \subset F_1^-(E_2')$. The set $X_2' = cQ^{2\prime}E_2'$ is compact (Remark 1.2.13) and therefore there exists a finite set $E_3' \subset E_3$ with $X_2' \subset F_2^-(E_3')$. By induction we define a finite set $E'_m \subset E_m$ for which $X'_{m-1} = cQ^{(m-1)\prime}E'_{m-1} \subset F^-_{m-1}(E'_m)$. Clearly, $X'_m = cQ^{m\prime}E'_m \subset X_m = F_m^-(E_1')$ holds. Let us adopt $E' = E_1' \times \dots \times E_m'$, $X' = X_1' \times \dots \times X_m'$ and $Q_n'(e,t) = Q_n^{1\prime}((p_1(e_1), \dots, p_1(e_n)),t) \times \dots \times Q_n^{m\prime}((p_m(e_1), \dots, p_m(e_n)),t)$ for $e = (e_1, \dots, e_n) \in (E')^n$, $t \in P^{n-1}$, $n \in N$ (see (1.10)). As it is seen (E',Q') is a weed in X' (Remark 3.1.1) and, in addition, X' is compact. For $F(x) = F_m(x_m) \times F_1(x_1) \times \dots \times F_{m-1}(x_{m-1})$, $x \in X'$ let us adopt $G(x) = X' \setminus F^-(y)$, $y \in E'$. All values of G are compact, closed and $\bigcap\{G(x): x\in E'\} = X' \setminus F^-(E') = \emptyset$. In view of Th. 1.4.5 G is not KKM. By Lemma 2.2.1 there exists an

$x \in cQ'F(x)$ and consequently $x \in cQ\,F(x) \subset T(x)$ (Remark 1.2.13). If $cQ\,F(x) \subset F(x)$, e.g. if all values of F_i are overhulls, $i = 1, \dots, m$, then $x_1 \in (F_m \circ \dots \circ F_1)(x_1)$. □

In particular, from Th. 3.1.3 ($X_i = E_i$, $i = 1, \dots, m$) we obtain

3.1.4. __Theorem__. *Let* $X_1, \dots, X_m$ *be convex sets in linear spaces,* $F_i\colon X_i \to 2^{X_{i+1}}$, $F_m\colon X_m \to 2^{X_1}$, $i = 1, \dots, m-1$ *mappings with convex values and such that all values of* F_i^- *are compactly open,* $i = 1, \dots, m$. *If for an index* i (3.2) *is satisfied* ($E_j = X_j$), *then* $F_m \circ \dots \circ F_1$ *has a fixed point.*

Let us prove a theorem which brings together 2.2.8, 2.2.11 and 2.2.14.

3.1.5. __Theorem__. *Assume* 3.1.2. *Let* $(X_i, \mathcal{U}_i)$ *be uniform spaces and let mappings* T_i *satisfy* (3.1) *for* $E = X$. *If* $\overline{T}_i(X_i)$ *are compact and for each* $i = 1, \dots, m$ *separately, one of the following is satisfied*

(a) X_i *is compact and for each* $x_i \in X_i$, $W \in \mathcal{U}_j$ *there exists a* $U \in \mathcal{U}_i$ *such that* $cQ^j(F_i \circ U)(x_i) \subset (W \circ T)(x_i)$ (*cp.* 2.2.8),

(b) F_i *is usc* $(E_j, \mathcal{V}_j)$ *is a uniform space,* $F_i(X_i)$ *is precompact and for each* $W \in \mathcal{U}_j$ *there exists a* $V \in \mathcal{V}_j$ *such that for each* $x_i \in X_i$ *we have* $cQ^j(V \circ F_i)(x_i) \subset (W \circ T_i)(x_i)$ (cp. 2.2.11),

(c) *for each* $U \in \mathcal{U}_i$ *there exists a finite set* $E_j \subset F(X_i)$ *such that* $X_i = (U \circ F_i^-)(E_j)$ *and for each* $W \in \mathcal{U}_j$ *there exists a* $U \in \mathcal{U}_i$ *such that* $cQ^j(F_i \circ U)(x_i) \subset (W \circ T_i)(x_i)$, $x_i \in X_i$ (*cp.* 2.2.14),

then $\overline{T}$ *has a fixed point, i.e. there exists an* $x_1 \in X_1$ *such that* $x_1 \in (\overline{T}_m \circ \dots \circ \overline{T}_1)(x_1)$.

Proof. Let us write $H_i = F_i \circ U_i$ whenever F_i satisfies (a) or (c) and $H_i = V_j \circ F_i \circ U_i$ in case (b). Mapping $H = H_m \times H_1 \times \dots \times H_{m-1}$ fulfils the requirements of Lemma 2.2.2 (see the proofs of Theorems 2.2.11, 2.2.14). Hence $A(U) = \{x \in X\colon x \in cQ\,H(x)\} \neq \emptyset$, $U \in \mathcal{U}$ ($\mathcal{U} = P\{\mathcal{U}_i\colon i = 1, \dots, m\}$). Now we state that $\bigcap\{\overline{A(U)}\colon U \in \mathcal{U}\} \neq \emptyset$ which is a little more complicated for (b), where one should remember that $\overline{A_j(U)} \cap Q_n^j(p_j(e))$ is a compact set (cp. the proof of 2.2.11). Now we consider the sets $B(V)$ of fixed points of $\overline{cQ} \circ H$

for $U_i = id_{X_i}$, $i = 1, \ldots, m$ and then we follow the remaining part of the proof of Th. 2.2.11. □

3.1.6. Remark. *If* $(X_i, \mathcal{U}_i)$ *is a paracompact space, all values of* F_i^- *are open and* $(F_i(X_i), Q^j)$ *is a weed in* X_j, *then* $cQ^j \circ F_i$ *admits a selection* (Th. 7.2). *It enables us to mix up* Theorems 3.1.3 *and* 3.1.5 *if such selections satisfy the assumptions of* Th. 3.1.5.

Since the case (c) in 3.1.5 seems to be of less importance, it will be disregarded in further considerations.

Theorems 2.2.8, 2.2.11 were followed by some special results. Let us formulate the respective cross theorems for overhulls.

3.1.7. Theorem. *Assume* 3.1.2 *for* $X = E$ *and let* $(X_i, \mathcal{U}_i)$ *be uniform spaces,* $F_i^-(X_i)$ *compact sets and all values of* F_i *closed overhulls,* $i = 1, \ldots, m$. *If for each* i *separately one of the following conditions is satisfied*

(a) X_i *is compact and* $\overline{cQ^j} \circ F_i$ *is usc (cp.* 2.2.9*),*

(b) F_i *is usc and for each* $W \in \mathcal{U}_j$ *there exists a* $V \in \mathcal{U}_j$ *such that for each* $x_i \in X_i$ $cQ^j(V \circ F_i)(x_i) \subset (W \circ F_i)(x_i)$ *holds (cp.* 2.2.12*),*

then $F_m \circ \ldots \circ F_1$ *has a fixed point.*

From 3.1.7 we obtain (cp. Corol. 2.1.16)

3.1.8. Theorem. *Let* $X_1, \ldots, X_m$ *be convex sets in locally convex spaces and* $F_i: X_i \to 2^{X_{i+1}}$, $F_m: X_m \to 2^{X_1}$, $i = 1, \ldots, m-1$ *compact mappings with closed convex values. Then* $F_m \circ \ldots \circ F_1$ *has a fixed point.*

The case $m = 2$ is of particular interest. The notion "cross function" is then motivated by $F(x_1, x_2) = F_2^-(x_2) \times F_1(x_1)$ and the formula $(x_1, x_2) \in F_2(x_2) \times F_1(x_1)$ explains the sense of "cross theorems".

3.2 COINCIDENCES, CASE $T, H: X \to 2^Y$

3.2.2 which enables us to prove, e.g. Theorems 4.1.7, 4.3.2 is probably the main theorem of this section.. As it is seen, condition $(x,y) \in H^-(y) \times T(x)$ is satisfied if and only if $y \in T(x) \cap H(x)$. Therefore cross theorems for $m = 2$ are equivalent to theorems on coincidences. We will profit from this dependence in what follows, without explicit mentioning it.

In particular, from Th. 3.1.3 we obtain

3.2.1. __Theorem__. *Let* X,Y *be spaces,* $T: X \to 2^Y$, $J = H^-: Y \to 2^X$ *mappings and* (J(Y),Q), (T(X),S) *weeds, respectively in* X,Y. *If the following conditions are satisfied*

(i) *all values of* T,J *are overhulls,*

(ii) *all values of* T^-, J^- *are compactly open,*

(iii) $Y \backslash J^-(K) = Y \backslash H(K)$ *is compact for a c-compact set* $K \subset X$.

Then there exists a point $(x,y) \in J(y) \times T(x)$, *i.e. such that* $y \in T(x) \cap H(x)$.

It can be seen that 3.2.1 extends [36, Th. 1.10 p. 168], which was proved in a more sophisticated way.

If X or Y is compact, then (iii) is superfluous and 3.2.1 becomes a direct generalization of a Fan's theorem [14, (4.1) p. 77].

The theorem to follow seems to be very convenient from the point of view of applications of the fixed point theory. The very formulation of it employs an approximation technique which usually comes together with the fixed theory methods in proofs of some advanced results. Our theorem is of importance for Chapter 4.

3.2.2. __Theorem__. *Assume that* (E,Q), (F,S) *are weeds in* X,Y, *respectively,* Z *is a space,* $G,H: X \times Y \to 2^Z$ *are mappings and* $\mathcal{U},\mathcal{V}$ *families of mappings* $U,V: X \times Y \to 2^Z$ *such that every* $V \in \mathcal{V}$ *contains a* $U \in \mathcal{U}$, *for each* $x \in E$, $y \in B$ $\mathcal{V}(x,y)$ *is a filter-base and each neighbourhood of* H(x,y) *contains a* $V(x,y) \in \mathcal{V}(x,y)$. *For each* $x \in X$, $y \in B$ *let* G(x,y) *be closed or* G(x,y) *compact and* Z *a* T_1*-space, or* G(x,y) *compact* H(x,y) *closed and* Z *regular. If the following conditions are satisfied*

(3.3) $\emptyset \neq T(x) \subset \{y \in F: G(x,y) \cap U(x,y) \neq \emptyset\}$, $x \in X$ *(thus for each* $x \in X$ *there exists a* $y \in F$ *such that* $G(x,y) \cap U(x,y) \neq \emptyset$*),*

(3.4) *all values of* T *are S-overhulls,*

(3.5) $T^-(y)$ *is compactly open,* $y \in F$,

(3.6) $J(y) = \{x \in E: G(x,y) \cap V(x,y) = \emptyset\}$ *is a Q-overhull,* $y \in Y$, (3.7) $J^{-}(x)$ *is compactly closed*, $x \in E$,

(3.8) $B = \{y \in Y\colon G(x,y) \cap W(x,y) \neq \emptyset,\ x \in K\}$ *is compact for a c-compact set* $K \subset E$ *and a* $W \in \mathscr{V}$,

then $G(x,y) \cap H(x,y) \neq \emptyset$ *for a* $y \in B$ *and all* $x \in E$.

Proof. Condition (3.3) means that $T\colon X \longrightarrow 2^F$ is a mapping. Conditions (3.4) - (3.7) are equivalent to (i), ii) of Th. 3.2.1. What is mcre, for $V \subset W$ we have $Y \backslash J^{-}(K) = \{y \in Y\colon G(x,y) \cap V(x,y) \neq \emptyset,\ x \in K\} \subset B$, which means that $Y \backslash J^{-}(K)$ is compact as being a closed subset of B which is compact (see (3.8)). Hence, (iii) of Th. 3.2.1 holds. For $U \subset V$ let us suppose that $J\colon Y \longrightarrow 2^F$ is a mapping. Hence, by 3.2.1 there exists a point $(x,y) \in J(y) \times T(x)$, i.e. such that $\emptyset \neq G(x,y) \cap U(x,y) \subset G(x,y) \cap V(x,y) = \emptyset$. Thus, J cannot be a mapping and for a $y \in Y$ we have $J(y) = \emptyset$. consequence, $C_V = \{y \in Y\colon G(x,y) \cap V(x,y) \neq \emptyset,\ x \in E\}$ is nonempty for each $V \in \mathscr{V}$. By assuming $V \subset W$ we state that $C_V \subset B$ and, consequently, C_V which is closed (see (3.7)) is compact. What is more, the family $\{C_V\colon V \in \mathscr{V}\}$ has the finite intersection property, $\mathscr{V}(x,y)$ being a filter base [15, p. 76], $x \in E$, $y \in B$. Hence, $C = \bigcap\{C_V\colon V \in \mathscr{V}\}$ is a nonempty set and obviously $C \subset B$. If $y_o \in C$, then $G(x,y_o) \cap V(x,y_o) \neq \emptyset$, $V \in \mathscr{V}$, $x \in E$. Let $G(x,y_o)$ be closed and suppose $G(x,y_o) \cap H(x,y_o) = \emptyset$. Then $Z \backslash G(x,y_o)$ is a neighbourhood of $H(x,y_o)$ and hence for a $V \in \mathscr{V}$ we have $\emptyset \neq G(x,y_o) \cap V(x,y_o) \subset G(x,y_o) \cap (Z \backslash G(x,y_o)) = \emptyset$ which is a contradiction. If Z is a T_1-space, then for each $z \notin A = H(x,y_o)$, $Z \backslash \{z\}$ is a neighbourhood of A and, therefore, $\bigcap \mathscr{V}(x,y_o) \subset H(x,y_o)$. The same results for closed $H(x,y_o)$ and regular Z [31, p. 113]. Assume $z_V \in G(x,y_o) \cap V(x,y_o)$, $V \in \mathscr{V}$ and let $G(x,y_o)$ be compact. Then $G(x,y_o)$ contains a cluster point of net $(z_V)_{V \in \mathscr{V}}$ [31, Th. 2 p. 136]. Every neighbourhood of $H(x,y_o)$ contains point z and, therefore, $z \in H(x,y_o)$. □

3.2.3. <u>Remark</u>. Theorem 3.2.2 *remains valid, e.g. for* $\mathscr{U} = \{U\}$, $\mathscr{V} = \{V,W\}$ *while* $U \subset V \subset H$, $V \subset W$. *In such a case* Z *need not be a space and in the respective proof we do not consider the intersection of* C_V.

Let us quote a simplified version of 3.2.2

3.2.4. <u>Theorem</u>. *Let* (E,Q), (F,S) *be weeds in* X,Y *respectively and* $G,H: X \times Y \longrightarrow 2^Z$ *mappings satisfying*

(3.3') $\emptyset \neq T(x) \subset \{y \in F: G(x,y) \cap H(x,y) \neq \emptyset\}$, $x \in X$,

(3.4') *all values of* T *are S-overhulls,*

(3.5') *all values of* T^- *are compactly open,*

(3.6') $J(y) = \{x \in E: G(x,y) \cap H(x,y) = \emptyset\}$ *is a Q-overhull* $y \in Y$,

(3.7') *all values of* J^- *are compactly open,*

(3.8') $B = \{y \in Y: G(x,y) \cap H(x,y) \neq \emptyset,\ x \in K\}$ *is compact for a c-compact set* $K \subset E$.

Then $G(x,y) \cap H(x,y) \neq \emptyset$ *for a* $y \in B$ *and all* $x \in E$.

Theorem 3.2.1 is a consequence of 3.1.3 and we derive the following in a similar way from Th. 3.1.7.

3.2.5. <u>Theorem</u>. *Let* $(X,\mathcal{U})$, $(Y,\mathcal{V})$ *be uniform spaces,* $T: X \longrightarrow 2^Y$, $J = H^-: Y \longrightarrow 2^X$ *mappings and* $(J(Y),Q)$, $(T(X),S)$ *weeds in* X, Y, *respectively. If* $\overline{T}(X)$, $\overline{J}(Y)$ *are compact, all values of* T, J *are closed overhulls and* T, J *separately satisfy at least one of the following conditions (with the respective substitutions)*

(a) X_1 *is compact and* $\overline{cQ^2} \circ F_1$ *is usc,*

(b) F_1 *is usc and for each* $W \in \mathcal{U}_2$ *there exists a* $V \in \mathcal{U}_2$ *such that for each* $x_1 \in X_1$ *we have* $cQ^2(V \circ F_1)(x_1) \subset \subset (W \circ F_1)(x_1)$,

then there exists a point $(x,y) \in J(y) \times T(x)$, *i.e. for an* $x \in X$ $T(x) \cap H(x) \neq \emptyset$ *holds.*

Theorem 3.2.5 leads to

3.2.6. <u>Theorem</u>. *Let* $(X,\mathcal{U})$, $(Y,\mathcal{V})$ *be uniform spaces,* $G,H: X \times Y \longrightarrow 2^Z$ *mappings and let* $T: X \longrightarrow 2^Y$ *be a mapping such that* $T(x) \subset \{y \in Y: G(x,y) \cap H(x,y) \neq \emptyset\}$. *If we require* T,J, *where* $J(y) = \{x \in X: G(x,y) \cap H(x,y) = \emptyset\}$, $y \in Y$, *to satisfy all assumptions of Th.*

3.2.5 *but* J *being a mapping, then there exists a point* $y \in Y$ $(y \in Y \setminus J^-(X))$ *such that* $G(x,y) \cap H(x,y) \neq \emptyset$, $x \in X$.

The above theorem contains strong requirements relative to the continuity of J, T. It is easier to verify that values of J^-, T^- are compactly open (see 3.2.4).

As regards the coincidence theorems, the assumptions on J can be formulated in another way. In place of $J = H^-: Y \to 2^X$ being a mapping and $(J(Y),Q)$ a weed one may assume (E,Q) to be a weed in X and $H: X \to 2^Y$ such that $H(E) = Y$ (H need not be a mapping).

3.3. COINCIDENCES, CASE $h: X \to Y$, $H: X \to 2^Y$

In order to obtain coincidence results for h,H as above, we apply more effective Theorems 1.4.5, 1.4.7. The theorems of the preceding section contain the case $T = h: X \to Y$, nevertheless they are not sufficiently general. Theorem 3.3.4 can be a useful tool in proving inequalities (see e.g. Th. 4.1.11).

3.3.1. <u>Theorem</u>. *Let* (E,Q) *be a weed in* X, Y *a space and* $J = H^-: Y \to 2^E$ *a mapping such that all values of* J^- *are compactly open. If* X *is a compact space and* $h \in C(X,Y)$ *or if* Y *is compact and* $h \in FC(X,Y)$, *then* $G = h^- \circ (Y \setminus J^-)$ *is not KKM and, consequently,* $cQ \circ J \circ h$ *has a fixed point. If, in addition,* $J(y)$ *is an overhull for each* $y \in h(X)$, *then* $J \circ h$ *has a fixed point, i.e.* $h(x) \in H(x)$ *for an* $x \in E$.

Proof. Let us adopt $L = Y \setminus J^-$. As it is seen all values of L are compactly closed. What is more, $\bigcap\{L(x): x \in E\} = Y \setminus J^-(E) = \emptyset$. By Th. 1.4.5 $G = h^- \circ L = h^-(Y) \setminus (h^- \circ J^-) = X \setminus (J \circ h)^-$ is not KKM. Now in view of Lemma 2.2.1 $cQ \circ J \circ h$ has a fixed point. □

Let us point out that if all values of J are overhulls, i.e. $cQ\, J(y) \subset J(y)$, $y \in Y$, then $J(Y) \subset X$ and we may assume $E \subset X$. The same concerns the theorem to follow which extends [36, Th. 1.1 p. 158].

3.3.2. <u>Theorem</u>. *Let* (E,Q) *be a weed in* X, Y *a space and* $J = H^-: Y \to 2^E$ *a mapping, satisfying*

(i) *all values of* J^- *are compactly open,*

(ii) $Y \setminus J^-(K)$ *is compact for a c-compact set* $K \subset E$.

Then for each $h \in C(X,Y)$ $G = h^- \circ (Y \setminus J^-)$ *is not KKM and,* *consequently,* $cQ\circ J\circ h$ *has a fixed point. If, in addition, for each* $y \in h(X)$ $J(y)$ *is an overhull, then* $J\circ h$ *has a fixed point in* E.

Proof. This time we apply Th. 1.4.7 to $L = Y\setminus J^{-}$. As it is seen $\bigcap\{L(x): x\in K\} = Y\setminus J^{-}(K)$ and (ii) is equivalent to (iii) of Th. 1.4.7. □

3.3.3. Remark. Theorems 3.3.1, 3.3.2 *are equivalent to* Theorems 1.4.5, 1.4.7, *respectively. Namely, from the assumption that* $G = h^{-}\circ(Y\setminus J^{-})$ *is KKM and* 3.3.1 *or* 3.3.2 *it may follow that* J *is not a mapping, i.e.* $\emptyset \neq Y\setminus J^{-}(E) = \bigcap\{L(x): x\in E\}$.

In accordance with Remark 1.4.8, condition (ii) of Th. 3.3.2 is superfluous, if X or Y is compact.

Now let us present an analog of Th. 3.2.2.

3.3.4. Theorem. *Let* (E,Q) *be a weed in* X, Y *a space and for* $Z,G,H,\mathcal{U},\mathcal{V}$ *as in* 3.2.2 *let* (3.6), (3.7), (3.8) *and the following be satisfied*

(3.9) *for each* $U\in\mathcal{U}$ *there exists an* $h \in C(X,Y)$ $(h \in FC(X,Y)$ *if* Y *is compact) such that* $G(x,h(x)) \cap U(x,h(x)) \neq \emptyset$, $x\in X$.

Then $G(x,y) \cap H(x,y) \neq \emptyset$ *for a* $y\in B$ *and all* $x\in E$.

Proof. If $J(y) \neq \emptyset$, $y\in Y$, then in view of Th. 3.3.2 (3.3.1) for $U \subset V$ and h as in (3.9) there exists an $x\in E$ with $G(x,h(x)) \cap V(x,h(x)) = \emptyset$ which contradicts (3.9). Now we follow the remaining part of the proof of Th. 3.2.2. □

According to Remark 3.2.2 the above theorem has simpler versions. Let us quote the following (cp. Th. 3.2.4)

3.3.5. Theorem. *Let* (E,Q) *be a weed in* X, Y *a space and for* G,H *as in* Th. 3.2.4 *let* (3.6'), (3.7'), (3.8') *and the following be satisfied*

(3.9') *there exists a* $h \in C(X,Y)$ $(h \in FC(X,Y)$ *if* Y *is compact) such that* $G(x,h(x)) \cap H(x,h(x)) \neq \emptyset$, $x\in X$.

Then $G(x,y) \cap H(x,y) \neq \emptyset$ *for a* $y\in B$ *and all* $x\in E$.

For $H(x,y) = Z\setminus A$, $x\in X$, $y\in Y$ where $A \subset Z$ is a fixed set and $G = g: X \times Y \to Z$, the above theorem extends [36, Th. 1.1' p. 159]. We do not quote this theorem explicitely. In Lassonde's paper it was used as a tool in proving some other results. We will apply Theorems 3.2.2, 3.2.4, 3.3.4 and 3.3.5.

3.3.6. <u>Remark</u>. *According to the technique of the proof of* Th. 3.2.2 (3.2.4) *and of* 3.3.4 (3.3.5), *conditions* (3.3) - (3.9) *can be replaced by the following system*

(3.3'') $\emptyset \neq T(x) \subset \{y\in F: G(x,y) \cap U_1(x,y) = \emptyset\}$, $x\in X$,

(3.4'') *all values of* T *are S-overhulls,*

(3.5'') $T^-(y)$ *is compactly open,* $y\in F$,

(3.6'') $J(y) = \{x\in E: G(x,y) \cap V_1(x,y) \neq \emptyset\}$ *is a Q-overhull,* $y\in Y$,

(3.7'') $J^-(x)$ *is compactly open,* $x\in E$,

(3.8'') $B = \{y\in Y: G(x,y) \cap W_1(x,y) = \emptyset, x\in K\}$ *is compact for a c-compact set* $K \subset E$ *and a* $W_1 \in \mathcal{V}_1$,

(3.9'') *for each* $U_1 \in \mathcal{U}_1$ *there exists a* $h \in C(X,Y)$ ($h \in FC(X,Y)$ *if* Y *is compact) such that* $G(x,h(x)) \cap U_1(x,h(x)) = \emptyset$, $x\in X$.

The families $\mathcal{U}_1,\mathcal{V}_1$ *consist of mappings in the form* $U_1 = Z\setminus U$, $V_1 = Z\setminus V$, $U\in\mathcal{U}$, $V\in\mathcal{V}$ *and* $W_1 = Z\setminus W$, *where* $\mathcal{U},\mathcal{V},W$ *are as in* Th. 3.2.2. *The respective proof, e.g. of* Th. 3.2.2 *leads to the following contradiction* $\emptyset \neq G(x,y) \cap V_1(x,y) \subset G(x,y) \cap U_1(x,y) = \emptyset$.

The above remark is trivial if $G = g: X \times Y \to Z$. For $G: X \times Y \to 2^Z$ conditions (3.3'') - (3.9'') are not the complements of (3.3) - (3.9).

Now we present some theorems for usc mappings

3.3.7. <u>Theorem</u>. *Let* $(X,\mathcal{U})$ *be a uniform space,* (E,Q) *a weed in* X, Y *a space and* $J = H^-: Y\to 2^E$ *a mapping such that all values of* J *are closed overhulls. If one of the following is satisfied*

(i) X *is compact and* $\overline{cQ}\circ J$ *is usc,*

(ii) J *is compact and for each* $W\in\mathcal{U}$ *there exists a* $V\in\mathcal{U}$ *such that for each* $y\in Y$ $cQ(V\circ J)(y) \subset (W\circ J)(y)$,

then for each $h \in C(X,Y)$ $J\circ h$ *has a fixed point, i.e. there exists an* $x\in E$ *with* $h(x) \in H(x)$.

Proof. In place of F in 2.2.9 or 2.2.12 we consider $J \circ h$ (see Lemma 2.1.4). □

In particular (see 2.2.13) 3.3.7 implies

3.3.8. Theorem. *Let* X *be a convex set in a locally convex space and* Y *a space. If* $J: Y \to 2^X$ *is compact with closed convex values, then for each* $h \in C(X,Y)$ $J \circ h$ *has a fixed point.*

From 3.3.8 we obtain the following generalization of [36, Corol. 1.19 p. 173] and [30, Th. 1 p. 501] for X being a convex subset of a locally convex space .

3.3.9. Theorem. *Let* (E,Q) *be a relatively compact weed in a uniform space* $(X,\mathcal{U})$, Y, Z *spaces and* $C \subset Z$ *a closed set. If mapping* $G: E \times Y \to 2^Z$ *is usc,* $J(y) = \{x \in E: G(x,y) \cap C \neq \emptyset\}$ *is nonempty for each* $y \in Y$ *and the following is satisfied*

(3.10) *for each* $W \in \mathcal{U}$ *there exists a* $V \in \mathcal{U}$ *such that for every* $y \in Y$ $cQ(V \circ J)(y) \subset (W \circ J)(y)$,

then for each $h \in C(X,Y)$ *there exists an* $x \in E$ *for which* $G(x,h(x)) \cap C \neq \emptyset$.

Proof. The set $G^{-}(C)$ is closed (Lemma 2.1.5), G being usc and C closed. Hence, $G_J = \{(y,x) \in Y \times X: G(x,y) \cap C \neq \emptyset\}$ is a closed set and in view of Lemma 2.1.8 (E is relatively compact) J is compact with closed compact values. What is more, by Lemma 2.1.15 all values of J are overhulls. Now in view of Th. 3.3.7 (ii) $J \circ h$ has a fixed point. □

Theorem 3.3.9, in turn, leads to an extension of [36, Prop. 1.7 p. 164] and of [20, Th. 10 p. 197] (cf. Ex. 1.2.8, Corol. 2.1.16)

3.3.10. Theorem. *Let* (E,Q) *be a relatively compact weed in a uniform space* $(X,\mathcal{U})$, Y,Z *spaces,* Z *normal and* $C \subset Z$ *a closed set. If* $G: E \times Y \to 2^Z$ *is an usc mapping with closed values which satisfies* (3.10) *and such that for a base* $\mathcal{V}$ *of neighbourhoods of* C *the set* $J_V(y) = \{x \in E: G(x,y) \cap V \neq \emptyset\}$ *is a nonempty overhull,* $y \in Y$, *then for each* $h \in C(X,Y)$ *there exists an* $x \in E$ *such that* $G(x,h(x)) \cap C \neq \emptyset$.

Proof. It suffices to show that $J(y) = \{x \in E: G(x,y) \cap C \neq \emptyset\}$ is nonempty for each $y \in Y$ and to apply Th. 3.3.9. The intersection of overhulls $\bigcap\{J_V(y): V \in \mathcal{V}\}$ is an overhull, $y \in Y$ (see Lemma 1.2.4). In addition $\bigcap \overline{J_V}(y)$ is nonempty as being an intersection of closed compact sets. If $x \notin J_V(y)$, then $G(x,y) \cap V = \emptyset$ and there exists a neighbourhood of the closed set $G(x,y)$ which is disjoint with a $W \in \mathcal{V}$ (Z is normal). As $G(\cdot,y)$ is usc, there exists a U_x such that $G(U_x,y) \cap \overline{W} = \emptyset$, i.e. $x \notin J_V(y)$. Hence $J(y) = \bigcap J_V(y) = \bigcap \overline{J_V}(y) \neq \emptyset$, $y \in Y$. □

3.3.11. Remark. *If* $G = g \in C(E \times Y, Z)$ *in* 3.3.10, *then it suffices to assume* Z *to be regular in place of being normal, the values of* g *need not be closed. In this connection the assumption of* Y *being Hausdorff in* [36, Corol. 1.8 p. 165] *is superfluous.*

From 3.3.7 we obtain (cp. Th. 3.2.6)

3.3.12. Theorem. *Let* (E,Q) *be a weed in a uniform space* $(X,\mathcal{U})$, Y *a space and* $G,H: X \times Y \longrightarrow 2^Z$ *mappings. If we require* $J: Y \longrightarrow 2^E$, $J(y) = \{x \in E: G(x,y) \cap H(x,y) = \emptyset\}$, $y \in Y$ *to satisfy all assumptions of* Th. 3.3.7 *but* J *being a mapping, and there exists an* $h \in C(X,Y)$ *such that for every* $x \in X$ $G(x,h(x)) \cap H(x,h(x)) \neq \emptyset$, *then there exists a* $y \in Y$ *such that* $G(x,y) \cap H(x,y) \neq \emptyset$, $x \in E$.

4. Examples of applications

4.1. INEQUALITIES

This section is devoted to inequalities for numeric functions. The main results are 4.1.7, 4.1.11, 4.1.12 (see Remark 4.1.13). First we present some properties of numeric functions which are related to the requirements of Chapter 3.

4.1.1. Definition. *Let* (X,Q) *be a weed. Then* $g\colon X \to \bar{R}$ *is quasiconvex if* $A_r = \{x \in X\colon g(x) \le r\}$ *is an overhull,* $r \in R$; g *is quasiconcave if* $-g$ *is quasiconvex.*

4.1.2. Lemma. *Let* (X,Q) *be a weed. Then* $g\colon X \to \bar{R}$ *is quasiconvex if and only if* $B_r = \{x \in X\colon g(x) < r\}$ *is an overhull,* $r \in R$. *If* g *is quasiconcave, then* $A_{-\infty}$, B_{∞} *are overhulls.*

Proof. It can be seen that $A_r = \bigcap\{B_{r+p}\colon p > 0\}$. If B_{r+p} is an overhull, $p > 0$, then A_r is an overhull (Lemma 1.2.4). Similarly, from $B_r = \bigcup\{A_{r-p}\colon p > 0\}$ and by the fact that $\{A_{r-p}\colon p > 0\}$ is directed by inclusion $\supset$ it follows (see Lemma 1.2.6) that B_r is an overhull. What is more, $A_{-\infty} = \bigcap\{A_r\colon r \in R\}$, $B_{\infty} = \bigcup\{B_r\colon r \in R\}$ and similar reasonings prove that these sets are overhulls. □

4.1.3. Definition. *Let* (X,Q) *be a weed. Then* $g\colon X \to \bar{R}$ *is convex if* $g(Q_n(x,t)) \le \sum t_i g(x_i)$, $x \in X^n$, $t \in P^{n-1}$, $n \in N$; g *is concave whenever* $-g$ *is convex (considering* Def. 1.2.1 *we adopt* $0 \cdot \infty = 0$).

4.1.4. Lemma. *Each convex mapping is quasiconvex. Nonnegative combination of convex mappings (on X) is convex. The least upper bound of convex (quasiconvex) mappings is convex (quasiconvex).*

Proof. Let $g\colon X \to \bar{R}$ be convex and let $x_1, \ldots, x_n \in A_r$ be arbitrary. Then we have $g(Q_n(x,t)) \le \sum t_i g(x_i) \le \sum t_i r = r$, which means that $Q_n(A_r) \subset A_r$ and $cQ\, A_r \subset A_r$, $n\in N$ being arbitrary. The second part of our lemma is obvious. If each $g \in \mathfrak{G}$ is convex, $g: X \to \overline{R}$, then $\sup \mathfrak{G}(Q_n(x,t)) \le \sup \{\sum t_i g(x_i): g\in\mathfrak{G}\} \le \sum t_i \sup \mathfrak{G}(x_i)$. Similarly, $\{x\in X: \sup \mathfrak{G}(x) \le r\} = \bigcap\{x\in X: g(x) \le r,\ g\in\mathfrak{G}\}$ and Lemma 1.2.4 show that $\sup \mathfrak{G}$ is quasiconvex, all $g\in\mathfrak{G}$ being quasiconvex. □

The requirement of $J^-(x)$ to be open (see (3.7)) associates with semicontinuity. Now we recall the respective definitions and some of the properties.

4.1.5. Definition. *Let* X *be a space. Then* $f: X \to \overline{R}$ *is scs (semi-continue supérieurement) whenever* $\{x\in X: f(x) < r\}$ *is an open set for each* $r\in R$; f *is sci (semi-continue inférieurement) if* $-f$ *is scs.*

4.1.6. Properties (cp. [15, p. 87]). *Let* $\mathfrak{G}$ *be a family of scs mappings of the form* $g: X \to \overline{R}$. *Then* $\inf \mathfrak{G}$ *is scs and for each* $f,g \in \mathfrak{G}$ $f + g$ *is scs. If* $f,g \in \mathfrak{G}$ *and* $f,g: X \to [0,\infty[$ *or* $f,g: X \to]0,\infty]$, *then* $f\cdot g$ *is scs. If, in addition,* X *is compact, then each* $g\in\mathfrak{G}$ *attains its minimum in a point of* X.

Let us present some applications of the theorems of Chapter 3.

If X,Y are sets and $f: X \times Y \to \overline{R}$ is a mapping then, clearly, $\inf f(x,Y) \le f(x,y)$ and, hence, $\sup_{x\in X} \inf f(x,Y) \le \sup f(X,y)$ which, in turn, implies

$$\text{(4.1)}\quad \sup_{x\in X} \inf f(x,Y) \le \inf_{y\in Y} \sup f(X,y).$$

The theorem to follow is a generalization of [36, Th. 1.11 p. 168]

4.1.7. Theorem. *Let* (X,Q), $(Y,\mathfrak{S})$ *be weeds and* $f: X \times Y \to \overline{R}$ *a mapping satisfying*

(i) $f(x,\cdot)$ *is quasiconvex and sci on compacta,* $x\in X$,

(ii) $f(\cdot,y)$ *is quasiconcave and scs on compacta,* $y\in Y$,

(iii) $B = \{y\in Y: \sup f(K,y) \le q\}$ *is compact for a c-compact set* $K \subset X$ *and a* q *such that* $b = \sup_{x\in X} \inf f(x,Y) < q$.

Then the following holds

$$\text{(4.1)}\quad \min_{y\in B} \sup_{x\in X} f(x,y) = \sup_{x\in X} \inf_{y\in Y} f(x,y).$$

Proof. In view of (4.1) it suffices to show that $\min_{y\in B}\sup f(X,y) \le b$ while $b < \infty$ (the case of $b = \infty$ is trivial, anyway). In Theorem 3.2.2 we adopt $Z = \bar{R}$, $G(x,y) = f(x,y)$, $H(x,y) = [-\infty,b]$, $U(x,y) = [-\infty,p[$, $V(x,y) = [-\infty,p]$, $p > b$, $W(x,y) = [-\infty,q]$ and $T(x) = \{y\in Y: f(x,y) < p\}$. Clearly, for each $x\in X$ there exists a $y\in Y$ such that $f(x,y) < p$ and so condition (3.3) is satisfied. In view of Lemma 4.1.2 conditions (i), (ii) are equivalent to (3.4) - (3.7). In addition, $\{y\in Y: G(x,y) \cap W(x,y) \ne \emptyset,\ x \in K\} = \{y\in Y: f(x,y) \le q,\ x\in K\} = \{y\in Y: \sup f(K,y) \le q\}$, i.e. (iii) is equivalent to (3.8). Now by Theorem 3.2.2 $f(x,y) \le b$ holds for a $y\in B$ and all $x\in X$ which means that $\min_{y\in B}\sup f(X,y) \le b$ (see 4.1.6). □

4.1.8. Remark. *In view of the semicontinuity assumptions, in the above we may replace inf, sup by min, max, respectively whenever the space under consideration is compact. In particular, from 4.1.7 for compact X,Y follows a generalization of Sion's theorem* [43, Th. 3.4 p. 174].

4.1.9. Remark. *If $b = 0$ in 4.1.7 (i.e. if for each $x\in X$, $p > 0$ there exists a $y \in Y$ such that $f(x,y) < p$), then we obtain the inequality $f(x,y) \le 0$ for a $y\in B$ and all $x\in X$.*

Theorem 4.1.7 leads in a natural way to the following extension of [17, Th. 1. p. 205] (cp. [36, Corol. 1.12 p. 169])

4.1.10. Theorem. *Let (Y,S) be a weed and $\{f_j: j\in J\}$ a family of mappings $f_j: Y \to \bar{R}$, $j\in J$ which are convex and sci on compacta. Assume that for a $q > 0$ there exists a finite set $K_1 \subset J$ such that $B = \{y \in Y: f_j(y) \le q,\ j \in K_1\}$ is compact. Then there exists a $y\in B$ such that for each $j\in J$ we have $f_j(y) \le 0$ if and only if for each $F_n = \Sigma\{t_i f_{j_i}: i = 1, \ldots, n\}$, $t \in P^{n-1}$, $\{j_1, \ldots, j_n\} \subset J$, $n \in N$, there exists a $z\in Y$ such that $F_n(z) \le 0$.*

Proof. Let X be a space of finite convex combinations of f_j, $j\in J$, while each set $\{\Sigma\, t_i f_{j_i} : t \in P^{n-1}\}$ is topologically isomorphic with P^{n-1}. It is seen that X with the sequence of convex combinations is a weed and for $f(x,y) = x(y)$, $x \in X$, $y \in Y$ the requirements of Th. 4.1.7 are satisfied ($K = \{f_j: j \in K_1\}$ is c-compact, see 1.4.8). If for each $x\in X$ there exists a $z\in Y$ such that $x(z) \le 0$, i.e. $\sup_{x\in X} \inf f(x,Y) \le 0$, then (see (4.2)) $\min_{y\in B} \sup f(X,y) \le 0$ and, in particular, $f_j(y) \le 0$ for a $y\in B$ and all $j\in J$. The converse is obvious. □

Now we will prove some theorems on inequalities. If $f: X \times Y \to \overline{R}$ is a mapping, then considering constant mappings we note that $\inf_{h\in F(X,Y)} \sup_{x\in X} f(x,h(x)) \le \sup f(X,y)$ holds. Therefore, the following is satisfied

$$(4.3)\qquad \inf_{h\in F(X,Y)} \sup_{x\in X} f(x,h(x)) \le \inf_{y\in Y} \sup f(X,y).$$

4.1.11. <u>Theorem</u> (cp. [36, Th. 1.2 p. 160]). *Let* (X,Q) *be a weed,* Y *a space and* $f: X \times Y \to \overline{R}$ *a mapping satisfying*

(4.4) $f(\cdot,y)$ *is quasiconcave,* $y\in Y$,

(4.5) $f(x,\cdot)$ *is sci on compacta,* $x\in X$,

(i) $B = \{y\in Y: \sup f(K,y) \le \sup_{x\in X} f(x,h_0(x)) = q\}$ *is compact for a c-compact set* $K \subset X$ *and an* $h_0 \in C(X,Y)$.

Then we have $\min_{y\in B} \sup_{x\in X} f(x,y) = \inf_{h\in C(X,Y)} \sup_{x\in X} f(x,h(x))$ $(= b)$.

Proof (Th. 3.3.4). In view of (4.3) it suffices to show that $\min_{y\in B} \sup f(X,y) \le b$, where $b < \infty$. Let us consider $Z = \overline{R}$, $\tilde{G}(x,y) = f(x,y)$, $H(x,y) = [-\infty,b]$, $U = V = [-\infty,p]$, $W = [-\infty,q]$, $b < p \le q < \infty$. From the way b was defined it follows that there exists an $h \in C(X,Y)$ such that $\sup_{x\in X} f(x,h(x)) \le p$, i.e. (3.9) holds. For $J(y) = \{x \in X: p < f(x,y)\}$ (4.4) implies (3.6), and (3.7) is a consequence of (4.5). Condition (i) is equivalent to (3.8). Hence in view of Th. 3.3.4 $\sup f(X,y) \le b$ for a $y\in B$. □

It is worth noting that for $h = id_X$, $Y = X$ the above extends a well known inequality of Fan [22, Th. 1 p. 103] which generalizes [18, Lemma 4 p. 309].

4.1.12. <u>Theorem</u>. *Let* (X,Q) *be a weed,* Y *a space and* $f: X \times Y \to \overline{R}$ *a mapping satisfying* (4.4), (4.5) *and*

(i) $\inf_{h\in C(X,Y)} \sup_{x\in X} f(x,h(x)) \le 0$,

(ii) $B = \{y\in Y: \sup f(K,y) \le 0\}$ *is compact for a c-compact set* $K \subset X$.

Then $f(x,y) \le 0$ *for a* $y \in B$ *and all* $x \in X$.

Proof. Condition (i) means that $b \le 0$ (see Th. 4.1.11). □

4.1.13. Remark. *Condition* (ii) *of* Th. 4.1.12 *can be formulated as follows*

(ii') *there exists a nonempty compact set* $B \subset Y$ *and a c-compact set* $K \subset X$ *such that for each* $y \in Y \setminus B$ *there exists an* $x \in K$ *with* $f(x,y) > 0$.

Hence Th. 4.1.12 *extends* [1, Th. 2 p. 3].

From 4.1.12, Lemma 4.1.4 and 4.1.6 we obtain

4.1.14. Theorem (cp. [36, Prop. 1.4 p. 162]). *Let* (X,Q) *be a weed,* Y *a space,* $g: X \times Y \to \bar{R}$ *a mapping satisfying* (4.5) *and such that* $g(\cdot,y)$ *is concave,* $y \in Y$. *If* $l: X \to \bar{R}$ *is a convex sci mapping satisfying*

(i) $\inf_{h \in C(X,Y)} \sup_{x \in X} [g(x,h(x))+l(h(x))-l(x)] \le 0$,

(ii) $B = \{y \in Y: \sup_{x \in X} [g(x,y)+l(y)-l(x)] \le 0\}$ *is compact for a c-compact set* $K \subset X$,

then the inequality $g(x,y) + l(y) \le l(x)$ *holds for a* $y \in B$ *and all* $x \in X$.

Let us recall (see Remark 1.4.8) that condition concerning B in Theorems 4.1.11, 4.1.12, 4.1.14, is superfluous if X or Y is a compact space (if only X is compact then we must write $\inf_{y \in Y}$ in place of $\min_{y \in B}$).

Two theorems to follow are generalizations of some auxiliary results in the approximation theory (for $f(x,y) = \inf p(x-F(y))$) and monotone operators, respectively.

4.1.15. Theorem. *Let* (E,Q) *be a weed in* X, *and* $f: E \times X \to \bar{R}$ *a mapping satisfying*

(i) $f(\cdot,y): E \to \bar{R}$ *is quasiconvex,* $y \in X$,

(ii) $y \mapsto f(y,y)-f(x,y)$ *is sci on compacta,* $x \in E$,

(iii) $B = \{y \in X: f(y,y) \le \inf f(K,y)\}$ *is compact for a c-compact set* $K \subset E$.

Then there exists a $y \in B$ *such that* $f(y,y) = \min f(E,y)$.

Proof. Suppose that for each $y \in X$ inequality $\inf f(E,y) < f(y,y)$ holds. Let us write $F(y) = \{x \in E: f(x,y) < f(y,y)\}$, $y \in X$. As it is seen, (ii) is equivalent to condition (i) of Th. 2.2.5. On the other hand, we have $B = X \setminus F^{-}(K)$. What is more, all values of F are overhulls. Now, in view of Th. 2.2.5 F has a fixed point, i.e. we have $f(y,y) < f(y,y)$ for a $y \in X$. This contradiction proves that $f(y,y) \le \inf f(E,y)$ holds for a $y \in X$ and by (iii) $y \in B$. □

4.1.16. <u>Definition</u> (cp. [38, Def. 2.1 p. 175]). *Let* (X,Q) *be a weed and* $E \subset X$. *A mapping* $f: X \times E \times X \longrightarrow \bar{R}$ *is semi-monotone if the following are satisfied*

(i) $f(z,x,y)+f(z,y,x) \ge 0$, $z \in X$, $x,y \in E$,

(ii) $f(z,x,\cdot)$ *is concave*, $z \in X$, $x \in E$,

(iii) $y \mapsto f(y,x,y)$ *is scs on compacta*, $x \in X$.

4.1.17. <u>Theorem</u> (cp. [38, Th. 2.2 p. 176]). *Let* (X,Q) *be a weed*, $E \subset X$ *and* $f: X \times E \times X \longrightarrow \bar{R}$ *a semi-monotone mapping. If in addition the following is satisfied*

(iv) $\{y \in Y: f(y,x,y) \ge 0, x \in K\}$ *is compact for a c-compact set* $K \subset E$,

then there exists a $y_o \in X$ *such that* $f(y_o,x,y_o) \ge 0$, $x \in E$.

Proof (cp. [38, p. 176]). Let $G(x) = \{y \in X: f(y,x,y) \ge 0\}$, $x \in E$. It suffices to show that $\bigcap\{G(x): x \in E\} \ne \emptyset$. We are going to apply Theorem 1.4.7 for $h = id_X$. First let us show that $G: E \longrightarrow 2^X$ is KKM. For $\{e_1, \ldots, e_n\} \subset E$ let us consider $y = Q_n(e,t)$ $(e = (e_1, \ldots, e_n))$. It follows from (i) that $f(y,e_i,e_j) + f(y,e_j,e_i) \ge 0$, $i,j = 1, \ldots, n$. Therefore, $\sum \{t_i t_j [f(y,e_i,e_j) + f(y,e_j,e_i)]: i,j = 1, \ldots, n\} \ge 0$ holds. In view of (ii) we obtain $\sum \{[t_i f(y,e_i,y) + t_j f(y,e_j,y)]: i,j = 1, \ldots, n\} \ge 0$, i.e. $y \in G(e_i)$ and G is KKM. Condition (iii) means that all values of G are compactly closed, and (iv) implies condition (iii) of Th. 1.4.7. □

4.2. SPECIAL INEQUALITIES FOR LINEAR SPACES

Theorem 1.4.9 enables us to obtain some results which for linear spaces are stronger than the respective theorems of Section 4.1. We are mainly interested in the theorems on the nearest points (see, e.g. 4.2.5). The final theorem (4.2.8) is a tool in proving fixed point theorems for inward mappings of Section 4.3. Let us point out that space $Y = X$ in Theorems 4.2.1, 4.2.3, 4.2.4 and 4.2.8 may be such as in Example 1.2.8, if X is regular.

First let us state an auxiliary result. It is related to Th. 2.2.5 (see the final comment of Section 1.4).

4.2.1. <u>Theorem</u>. *Let* E *be a subset of a convex set* Y *in a linear space and* $F: Y \to 2^E$ *a mapping such that*

(i) *all values of* F *are convex.*

(ii) *all values of* F^- *are compactly open.*

(4.6) $C = \{y \in Y: F(y)$ [redacted] $\cap \operatorname{conv}(K \cup \{y\}) = \emptyset\}$ *is relatively compact for a c-compact set* $K \subset E$.

Then F *has a fixed point.*

Proof. Let $L(y) = Y \setminus F^-(y)$, $y \in E$. As it is seen all values of F are compactly closed and what is more $C = \{y \in Y:$ if $x \in E \cap \operatorname{conv}(K \cup \{y\})$, then $x \notin F(y)\} = \{y \in Y:$ if $x \in E \cap \operatorname{conv}(K \cup \{y\})$, then $y \in (Y \setminus F^-)(x)\}$, i.e. (iii) of Th. 1.4.9 is satisfied. We have $\bigcap\{L(y): y \in E\} = Y \setminus F^-(E) = \emptyset$ and, therefore, (Th. 1.4.9) $L: E \to 2^Y$ is not KKM. Now it suffices to apply Lemma 2.2.1. □

4.2.2. <u>Remark</u>. *In the above we could require* K *to be a closed convex and nonempty subset of* Y *such that* $K \cap E \neq \emptyset$. *Indeed, according to* Def 1.4.2 $\operatorname{conv} K$ *is relatively compact.* K *being c-compact. From the fact that* Y *is a regular space it follows that* $\overline{\operatorname{conv}} K$ *is compact* [31, B (b) p. 161] *and in view of* [32, 13.1 (iii) p. 111] $\overline{\operatorname{conv}}\,[Z \cup \overline{\operatorname{conv}}\,K]$ *is compact for each finite set* $Z \subset Y$. *Hence* $\overline{\operatorname{conv}}\,K$ *is c-compact.*

The subsequent theorem is a specialization of Th. 4.1.15

4.2.3. <u>Theorem</u>. *Let* E *be a subset of a convex set* Y *in a linear space and* $f: Y \times Y \to \overline{R}$ *a mapping satisfying*

(4.7) $f(\cdot,y): E \to \overline{R}$ *is quasiconvex*, $y \in Y$,

(4.8) $y \mapsto f(y,y) - f(x,y)$ *is sci on compacta*, $x \in E$,

(4.9) $C = \{y \in Y: f(y,y) \leq \inf f(E \cap \operatorname{conv}(K \cup \{y\}), y)\}$ *is relatively compact for a c-compact set* $K \subset E$.

Then there exists a $y \in \overline{C}$ *such that* $f(y,y) = \min f(E,y)$.

Proof. Suppose that for each $y \in Y$ inequality $\inf f(E,y) < f(y,y)$ holds. Let us write $F(y) = \{x \in E: f(x,y) < f(y,y)\}$, $y \in Y$. As it is seen, conditions (i), (ii) of Th. 4.2.1 are satisfied. What is more, $\{y \in Y: F(y) \cap E \cap \mathrm{conv}(K \cup \{y\}) = \emptyset\} = C$ (see (4.9)). Hence F has a fixed point, i.e. $f(y,y) < f(y,y)$ holds for a point $y \in Y$ – a contradiction. □

Theorems 4.2.1, 4.2.3 can be modified by transferring the negation of (4.6), (4.9) to the respective propositions which become alternatives. For example, the negation of (4.9) is of the form (cp. 4.2.2)

(i) *for each compact closed convex set* $K \subset Y$ *such that* $K \cap E \neq \emptyset$ *the set* C *as in* (4.9) *is not relatively compact.*

In particular, (i) means that $C \subset K$ cannot be satisfied. Hence, (i) implies

(ii) *for each compact closed convex set* $K \subset Y$ *such that* $K \cap E \neq \emptyset$ *there exists a* $y \in Y \setminus K$ *such that* $f(y,y) \leq \inf f(E \cap \mathrm{conv}\,(K \cup \{y\}),y)$.

For $E = Y$ the following is a consequence of 4.2.3.

4.2.4. <u>Theorem</u>. *Let* Y *be a convex set in a linear space and* $f: X \times Y \longrightarrow \bar{R}$ *a mapping satisfying* (4.7), (4.8) $(E = Y)$. *Then for each compact closed convex and nonempty set* $K \subset Y$ *at least one of the following conditions is satisfied*

(4.10) *there exists a* $y \in Y \setminus K$ *such that* $f(y,y) = \min f(\mathrm{conv}[K \cap \{y\}],y)$,

(4.11) *there exists a* $y \in K$ *such that* $f(y,y) = \inf f(Y,y)$.

The proof of the above lies in the negation of (4.10), which by the earlier reasoning, implies (4.9) and we apply 4.2.3. From the negation of (4.10) it additionally follows that y such that $f(y,y) = \inf f(Y,y)$ is an element of K.

In view of the preceding considerations the theorem to follow is a generalization of [36, Prop. 3.4 p. 189]. A simpler version can be obtained from 4.2.4 for $f(x,y) = \inf p(x-F(y))$.

For quasiconvex mapping $p: Y \longrightarrow R$ (Y – convex) let us write $d_p(A,B) = \inf \{p(x-y): x \in A,\ y \in B\}$, $A,B \subset Y$, $A,B \neq \emptyset$.

4.2.5. Theorem. *Let* Y *be a convex set in a linear space* M, F: $Y \to 2^M$ *a mapping which is usc and lsc on compacta and with compact values,* p: Y → R *a continuous quasiconvex mapping. Then for each compact closed convex and nonempty set* $K \subset Y$ *at least one of the following is satisfied*

(4.12) $C = \{y \in Y: d_p(y,F(y)) = d_p(\mathrm{conv}[K \cup \{y\}],F(y))\}$ *is not relatively compact,*

(4.13) *there exists a* $y \in \bar{C}$ *such that* $d_p(y,F(y)) = d_p(Y,F(y))$.

Proof. Let us consider $f(x,y) = d_p(x-F(y))$, $x,y \in Y$. Suppose (4.12) is not satisfied, i.e. (4.9) holds. In view of Th. 4.2.3 it suffices to verify conditions (4.7), (4.8). If $f(x,y) \le r$ and $f(z,y) \le r$ hold, then by the compactness of F(y) and in view of p being sci there exist $u,v \in F(y)$ such that $p(x-u) \le r$ and $p(z-v) \le r$. Hence, $p(tx+[1-t]z-tu-[1-t]v) = p(t[x-u]+(1-t)[z-v]) \le \le r$, $t \in I$ p being quasiconvex. The fact that $tu+(1-t)v \in F(y)$ means that $d_p(tx+(1-t)z,F(y)) \le r$, $t \in I$, i.e. (4.7). If $f(y,y) = b > r$, then there exists an $u \in F(y)$ such that $\inf p(y-F(y)) = b$. The set $\{y \in Y: p(y-z) > (b+r)/2\}$ is open (p is sci) and in view of the way $d_p(y,F(y))$ was defined, it contains F(y). Hence, there exists a neighbourhood U of zero in M such that $p(U + y - F((U+y) \cap Y)) \subset](b+r)/2,\infty]$, F(y) being compact. This means that $f((U \cap Y) \times (U \cap Y)) \subset]r,\infty]$, and $y \mapsto f(y,y)$ is sci. If we have $f(x,y) = b < r$, then for a $v \in F(y)$ such that $b = p(x-v)$ the set $W = \{z \in Y: p(x-z) < (b+r)/2\}$ is a neighbourhood of v as p is scs. For a neighbourhood U of zero in M we have $F(y+z) \cap W \ne \emptyset$, $z \in U \cap Y$, F being lsc. Now it is clear that $f(x,(y+U) \cap Y) \subset [-\infty,r[$, i.e. $f(x,\cdot)$ is scs. Thus we have proved that $y \mapsto f(y,y) - f(x,y)$ satisfies (4.8). □

Theorem 4.2.5 enables us to obtain the following (cp. [4.2, Th. 1 p. 108]

4.2.6. Theorem. *Let* Y *be a convex set in a locally convex space* $(M,\mathcal{T})$ *and* f: $(Y,w) \to (M,\mathcal{T})$ *a mapping which is continuous on compacta (w – the weak topology). If* p *is continuous pseudo-norm and there exists a nonempty closed convex and compact (in w) set* $K \subset Y$ *such that*

(i) $C = \{y\in Y: p(y-f(y)) = \min p(\mathrm{conv}(K\cup\{y\})-f(y))\}$ *is relatively weakly compact.*

then there exists a $y\in\bar{C}$ *(in w) such that* $p(y-f(y)) = \min p(Y-f(y))$.

If in 4.2.5 we consider (Y,w), then the only trouble is to prove that $y \mapsto g(y) = p(y-f(y)) - p(x-f(y))$ is sci on weak compacta (cp. Th. 4.2.3). To obtain this result we extend a little [42, Lemma 1 p. 109]

4.2.7. <u>Lemma</u>. *If the requirements of* Th. 4.2.6 *are satisfied, then* $g: Y \to R$ *is sci on weak compacta.*

Proof (cp. [42, p. 109]). It suffices to show that $A = \{y \in Y:$ is compactly closed (in w). Let $D \subset Y$ be a weakly compact set and let $(y_j)_{j\in J}$ be a net of points of $A \cap D$ which converges weakly to a $y\in D$. Clearly, we have $p(x-f(y_j)) \to p(x-f(y))$. Let $x \in M^*$ be such that $|x(y-f(y))| = p(y-f(y))$ and $|x(z)| \le p(z)$, $z\in A$ [32, 14.1 (ii) p. 117]. As $y_j-f(y_j) \to y-f(y)$ in the weak topology, then we have $p(y - f(y)) - p(x - f(y)) = \lim [\,|x(y_j)-f(y_j))| - p(x-f(y_j))] \le$ $\le \lim [p(y_j-f(y_j)) - p(x-f(y_j))] \le r$. Hence follows $y\in A$. □

The subsequent theorem will be applied in 4.3. Its proof is based on Browder's idea of the proof of [5, Th. 3 p. 286].

4.2.8. <u>Theorem</u>. *Let* Y *be a paracompact convex set in a linear space and* $\mathcal{F}$ *a convex family of convex mappings* $f: Y \to R$ *which are sci on compacta. Assume that for each* $y\in Y$ *there exists a* U_y *and a* $f\in\mathcal{F}$ *such that* f *has property* A *on* U_y, *and* A *is such that for any* $n\in N$ *if* $f_1, \ldots, f_n \in \mathcal{F}$ *have property* A *in a point* $y\in Y$, *then* $h = \sum t_i f_i$ *satisfies* A *in* y *for each* $t \in P^{n-1}$. *If in addition, there exists a closed compact convex and nonempty set* $K \subset Y$ *such that*

$$(4.14)\quad \begin{array}{l} D = \{y\in Y: \textit{there exists an } f\in\mathcal{F} \textit{ such that } f(y) \le \\ \le \inf f(\mathrm{conv}(K\cup\{y\})) \textit{ and } f \textit{ satisfies } A \textit{ in } y\}, \\ \textit{is relatively compact (in } Y) \end{array}$$

holds, then there exists an $f_0 \in \mathcal{F}$ *and a* $y \in \bar{D}$ *such that* $f_0(y) = \min f_0(Y)$ *and* f_0 *has property* A *in* y.

Proof. There exists a locally finite partition of unity $\mathcal{G} = \{g_s: s\in S\}$, being a refinement of $\mathcal{U} = \{U_y: y\in Y\}$ [31, W p. 171, Corol. 32

p. 159]. Let $f(x,y) = \sum \{g_s(y)f_s(x): s\in S\}$, $x,y \in Y$ where $f_s \in \mathcal{F}$ is such that $g_s^{-}(]0,1]) \subset U \in \mathcal{U}$ and f_s has property A on U. Clearly, $f(\cdot,y) \in \mathcal{F}$, $y\in Y$ as $\mathcal{F}$ is convex. With the help of Th. 4.2.3 we will prove that there exists a $y \in Y$ such that $f(y,y) = \min f(Y,y)$. Condition (4.7) holds as $f(\cdot,y)$ is a convex combination of convex mappings (Lemma 4.1.4). The mapping $f(x,\cdot)$ is continuous, $x \in Y$ and $y \mapsto f(y,y)$ is sci on compacta being a positive combination of sci mappings on compacta (see 4.1.6). Thus we obtain (4.8). According to the way f was defined, $f(\cdot,y)$ has property A in y as $g_s(y) \neq 0$ means that f_s satisfies A in y. Therefore, we have $C \subset D$ (see (4.9)). In view of Th. 4.2.3 there exists a $y \in \bar{D}$ such that $f(y,y) = \min f(Y,y)$, i.e. $f_o = f(\cdot,y)$ is the desired mapping. □

4.3. FIXED POINT THEOREMS

In this section we present some fixed point and coincidence results for upper hemicontinuous mappings. They extend important theorems of Lassonde (Th. 4.3.2), Fan, Browder and Halpern.

As regards the notion of convexity, it is meaningless if a linear space under consideration is real or complex. On the other hand, the properties of continuous functionals we need, can be derived from the respective properties of their real parts (see [31, pp. 3, 7, Th. 5.4 (vi) p. 37]). For the simplicity of the notations we "restrict" our considerations to the real linear spaces, and in view of the above information the "general" complex results can be obtained by writing rx (the real part of functional x) in place of x.

First, let us quote a little modified definition of uhc mapping due to Lasry and Robert [35, p. 1435]

4.3.1. <u>Definition</u>. *Let* Y *be a subset of a real locally convex space* M. *Then* $F: Y \to 2^M$ *is uhc (upper hemicontinuous) if for each continuous linear functional* x *(i.e.* $x\in M^*$*) mapping* $\sup x(F(\cdot))$: $Y \to \bar{R}$ *is scs (Def. 4.1.5 p. 52).*

Let us recall Fan's definition [21, p. 236]: $F: Y \to 2^M$ *is upper demi-continuous if* $F(y_o) \subset H = \{y\in M: x(y) < p\}$ *(H is an open half-space in a real space M) implies* $F(U_{y_o}) \subset H$ for a $U_{y_o} \subset Y$.

Let us compare the above notions and the concept of usc mappings. If $F: Y \to 2^M$ is usc and M is a real locally convex space, then F is upper demi-continuous as every open half-space containing $F(y_o)$ is a neighbourhood of $F(y_o)$. Assume, in turn, that F is upper demi-continuous and $\sup x(F(y_o)) = b < p$. Then we have $F(y_o) \subset H = \{y \in M: x(y) < (b+p)/2\}$, $F(U_{y_o}) \subset H$ and hence $\sup x(F(U_{y_o})) \leq (b+p)/2 < p$. Thus, if F is upper demi-continuous, then F is uhc. The converse does not usually hold. For example, let us consider $F: R\backslash\{0\} \longrightarrow 2^R$ defined as follows

$$F(y) = \begin{cases}]-1,1[, & \text{for rationals,} \\ [-1,1], & \text{for irrationals.} \end{cases}$$

Clearly, F is uhc but there is no neigbourhood U_2 such that $F(U_2) \subset]-\infty,1[$ though $F(2) \subset]-\infty,1[$. Thus F is not upper demi-continuous.

4.3.2. <u>Theorem</u>. *Let* Y *be a bounded convex set in a real locally convex space* M *and* $F: Y \longrightarrow 2^M$ *a mapping with closed convex values which is uhc on compacta. If the following conditions are satisfied*

(4.15) $\inf x(Y) \leq \sup x(F(y))$, $x \in X = M^*$, $y \in Y$,

(4.16) $B = \{y \in Y: x(y) \leq \sup x(F(y)) + q, x \in K\}$ *is compact for a finite set* $K \subset X$ *and a* $q > 0$,

then F *has a fixed point in* B.

Proof (Th. 3.2.2). First we will show that there exists a $y \in Y$ such that for each $x \in X$ we have $x(y) \leq \sup x(F(y))$. Let us consider $Z = \bar{R}$, $G(x,y) = [-\infty, \sup x(F(y))]$, $H(x,y) = [x(y),\infty]$, $U(x,y) =]x(y) - p,\infty]$, $p > 0$, $V(x,y) = \bar{U}(x,y)$, $W(x,y) = [x(y) - q,\infty]$ and $T(x) = \{y \in Y: x(y) - p < \inf x(Y)\}$. We have $-\infty < \inf x(Y)$, Y being bounded, and, therefore, for each $x \in X$ $T(x)$ is nonempty. In view of (4.15) $T(x) \subset \{y \in Y: x(y) - p < \sup x(F(y))\} = \{y \in Y: G(x,y) \cap U(x,y) \neq \emptyset\}$. Thus, condition (3.3) is satisfied. The values of T are clearly convex. In order to show that $T^-(y)$ is open, $y \in Y$, we must equip X with the respective topology. Let every simplex in X be topologically isomorphic with P^{n-1} for an $n \in N$. Mapping $x \mapsto x(y) - \inf x(Y)$ is convex for fixed y [2, Th. 5 p. 199] and, therefore, it is continuous in each simplex [2, Th. 7 p. 201]. Thus, $T^-(y) = \{x \in X: x(y) - \inf x(Y) < p\}$ is an open set and (3.5) holds. All values of J, $J(y) = \{x \in X: \sup x(F(y)) < x(y) - p\}$ are convex. Namely, from the fact that $(tx+[1-t]z)(u) \leq tx(u) + (1-t)\sup z(F(y))$ for $u \in F(y)$, follows $\sup (tx + [1-t]z)(F(y)) \leq \sup tx(F(y)) + \sup (1-t)z(F(y))$, $x,z \in X$, $t \in I$. Hence, if $x,z \in J(y)$, then tx+

$+(1-t)z \in J(y)$. Condition (3.7) follows directly from the fact that F is uhc on compacta. Each finite set in X is c-compact (Remark 1.4.8) and from (4.16) follows (3.8). Now in view of Th. 3.2.2 there exists a $y \in B$ such that for each $x \in X$ $x(y) \le \sup x(F(y))$ holds. In particular, we obtain $-x(y) \le \sup -x(F(y))$ and, consequently, $\inf x(F(y)) \le x(y) \le \sup x(F(y))$, $x \in X$. This means that y and $F(y)$ cannot be strongly separated by a linear functional. Now, in view of [32, Corol. 14.4 p. 119], we obtain $y \in F(y)$. □

Theorem 4.3.2 is more general than [36, Th. 1.16 p. 171]. We do not require Y to be T_2; if $Y \cap F(y) \ne \emptyset$, $y \in Y$ [36, (18) p. 171], then clearly (4.15) is satisfied.

In connection with (4.15) let us present the following

4.3.3. Lemma. *Let* Y *be a closed convex set in a real locally convex space* M *and* $F: Y \longrightarrow 2^M$ *a mapping. If* $\overline{\text{conv}}\, F(Y)$ *or* Y *is a compact set and the following is satisfied*

(i) $\inf x(Y) \le \sup x(F(Y))$, $x \in M^*$,

then $Y \cap \overline{\text{conv}}\, F(Y) \ne \emptyset$. *If all values of* F *are compact convex and* (4.15) *holds, then* $Y \cap \bar{F}(y) \ne \emptyset$, $y \in Y$. *If* $\overline{\text{conv}}\, F(Y)$ *is compact and the follwing is satisfied*

(ii) $x(y) \le \sup x(F(Y))$, $x \in X$, $y \in Y$ $(X = M^*)$,

then $y \in \overline{\text{conv}}\, F(Y)$, $y \in Y$.

Proof. As it is seen, for each $x \in X$ and $\emptyset \ne D \subset M$ we have $\text{conv}\, D \subset \{y \in M: \inf x(D) \le x(y) \le \sup x(D)\} =: D^x$. This last set is closed and, therefore, $D \subset \overline{\text{conv}}\, D \subset D^x$. Thus, $D^x = \{y \in M: \inf x(\overline{\text{conv}}\, D) \le x(y) \le \sup x(\overline{\text{conv}}\, D)\}$. Assume (i) and suppose that Y, $\overline{\text{conv}}\, F(Y)$ are disjoint. The initial part of this proof and [32, Corol. 14.4 p. 119] mean that there exists an $x \in X$ such that $[\inf x(Y), \sup x(Y)] \cap [\inf x(F(Y)), \sup x(F(Y))] = \emptyset$. In view of (i) this is the case when $\sup x(Y) < \inf x(F(Y))$. By considering $-x$ we obtain $\sup x(F(Y)) < \inf x(Y)$ which contradicts (i). The remaining parts of our lemma can be proved in a similar way. □

4.3.4. Remark. *If* Y *is a compact set, then* (4.16) *in* 4.3.2 *is superfluous. If* $\overline{\text{conv}}\, F(Y)$ *is compact, then in view of* 4.3.3 *it suffices to consider* $F: Y_1 \longrightarrow 2^M$, *where* $Y_1 = Y \cap \overline{\text{conv}}\, F(Y)$.

In connection with the remarks preceding Th. 4.2.3 the theorem to follow is a generalization of [23, Th. 5 p. 153]. Its proof is based on Th. 4.2.8.

4.3.5. Theorem. *Let* Y *be a paracompact convex set in a real locally convex space* M. *Assume* $F,H\colon Y \to 2^M$ *are uhc mappings with convex values and such that for each* $y \in Y$ *one of the sets* $F(y)$, $H(y)$ *is compact and another one is closed. If there exists a c-compact set* $K \subset Y$ *such that the following are satisfied*

$$D = \{y \in Y: \text{there exists an } x \in X = M^* \text{ such that } x(y) \le \inf x(K) \text{ and } \sup x(F(y)) < \inf x(H(y))\} \text{ is relatively compact,} \tag{4.17}$$

$$\text{for each } x \in X,\ y \in D \text{ if } x(y) = \min x(Y), \text{ then } \inf x(H(y)) \le \sup x(F(y)), \tag{4.18}$$

then there exists a $y \in Y$ *such that* $H(y) \cap F(y) \neq \emptyset$.

Proof. Let us prove that there exists a $y \in Y$ such that $\inf x(H(y) \le \sup x(F(y))$, $x \in X$. Suppose that this condition is not satisfied. Hence, for each $y \in Y$ there exists an $x \in X$ such that $\sup x(F(y)) < \inf x(H(y))$ (property A). From the fact that F,H are uhc it follows that for each $x \in X$ the set $\{y \in Y: \sup x(F(y) < < \inf x(H(y))\}$ is open and the family of such sets covers Y. What is more, if $\sup x_i(F(y)) < \inf x_i(H(y))$, $i = 1, \ldots, n$, then for each $t \in P^{n-1}$ we have $\sup \sum t_i x_i F(y)) \le \sum t_i \sup x_i(F(y)) < < \sum t_i \inf x_i(H(y)) \le \inf \sum t_i x_i(H(y))$. Thus, property A is as in Th. 4.2.8 (clearly, X is a convex family of convex maps). What is more, we note that the inequality $x(y) \le \inf x(K)$ is equivalent to $x(y) = \min x(\mathrm{conv}[K \cup \{y\}])$. Now, by Th. 4.2.8 there exists an $x \in X$ and a $y \in \bar{D}$ such that $x(y) = \min x(Y)$ and $\sup x(F(y)) < \inf x(H(y))$ which contradicts (4.18). Thus, there exists a $y \in Y$ such that for each $x \in X$ $\inf x(H(y)) \le \sup x(F(y))$ holds. In particular, for $-x$ we obtain $\inf x(F(y)) \le \sup x(H(y))$. These two inequalities imply $[\inf x(H(y)), \sup x(H(y))] \cap [\inf x(F(y)), \sup x(F(y))] \neq \emptyset$ which means that $H(y) \cap F(y) \neq \emptyset$ [32, Corol. 14.4 p. 119]. □

4.3.6. Remark. *By considering property A as in the above proof for* F *in place of* H *and vice versa, and the family* $\mathcal{X} = \{-x: x \in X\}$, *we state that conditions in Th. 4.3.5 can be replaced by*

$$D = \{y \in Y: \text{there exists an } x \in X = M^* \text{ such that } \sup x(K) \le x(y) \text{ and } \sup x(F(y)) < \inf x(H(y))\} \text{ is relatively compact,} \tag{4.17'}$$

(4.18') *for each* $x \in X$, $y \in D$ *if* $x(y) = \max x(Y)$, *then* $\inf x(H(y)) \le \sup x(F(y))$.

We have (see (4.17)) $Y \backslash D = \{y \in Y:$ for each $x \in X$ if $x(y) \le \le \inf x(K)$, then $\inf x(H(y)) \le \sup x(F(y))\}$. Let us recall (see Remark 4.2.2) that it is natural to assume K to be closed compact and convex. By assuming that $D \subset K$, from Th. 4.3.5 we obtain the following which is closer to [23, Th. 5 p. 153] but still stronger (clearly, we may use (4.17'), (4.18') in place of (4.17), (4.18))

4.3.7. Theorem. *Let* Y *be a paracompact convex set in a real locally convex space* M, *and* K *a nonempty closed compact and convex set in* Y. *Assume* $F,H: Y \longrightarrow 2^M$ *to be uhc mappigs with convex values and such that for each* $y \in Y$ *one of the sets* $F(y)$, $H(y)$ *is compact and another one closed. If the following are satisfied*

(i) *for each* $x \in M^*$, $y \in K$ *if* $x(y) = \min x(Y)$, *then* $\inf x(H(y)) \le \sup x(F(y))$,

(ii) *for each* $y \in Y \backslash K$, $x \in M^*$ *if* $x(y) \le \inf x(K)$, *then* $\inf x(H(y)) \le \sup x(F(y))$,

then there exists a $y \in Y$ *such that* $H(y) \cap F(y) \neq \emptyset$.

For $H(y) = y$, $y \in Y$ the above become fixed point theorems, e.g. from 4.3.5 (Remark 4.3.6) we obtain

4.3.8. Theorem. *Let* Y *be a paracompact convex set in a real locally convex space* M. *Assume that* $F: Y \longrightarrow 2^M$ *is a uhc mapping with closed convex values. If for a c-compact set* $K \subset Y$ *we have*

(i) $D = \{y \in Y:$ *there exists an* $x \in M^*$ *such that* $x(y) \le \le \inf x(K)$ $(\sup x(K) \le x(y))$ *and* $\sup x(F(y)) \le x(y)\}$ *is relatively compact*,

(ii) *for each* $x \in M^*$, $y \in D$ *if* $x(y) = \min x(Y)$ $(\max x(Y))$, *then* $x(y) \le \sup x(F(y))$,

then F *has a fixed point.*

Every compact subset of a linear space M is paracompact as M is regular. In addition, if we consider $K = Y$, then by (4.17), (4.18) we note that $D = \emptyset$. Then condition (4.18) is superfluous and (4.17) ((4.17')) for $y \in Y \backslash D = Y$ has the following form

(4.19) *for each* $x \in M^*$, $y \in Y$ *if* $x(y) = \min x(Y)$ $(\max x(Y))$, *then* $\inf x(H(y)) \le \sup x(F(y))$.

Therefore, Th. 4.3.5 implies the following generalization of [22, Th. 5 p. 108]

4.3.9. <u>Theorem</u>. *Let* Y *be a compact convex set in a real locally convex space* M *and* $F,H\colon Y \to 2^M$ *uhc mappings with convex values such that for each* $y \in Y$ *one of the sets* $F(y)$, $H(y)$ *is compact and another one is closed. If* (4.19) *is satisfied, then there exists a* $y \in Y$ *such that* $F(y) \cap H(y) \neq \emptyset$.

As it will be shown below, the theorem to follow extends [27, Th. 2 p. 88, Th. 3 p. 89]. It is clear that this theorem is a natural generalization of [22, Th. 6 p. 109].

4.3.10. <u>Theorem</u>. *Let* Y *be a compact convex set in a real locally convex space* M. *If* $F\colon Y \to 2^M$ *is a uhc mapping with closed convex values which satisfies*

(4.20) *for each* $x \in M^*$, $y \in Y$ *if* $x(y) = \min x(Y)$ $(\max x(Y))$, *then* $x(y) \le \sup x(F(y))$,

then F *has a fixed point.*

Let us point out that in place of (4.19) one may assume that F satisfies (4.20) and H (4.20) with max in place of min. By writing $-x$ in place of x for H we obtain then (4.19) and a generalization of [21, Th. 8 p. 239].

Let us compare (4.19) and [21, (10) p. 237]. If for each $y \in Y$ there exist $z \in Y$ and $u \in H(y)$, $v \in F(y)$, $t > 0$ such that $z-y = t(v-u)$ (cp. [21]), then from $x(y) = \min x(Y)$ it follows that $x(y) \le x(z)$ and $x(u) \le x(v)$ as $t > 0$. In consequence, we obtain $\inf x(H(y)) \le \sup x(F(y))$. Hence, (4.19) is much more general than [21, (10)].

The above reasoning, in the case of $H = id_Y$ means that (4.20) is more general than [21, (8) p. 236]. By applying the notation of [27, p. 88] which is equivalent to [21, (10)], we may say that $F\colon Y \to 2^M$ is inward if for each $y \in Y$ $F(y) \cap I_Y(y) \neq \emptyset$ holds, where $I_Y(y) = \{z \in M\colon (1-t)z+ty \in Y \text{ for a } t \in [0,1[\}$. Hence, if $x(y) = \min x(Y)$, then $I_Y(y) \subset \{z \in M\colon x(y) \le x(z)\}$. This last set is closed and, therefore, condition $F(y) \cap \overline{I_Y(z)} \neq \emptyset$ is not more general than (4.20). Thus, 4.3.10 is a generalization of [27, Th. 2 p. 88] as we consider uhc mappings.

Let us consider the outwardness condition [27, p. 88] and (4.20) (the version in brackets). If $F(y) \cap O_Y(y) \neq \emptyset$, then for a $t \in$ $\in [0,1[$ and a $u \in F(y)$ we have $[1-t](2y-u)+tu \in Y$. If in addition $x(y) = \max x(Y)$, then the above implies $x(y+[1-t](y-u)) \leq x(y)$, i.e. $x(y-u) \leq 0$ and $x(y) \leq \sup x(F(y))$. In consequence, Th. 4.3.10 extends [27, Th. 2 p. 89].

5. Nonempty intersections, measures of noncompactness

5.1. ON A THEOREM OF FAN

First we generalize a theorem of Lassonde [38, Th. 1.9 p. 166] which extends one of the well known theorems of Fan (on convex sections) [19, Th. 1 p. 3925]. Some applications of this theorem are included. The notations are as in the final part of 1.1.

5.1.1. Theorem. *Let* (X_i, Q^i), $i = 1, \ldots, m$, $m \geq 2$ *be weeds and* $A_1, \ldots, A_m \subset X = P\{X_i\colon i = 1, \ldots, m\}$ *the sets satisfying the following conditions*

(a) $A_i(\partial_i x) = \{y_i \in X_i\colon \langle y_i, \partial_i x\rangle \in A_i\}$ *is a nonempty overhull in* X_i, $i = 1, \ldots, m$, $x \in X$,

(b) $\partial_i A_i(y_i) = \{\partial_i x \in \partial_i X\colon \langle y_i, \partial_i x\rangle \in A_i\}$ *is compactly open in* $\partial_i X_i$, $i = 1, \ldots, m$, $y_i \in X_i$,

(c) $\partial_i X \backslash \partial_i A_i(K_i)$ *is compact for a c-compact set* $K_i \subset X_i$ *and for at least* $m-1$ *indices* i.

Then $A_1 \cap \ldots \cap A_m$ *is nonempty.*

Proof (cp. [38]). Assume that (c) holds for $i = 1, \ldots, m-1$. In view of (b), (c) there exist the finite sets $Z_i \subset X_i$ such that $\partial_i X = \partial_i A_i(K_i \cup Z_i)$, $i = 1, \ldots, m-1$. The sets $X_i' = (K_i)_Z \subset X_i$ (see Def. 1.4.6) are compact and by (b) there exists a finite set $K_m \subset X_m$ for which $X_1' \times \ldots \times X_{m-1}' \subset \partial_m A_m(K_m)$. Let us adopt $E = (K_1 \cup Z_1) \times \ldots \times (K_m \cup Z_m)$ $(Z_m = \emptyset)$ and $Q' = Q^{1\prime} \times \ldots \times Q^{m\prime}$ (see 3.1.1). Then (E, Q') is a weed in $X' = X_1' \times \ldots \times X_{m-1}' \times cQ^{m\prime}K_m$ and X' is a compact set (Remark 1.1.13). Let us consider $F\colon X' \to 2^E$ defined by $F(x) = (A_1(\partial_1 x) \times \ldots \times A_m(\partial_m x)) \cap E$, $x \in X'$. For each $x \in X'$, $i = 1, \ldots, m$ we have $\partial_i x \in \partial_i A_i(K_i \cup Z_i)$, i.e. $A_i(\partial_i x) \cap (K_i \cup Z_i) \neq \emptyset$. Hence, $F(x) \neq \emptyset$, $x \in X'$. In view of (b) $F^-(y) = \bigcap\{x \in X'\colon \partial_i x \in \partial_i A_i(y_i)\} \cap X'$ is open X' being compact. Now by Th. 2.2.3 there exists a point $x \in X'$ such that $x \in cQ'F(x)$. From $cQ'F(x) \subset cQ\,F(x) \subset F(x)$ ($F(x)$ is a Q-overhull) we obtain $x \in A_i$ for $i =$

$= 1, \ldots, m$. □

Theorem 5.1.1 enables us to extend the results which were obtainable from the classical theorem of Fan. We present here only general versions.

5.1.2. Theorem (cp. [19, Th. 2 p. 392]). *Let* (X_i, Ω^i), $i = 1, \ldots, m \geq 2$ *be weeds,* $X = P\{X_i: i = 1, \ldots, m\}$, *and let* $f_i: X \to \bar{R}$ *be mappings which for fixed* $t_i \in [-\infty, \infty[$ *and all* $i = 1, \ldots, m$ *satisfy the following conditions*

(a) *for each* $x \in X$, $f_i(\langle \cdot, \partial_i x\rangle)$ *is quasiconcave,*

(b) *for each* $x_i \in X_i$, $f_i(\langle x_i, \cdot\rangle)$ *is sci on compacta in* $\partial_i X$,

(c) $\{x \in X: f_i(\langle X_i, \partial_i x\rangle) \cap]t_i, \infty] = \emptyset\}$ *is nonempty,*

(d) *for at least* $m-1$ *indices* i *there exist compact sets* $K_i \subset X_i$ *for which* $\{\partial_i x \in \partial_i X: \sup f_i(\langle K_i, \partial_i x\rangle) \leq t_i\}$ *is compact.*

Then there exists an $x \in X$ *such that* $f_i(x) > t_i$, $i = 1, \ldots, m$.

Proof. Let $A_i = \{x \in X: f_i(x) > t_i\}$, $i = 1, \ldots, m$. As it is seen conditions (b) of 5.1.1 and 5.1.2 are equivalent. From (a) and (c) follows 5.1.1 (a) and, what is more, $\partial_i X \setminus \partial_i A_i(K_i) = \partial_i X \setminus \{\partial_i x \in \partial_i X: f_i(\langle s, \partial_i x\rangle) > t_i$ for an $s \in K_i\} = \{\partial_i x \in \partial_i X: \sup f(\langle K_i, \partial_i x\rangle) \leq t_i\}$, i.e. 5.1.1 (c) is satisfied. Now by 5.1.1 we have $\bigcap A_i \neq \emptyset$. □

Theorem 5.1.2 leads to the following generalized version of a theorem of Nash [41, Th. 1 p. 288]

5.1.3. Theorem. *Assume that for each* $i = 1, \ldots, m$ (X_i, Ω^i) *is a weed,* X_i *a compact set,* $g_i: X \to [-\infty, \infty[$ *a map and for each* $x \in X = P\{X_i: i = 1, \ldots, m\}$ $g_i(\langle \cdot, \partial_i x\rangle)$ *is quasiconcave. Then, there exists an* $x \in X$ *such that* $g_i(x) = \max g_i(\langle X_i, \partial_i x\rangle)$, $i = 1, \ldots, m$.

Proof. Let us consider $f_i(x) = g_i(x) - \max g_i(\langle X_i, \partial_i x\rangle)$ and suppose that $p < 0$ is such that for each $x \in X$ there exists an i for which $f_i(x) \leq p$ holds. As it is seen $f_i(\langle \cdot, \partial_i x\rangle)$ is quasiconcave. On the other hand, $\max g_i(\langle X_i, \cdot\rangle)$ is sci (4.1.6). Let us show that it is scs. Assume that $\max g_i(\langle X_i, y_o\rangle) = b < p$ ($y_o \in \partial_i X$). For each $s \in X_i$ there exist U_s, V_y such that $g_i(\langle U_s \times V_{y_o}\rangle) \cap [(b+p)/2, \infty] = \emptyset$ as g_i is continuous. We may choose a finite subcover $\{U_j\}$ of $\{U_s: s \in X_i\}$, X_i being compact. Then for $y \in V = \bigcap V_j$ we have $g_i(\langle s, y\rangle) < (b+p)/2$, $s \in X_i$ and hence $\max g_i(\langle X_i, y\rangle) \leq (b+p)/2 < p$, $y \in V$. Thus, $f_i(\langle x_i, \cdot\rangle)$ is continuous.

Clearly, condition (c) of Th. 5.1.2 is satisfied. Therefore, by Th. 5.1.2 there exists an $x \in X$ such that $f_i(x) > t_i$, $i = 1, \ldots, m$ - a contradiction. This means that $\sup_{x \in X} \min_i f_i(x) = 0$ and in view of the fact that X is compact and f_i continuous, theorem is established. □

Now we extend [16, Th. 2 p. 125].

5.1.4. Theorem. *Let* (X_i, Q^i) *be weeds and* $(X_j, \mathcal{U}_j)$ *uniform spaces. Assume that* $\{A_j : j \in J\}$ *is a family of closed sets in* $X = P\{X_j : j \in J\}$ *such that for each* $x \in X$, $j \in J$ *the set* $A_j(\partial_j x) = \{y_j \in A_j : \langle y_j, \partial_i x \rangle \in A_j\}$ *is a nonempty overhull and* $\bigcup\{A_j(\partial_j x) : x \in X\}$ *is relatively compact. If for each* $j \in J$ (2.3) p. 29 *is satisfied (for* $A = A_j(\partial_j x)$, $W, V \in \mathcal{U}_j$), *then* $\bigcap A_j \neq \emptyset$.

Proof (cp. [5, p. 297]). Let $F : X \to 2^X$ be defined by $F(x) = P\{A_j(\partial_j x) : j \in J\}$, $x \in X$. Clearly, all values of F are nonempty and closed. We will show that G_F (see p. 27) is a closed set. If $(x, y) \notin G_F$, then there exists a $j \in J$ such that $\langle y_j, \partial_j x \rangle \notin A_j$. The set A_j is closed and by [31, Th. 12 p. 142] there exist neighbourhoods $U \subset X_j$ of y_j and $V \subset \partial_j X$ of $\partial_j x$ such that $\langle U \times V \rangle \cap A_j = \emptyset$. Thus, $(\langle U \times \partial_j X \rangle) \times (\langle X_j \times V \rangle) \cap G_F = \emptyset$, i.e. G_F is closed. Now in view of Lemma 2.1.8 F is usc and the final part of 2.2.12 shows that F has a fixed point, i.e. $\bigcap A_j \neq \emptyset$. □

Now the following (cp. [5, Th. 15 p. 298]) can be derived from 5.1.4

5.1.5. Theorem. *Assume that for each* $j \in J$, (X_j, Q^j) *is a weed.* $(X_j, \mathcal{U}_j)$ *is a uniform space, for* $X = P\{X_j : j \in J\}$ $f_j : X_j \to \bar{R}$ *is a map such that* $\{y_j \in X_j : f_j(\langle y_j, \partial_j x \rangle) = \max f_j(\langle X_j, \partial_j x \rangle)\} = A_j(\partial_j x)$ *is a nonempty overhull,* $x \in X$. *If* $\bigcup\{A_j(\partial_j x) : x \in X\}$ *is relatively compact and* (2.3) *holds* ($A = A_j(\partial_j x)$, $V, W \in \mathcal{U}_j$). *Then there exists an* $x \in X$ *such that* $f_j(x) = \max f_j(\langle X_j, \partial_j x \rangle)$, $j \in J$.

Proof. For $A_j = \{x \in X : f_j(x) = \max f_j(\langle X_j, \partial_j x \rangle)\}$ the requirements of 5.1.1 are satisfied. □

Let us recall that if X_j in 5.1.4 or 5.1.5 are nonempty convex sets in locally convex spaces, then (2.3) is satisfied (Corol 2.1.16).

5.2. ON A THEOREM OF CELLINA

In the present section we are interested in theorems on nonempty intersections of families of sets which are not compact. These results are related to the measures of noncompactness. Main theorem is 5.2.3 though 5.2.2 - a generalization of a Kuratowski result is also worth noting.

5.2.1. Definition. *Let* X *be a space. A measure of noncompactness on* X *is a mapping* b: $2^X \to \overline{\mathbb{R}}$ *satisfying the following conditions*

(5.1) b(A) = 0 *if and only if* A ⊂ X *is relatively compact,*

(5.2) *if* A,B ⊂ X, A ⊂ B, *then* b(A) ≤ b(B).

First we present a modified version of a theorem of Kuratowski [34, Th. 1 p. 303].

5.2.2. Theorem. *Let* b *be a measure of noncompactness on* X *which satisfies*

(5.3) $b(A \cup \{x\}) = b(A)$, $A \subset X$, $x \in X$.

If for a family $\mathcal{B}$ *of compactly closed sets in* X, *such that* $\inf\{b(B)\colon B \in \mathcal{B}\} = 0$ *we have*

(i) *the family* $\mathcal{A}$ *of all compact sets belonging to* $\mathcal{B}$ *has the finite intersection property,*

(ii) *for each* $A,B \in \mathcal{B}$, *if* $b(A) < b(B)$, *then* $A \subset B$,

then $\bigcap\mathcal{B}$ *is nonempty.*

Proof. If there exists a compact set $A \in \mathcal{B}$, then in view of (i) $\bigcap\mathcal{A}$ is nonempty. What is more, by (ii) $b(B) > 0$ means that $A \subset B$. Hence we obtain $\emptyset \neq \bigcap\mathcal{A} \subset \bigcap\mathcal{B}$. Assume now that $b(B) > 0$, $B \in \mathcal{B}$. There exists a sequence (B_n) of members of $\mathcal{B}$ such that $\lim_{n\to\infty} b(B_n) = 0$. Then for $A = \{x_n\colon n \in \mathbb{N}\}$, $x_n \in B_n$ by (5.3) we obtain $0 \le b(A) = b(\{x_n\colon n \ge k\}) \le b(B_k)$, $k \in \mathbb{N}$. This means that $b(A) = 0$, i.e. A is a relatively compact set. Let C be a compact set containing A. Clearly, $\{B \cap C\colon B \in \mathcal{B}\}$ is a family of compact sets which are closed in C. For each finite family $\mathcal{D} \subset \mathcal{B}$ there exists a set B_n such that $b(B_n) < \min\{b(D)\colon D \in \mathcal{D}\}$, and in view of (ii) $\emptyset \neq B_n \cap C \subset \bigcap\{D \cap C\colon D \in \mathcal{D}\}$. This means that the family $\{B \cap C\colon B \in \mathcal{B}\}$ has the finite intersection property and consequently $\emptyset \neq \bigcap\{B \cap C\colon B \in \mathcal{B}\} \subset \bigcap\mathcal{B}$. □

The above theorem enables us to prove the following generalized version of a Cellina's theorem [6, Th. 1]

5.2.3. Theorem. *Let* (B_n) *be a decreasing sequence of nonempty sets such that for* $E = \bigcup B_n$, (E,Q) *is a weed in* X. *Assume that the following are satisfied*

$$(5.4)\quad b(A\cup B) = \max\{b(A),b(B)\},\quad A,B \subset X,$$

$$(5.5)\quad \textit{if}\ A_n \subset E\ \textit{and}\ \lim_{n\to\infty} b(A_n) = 0,\ \textit{then}\ \lim_{n\to\infty} b(\overline{cQ}\,A_n) = 0$$

If B_n *is a closed overhull*, $n\in N$, $\lim_{n\to\infty}(B_n\setminus B_{n+1}) = 0$, $b(B_1) < \infty$ *and*

$$(5.6)\quad \textit{for each}\ A \subset B_n,\ q \in]0,1[\ \textit{there exist}\ x\in A,\ t\in I\ \textit{such that}\ qb(A) = b(Q_2(\{x\} \times A,t)),$$

holds, then $\bigcap B_n \neq \emptyset$.

Proof. If $\inf\{b(B_n): n\in N\} = 0$, then we may apply Th. 5.2.2. Therefore, we assume that $0 < 3a \le b(B_n)$, $n\in N$. Without losing the generality we may demand $b(B_n\setminus B_{n+1}) < a$ to be satisfied for all $n \in N$. Hence, the sequence $(a_n)_{n\in N}$ defined as follows: $a_n = \sup\{b(B_p\setminus B_{p+1}): p\in N,\ n \le p\} + a/n$, satisfies condition $a_n < b(B_n)$, $n\in N$. What is more, (a_n) is decreasing and $\lim_{n\to\infty} a_n = 0$. Let $x_1 \in B_1$, $t_1 \in I$ and $Z_1 = Q_2(\{x_1\} \times B_1,t_1)$ be such that $b(Z_1) = a_1$ $(b(B_1) < \infty)$. Clearly, $Z_1 \subset B_1$ and $cQ\,Z_1 \subset B_1$, as B_1 is an overhull. From $b(Z_1) = \max\{b(Z_1\cap B_2),b(Z_1\cap(B_1\setminus B_2))\} = a_1$ together with $b(Z_1\cap(B_1\setminus B_2)) \le b(B_1\setminus B_2) < a_1$ it follows that $b(Z_1) = b(Z_1\cap B_2)$. Let $x_2 \in Z_1\cap B_2$, $t_2 \in I$ be such that for $Z_2 = Q_2(\{x_2\} \times (Z_1\cap B_2),t_2)$ we have $b(Z_2) = a_2$ $(a_2 < a_1)$. Clearly, we have $Z_2 \subset cQ\,B_2 \subset B_2$ and $b(Z_2) = \max\{b(Z_2\cap B_3),b(Z_2\cap(B_2\setminus B_3))\}$. From the inequality $b(Z_2\cap(B_2\setminus B_3)) \le b(B_2\setminus B_3) < a_2 = b(Z_2)$ follows $b(Z_2) = b(Z_2\cap B_3)$. What is more, we have $Q_2(\{x_2\} \times (Z_1\cap B_2),I) \subset Q_2(\{x_2\} \times Z_1,I) \subset Q_2(Z_1)$, i.e. $cQ\,Z_2 \subset cQ\,Z_1$. By induction, if Z_n is defined, then $x_{n+1} \in Z_n \cap B_{n+1}$, $t_{n+1} \in I$ are such that for $Z_{n+1} = Q_2(\{x_{n+1}\} \times (Z_n\cap B_{n+1}),t_{n+1})$ the following holds $a_{n+1} = b(Z_{n+1}) = \max\{b(Z_{n+1} \setminus B_{n+2}), b(Z_{n+1} \cap (B_{n+1} \setminus B_{n+2}))\}$. From the inequality $b(Z_{n+1} \cap (B_{n+1} \setminus B_{n+2})) \le b(B_{n+1} \setminus B_{n+2}) < a_{n+1}$ we obtain $b(Z_{n+1}) = b(Z_{n+1} \cap B_{n+2})$. The sequence $(cQ\,Z_n)$ is decreasing. What is more, $\lim_{n\to\infty} b(Z_n) = \lim_{n\to\infty} a_n = 0$, i.e. $\lim_{n\to\infty} b(\overline{cQ}\,Z_n) = 0$ (see (5.5)).

Now by, Th. 5.2.2 $\bigcap \overline{cQ}\, Z_n$ is nonempty and, clearly, $\bigcap B_n \neq \emptyset$. □

If (X,d) is a metric space, then for $x \in X$, $\emptyset \neq A \subset X$ we write $d(x,A) = d(A,x) = \inf\{d(x,y): y \in A\}$, $B(A,r) = \{x \in X: d(A,x) < r\}$. The Hausdorff metric in the family $C(X)$ of all closed and nonempty subsets of X is defined as follows $D(A,E) = \max\{\sup\{d(x,e): x \in A\}, \sup\{d(A,y): y \in E\}\}$.

With the help of Th. 5.2.3 we prove

5.2.4. Theorem. *Let* (B_n) *be a decreasing sequence of nonempty sets in a metric space* (X,d) *and for* $E = \bigcup B_n$ *let* (E,Q) *be a weed in* X. *Assume that* b *is a measure of noncompactness on* X *satisfying* (5.4), (5.5) *and the following*

(5.7) *for each sequence* (A_n) *of subsets of* E, *if* $\lim_{n\to\infty} D(A_n,A) = 0$, *then* $\lim_{n\to\infty} b(A_n) = b(A)$,

(5.8) *for each* $q > 0$ *there exists a* $p > 0$ *such that for each* $x,y \in E$, $s,t \in I$ *with* $|s-t| < p$, *we have* $d(Q_2((x,y),s), Q_2((x,y),t)) < q$.

If the sequence (B_n) *consists of closed overhulls,* $b(B_1) < \infty$ *and* $\lim_{n\to\infty} b(B_n \setminus B_{n+1}) = 0$, *then* $\bigcap B_n \neq \emptyset$.

Proof. Condition (5.7) means that $b: C(X) \longrightarrow \overline{R}$ is a map. From (5.8) it follows that for each $x \in E$, $A \in C(E)$ mapping $Q_2(\{x\} \times A, \cdot): I \longrightarrow (C(X),D)$ is continuous. By considering the composition of these maps we obtain (5.6) and now 5.2.4 becomes a consequence of Th. 5.2.3. □

5.2.5. Lemma [P2, Prop.]. *If a measure of noncompactness* b *on* X, *where* (X,d) *is a metric space, satisfies*

(i) *there exists a mapping* $f: [0,\infty[\longrightarrow R$ *such that* $A \subset X$ *implies* $b(B(A,r)) \leq b(A)+f(r)$, *and* $\lim_{t\to 0^+} f(t) = 0$,

then (5.7) *holds*.

Proof. Condition $D(A_n,A) < r$ is equivalent to $A_n \subset B(A,r)$ and $A \subset B(A_n,r)$. Hence we obtain $B(A_n) \leq b(A)+f(r)$ and $b(A) \leq b(A_n) + f(r)$ and, consequently, $b(A)-f(r) \leq b(A_n) \leq b(A)+f(r)$. □

5.3. CONDENSING MAPPINGS

We present here two types of mappings (Def. 5.3.3, 5.3.6) which are not compact, they have but nice properties as regards the fixed point theory (see Th. 5.3.5).

5.3.1. <u>Lemma</u>. *Let* X *be a space and* $F: 2^X \to 2^X$ *increasing (see* Def. 2.1.1 p. 26), $F(x) \neq \emptyset$, $x \in X$. *If for a nonempty compact set* $B \subset X$ *we have* $F(B) \subset B$, *then there exists a nonempty set* $Z \subset X$ *such that* $\overline{F}(Z) = Z$.

Proof. Let us consider $\mathcal{A} = \{A = \overline{A} \subset X\colon\ F(A) \subset A\}$. Clearly, $\mathcal{A}$ is nonempty as $X \in \mathcal{A}$. Let $\mathfrak{G} \subset \mathcal{A}$ be any maximal chain of sets such that $G \cap B \neq \emptyset$. Then $\overline{Z} = Z = \bigcap \mathfrak{G}$ is nonempty as the family $\{G \cap B\colon G \in \mathfrak{G}\}$ of closed compact sets has the finite intersection property. fom the fact that F is increasing we obtain $F(Z) = F(\bigcap \mathfrak{G}) \subset \bigcap F(\mathfrak{G}) \subset \bigcap \mathfrak{G}$, i.e. $\overline{F}(Z) \subset Z$. Hence, $F(\overline{F}(Z)) \subset F(Z) \subset \overline{F}(Z)$ and what's more $\emptyset \neq F(Z \cap B) \subset F(Z) \cap F(B) \subset F(Z) \cap B$. These facts mean that $\overline{F}(Z) \in \mathfrak{G}$. Therefore, we have $\overline{F}(Z) = Z$, $\mathfrak{G}$ being maximal. □

From the above and Lemma 2.1.2 we obtain

5.3.2. <u>Corollary</u> [P6, Lemma 3.11]. *If* X *is a space,* $F: X \to 2^X$ *a mapping,* $(F(X), Q)$ *a weed in* X, F(x) are [illegible] *and* $B \subset X$ *a nonempty compact set such that* $F(B) \subset B$, *then there exists a nonempty set* $Z \subset X$ *with* $\overline{cQ}\ F(Z) = Z$.

The definition to follow was suggested by the notion of α-generalized concentrative mapping [9, Def. 3 p. 123]. Yet it is much more general if $\alpha = cQ$.

5.3.3. <u>Definition</u>. *A mapping* $F: X \to 2^X$ *is compacting (for* Q*) if* $(F(X), Q)$ *is a weed in* X *and the following are satisfied*

(5.9) *for each* $B \subset X$, *if* $F(B) \subset B$ *and* $\mathrm{card}(B \setminus \overline{F}(B)) \leq 1$, *then* $\overline{F}(B)$ *is a compact set.*

(5.10) *for each nonempty set* $Z \subset X$, *if* $Z = \overline{cQ}\ F(Z)$, *then* $\overline{F}|Z$ *is usc,* $F(Z)$ *is precompact and* $\overline{\bigcup\{cQ\ F(z)\colon z \in Z\}}$ *is a compact set.*

The subsequent lemma will help us to prove Th. 5.3.5.

5.3.4. Lemma. *Let* $F: X \to 2^X$ *be a mapping and* $(F(X), Q)$ *a weed in* X. *If* F *is compacting,* its values are underhulls *and* $\{x_o\}$ *is closed for an* $x_o \in X$, *then there exists a nonempty set* $Z = \overline{cQ}\, F(Z) \subset X$.

Proof. Let $\mathfrak{G} = \{G = \overline{G} \subset X\colon x_o \in G$ and $F(G) \subset G\}$. Clearly, $X \in \mathfrak{G}$ which is thus nonempty. For $B = \bigcap \mathfrak{G}$ we have $x_o \in B \neq \emptyset$ and $\overline{F}(B) = \overline{F}(\bigcap \mathfrak{G}) \subset \bigcap \overline{F}(\mathfrak{G}) \subset \bigcap \mathfrak{G} = B$. What is more, $F(\overline{F}(B) \cup \{x_o\}) = F(\overline{F}(B)) \cup F(x_o) \subset F(B) \subset \overline{F}(B) \cup \{x_o\}$. From (5.9) it follows that B is a compact set and we apply 5.3.2. □

5.3.5. Theorem. *Let* $(X, \mathcal{U})$ *be a uniform space,* $\{x_o\} \subset X$ *a closed set and* $F: X \to 2^X$ *a compacting mapping* with values being underhulls *which satisfies*

(i) *for each* $W \in \mathcal{U}$ *there exists a* $V \in \mathcal{U}$ *such that for each* $x \in X$ *we have* $cQ\, V(F(x)) \subset W(cQ\, F(x))$.

Then $\overline{cQ} \circ F$ *has a fixed point.*

Proof. In view of Lemma 5.3.4 it suffices to consider $F: Z \to 2^Z$ for a set Z satisfying (5.10). Clearly, $(F(Z), Q)$ is a weed in $\overline{cQ}\, F(Z)$ and in view of Lemma 2.1.13 $F|Z$ satisfies condition (2.1) for $X = Z$, $(E, \mathcal{V}) = (Z, \mathcal{U}|Z)$. Now by (5.10) and Th. 2.2.12 $\overline{cQ} \circ F$ has a fixed point in Z. □

5.3.6. Definition. *A mapping* $F: X \to 2^X$ *is* b-*condensing if* $(F(X), Q)$ *is a weed in* X, b *is a measure of noncompactness on* X *and the following is satisfied*

(5.11) *for each nonempty set* $Z \subset X$ *such that* $F(Z) \subset Z$ *and* $b(Z) \le b(\overline{cQ}\, F(Z))$, $\overline{F}|Z$ *is compact and* $\overline{\bigcup\{cQ\, F(z)\colon z \in Z\}}$ *is a compact set.*

The following shows that 5.3.6 has much in common with Def. 5.3.3

5.3.7. Lemma. *If* $F: X \to 2^X$ *is* b-*condensing, all values of* F *are underhulls and* (5.3) *holds, then* F *is compacting.*

Proof. If $F(B) \subset B \neq \emptyset$ and $\mathrm{card}(B \setminus \overline{F}(B)) \le 1$, then from (5.3) and $F(B) \subset \bigcup\{cQ\, F(x)\colon x \in B\} \subset cQ\, F(B)$ (Lemma 1.2.4) follows $b(B) \le$ $\le b(\overline{F}(B) \cup \{x\}) = b(\overline{F}(B)) \le b(\overline{cQ}\, F(B))$. By (5.11) set $\overline{F}(B)$ is compact and we obtain (5.9). In turn, if $Z = \overline{cQ}\, F(Z)$, then (5.11) implies (5.10). □

Definition 5.3.6 has something in common with the notion of

"(Ψ)-kondensirend Abbildung" of Hahn [28, p. 7]. Let us recall that Ψ: $\mathcal{A} \to \bar{R}$ (for $\mathcal{A} \subset 2^E$) is a measure of noncompactness (in the sense of Hahn, see [28, p. 6]) if E is a locally convex space and

(a) $\Psi(\overline{\text{conv}}\ A) = \Psi(A)$, $A \in \mathcal{A}$,

(b) $\Psi(A) \le \Psi(B)$, $A \subset B$, $B \in \mathcal{A}$.

Roughly speaking, an usc mapping $F: X \to 2^X$ with closed convex values is "(Ψ)-kondensirend" if $\Psi(A) \le \Psi(F(A))$ implies the compactness of $\overline{F}(A)$.

As it is seen, by the above conditions from $\Psi(Z) \le \Psi(\overline{\text{conv}}\ F(Z))$ follows the compactness of $\overline{F}(Z)$ (see (5.11)). This fact with the assumption that F is usc means that $F|Z$ is compact. Thus Def. 5.3.6 is more general. Nevertheless, it should be pointed out that the original definition of Hahn concerns mappings $F: X \times \Gamma \to 2^X$, where Γ is a compact space.

6. Retractions

6.1. GENERAL THEOREMS

We present here some existence theorems on retractions and continuous extensions. Theorems 6.1.3, 6.1.4, 6.1.6, 6.1.12 and 6.2.3 were proved in [P6] for the definitions less general than 1.2.1 and 1.2.7 (cp. Remark 1.2.13).

Let us recall the notations concerning metric spaces. They will be applied here and in Chapter 7. If (X,d) is a metric space, $A,E \subset X$ are nonempty sets, $x \in X$ and $r > 0$, then we write

$$\begin{aligned} &\operatorname{dia} A = \sup \{d(x,y)\colon x,y \in A\}, \quad d(x,A) = d(A,x) = \\ &= \inf \{d(x,y)\colon y \in A\}, \quad B(A,r) = \{y \in X\colon d(A,y) < r\}, \\ &d(A,E) = \inf \{d(x,y)\colon \ x \in A, \ \ y \in E\}, \quad D(A,E) = \\ &= \max \{\sup\{d(x,E)\colon x \in A\}, \sup\{d(A,y)\colon y \in E\}\}. \end{aligned} \tag{6.1}$$

As it is known, D is the Hausdorff metric in the family $C(X)$ of all nonempty closed subsets of X (if X is bounded).

The subsequent theorem is an auxiliary result, though it seems to be of importance.

6.1.1. <u>Theorem</u>. *Let* A *be a closed set in a metric space* (X,d) *and* $p\colon X\setminus A \longrightarrow]0,\infty[$ *an sci mapping* (Def, 4.1.5 p. 52). *Then for each nonempty set* $C \subset A$ *there exists a locally finite open cover* $\mathcal{U} = \{U_j\colon j \in J\}$ *of* $X\setminus A$ *and a set* $\{y_j\colon j \in J\} \subset C$ *such that*

(6.2) *for each* $x \in X\setminus A$, $j \in J$ *if* $x \in St(U_j,\mathcal{U})$, *then* $d(x,y_j) < p(x) + d(x,C)$.

Proof. Let us write $h(x) = p(x)/10$, $x \in X \setminus A$. Clearly, $h\colon X\setminus A \longrightarrow]0,\infty[$ is sci and, therefore, $Z(q) = h^{-}(]q/2,\infty[)$ is an open set for each $q \in R$. We have $Z(h(x)) = \{y \in X\setminus A\colon h(x)/2 < h(y)\}$ and hence $x \in Z(h(x))$, $x \in X\setminus A$. Therefore, $\mathcal{B} = \{B(x,h(x)) \cap Z(h(x))\colon x \in X\setminus A\}$ is an open cover of $X\setminus A$. There exists a locally finite star refinement $\mathcal{U}$ of $\mathcal{B}$ [15, Th. 5.1.12, Lemma 5.1.15 p. 377] as $(X\setminus A,d)$ is a paracompact space. For $j\in J$ let $z_j \in X\setminus A$ be such that $St(U_j,\mathcal{U}) \subset B(z_j,h(z_j)) \cap Z(h(z_j))$ and for $x_j \in U_j$ let $y_j \in B(x_j,d(x_j,C)+h(z_j)) \cap C$. Then for $x \in St(U_j,\mathcal{U})$ we have $d(x,y_j) \le d(x,x_j)+d(x_j,y_j) \le \mathrm{dia}\ St(U_j,\mathcal{U}) + d(x_j,C) + h(z_j) \le 3h(z_j)+d(x_j,C) \le 3h(z_j)+d(x_j,x)+d(x,C) \le 5h(z_j) + d(x,C)$. From the fact that $x \in Z(h(z_j))$, i.e. $h(z_j)/2 < h(x)$ we obtain $d(x,y_j) < 10h(x) + d(x,C) \le p(x) + d(x,C)$. □

6.1.2. <u>Definition</u> (cp. [P8, Def. 1.9]). *We write* $X \in W2(E)$ *if there exists a* Q *such that* (E,Q) *is a weed in* X *and the following is satisfied*

(6.3) *for each* $z\in E$, $V_z \subset X$, *there exists a* $U_z \subset X$ *such that* $cQ\ U_z \subset V_z$.

In place of $X \in W2(X)$ *we write* $X \in W2$.

The sense of $X \in W2$ agrees with the notion of the local simplicial convexity [3, Def (1.9) p. 160] as by (6.3) we have $\{x\} = cQ\ \{x\}$, $x\in X$ (see Lemma 6.2.11).

In connection with the subsequent theorem we call reader's attention to the fact that if $Fr\ A$ (the boundary of A) is emtpy and A is closed and nonempty, then the equalities $r(x) = x$, $x\in A$, $r(X\setminus A) = x_0 \in A$ define a continuous retraction $r: X \to A$.

6.1.3. <u>Theorem</u>. *If* A *is a closed set in a metric space* (X,d) *and* $A \in W2(Fr\ A)$, *then* A *is a retract of* X.

Proof. For the notations as in Th. 6.1.1 let $C = Fr\ A$, and $p(x) = d(x,Fr\ A)$ while J is a linearly ordered set of indices. For each $x \in X\setminus A$ there exists a W_x such that $J(x) = \{j\in J: U_j \cap W_x \neq \emptyset\}$ is a finite set. Let $J(x) = \{j_1, \dots, j_m\}$ $(m = m(x))$ be ordered as follows $j_1 < \dots < j_m$. Then for $t_j(x) = d(x,X\setminus U_j)$, $j\in J$ let us write $t(x) = (t_{j_1}(x), \dots, t_{j_m}(x))$, $y = (y_{j_1}, \dots, y_{j_m})$, $x \in X\setminus A$. We will prove that for Q as in 6.1.2

$$(6.4)\quad r(x) = \begin{cases} x, & x\in X, \\ Q_m(y,t(x)), & x \in X\setminus A \end{cases}$$

defines a retraction. The values of $Q_m(y,t(x))$ depend only on y_j for which $t_j(x) \neq \emptyset$ (see (1.1)) and hence $r: X \to A$ is well defined and it is a mapping as $\mathcal{U}$ covers $X\setminus A$. By the continuity assumptions of Def. 1.2.7 $r|W_x$ is continuous t being continuous on W_x. Hence follows the continuity of r on $X\setminus A$. Now it suffices to show that r is continuous in points of Fr A. Condition (6.2) implies the inclusion $r(x) = Q_m(y,t(x)) \in cQ[B(x,p(x)+d(x,\mathrm{Fr}\ A)) \cap \cap \mathrm{Fr}\ A]$, $x \in X\setminus A$. If $x \to z \in \mathrm{Fr}\ A$ $(x\notin A)$, then by (6.3) and the fact that $p(x) \to 0$ if $x \to z$, then we have $r(x) \to z$. □

The method presented in the above proof enables us to prove the following generalization of the well known theorem of Dugundji [10, Th. 4.1 p. 357].

6.1.4. Theorem. *Let* A *be a closed set in a metric space* (X,d) *and* $g: A \to Y$ *a map. If* $Y \in W2(g(\mathrm{Fr}\ A))$, *then* g *can be extended to a map* $f: X \to Y$ *in such a way that* $f(X) \subset g(A) \cup cQ\ g(\mathrm{Fr}\ A)$ (Q *as in* 6.1.2).

Proof. Let t,y be as in the proof of Th. 6.1.3. Then for $g(y) = (g(y_1), \ldots, g(y_m))$ the desired extension is defined as follows

$$(6.5)\quad f(x) = \begin{cases} g(x), & x\in A, \\ Q_m(g(y),t(x)), & x \in X\setminus A. \end{cases}$$

The continuity of f on Fr A follows from the inclusion $f(x) = = Q_m(g(y),t(x)) \in cQ\ g([B(x,p(x)+d(x,\mathrm{Fr}\ A)) \cap \mathrm{Fr}\ A])$ and from (6.3) for Y. □

In particular, if $A \in W2$ then from the above we obtain [3, Th. (2.1) p. 162] as $f(X) \subset g(A) \cup cQ\ g(\mathrm{Fr}\ A) \subset cQ\ g(A)$.

6.1.5. Definition (cp. [P6, Def. 1.9]). *We write* $X \in LW2(E)$ *if there exists a* Q *such that* (E,Q) *is a weed in the set* X (Def. 1.2.1), X *is a space*, (6.3) *and*

$$(6.6)\quad \begin{array}{l} \textit{for each } z\in E \textit{ there exists a } U_z \subset X \textit{ such that for} \\ \textit{every } m \in N,\ \{e_1, \ldots, e_m\} \subset U_z \cap E,\ Q_m(e,\cdot): \\ P^{m-1} \to X \textit{ is continuous } (e = (e_1, \ldots, e_m)) \end{array}$$

hold. In place of $X \in LW2(X)$ *we write* $X \in LW2$.

Condition (6.6) can be expressed in a simpler way: for each $z \in E$ there exists a $U_z \subset X$ such that $(U_z \cap E, Q)$ is a weed in X (Def. 1.2.7).

6.1.6. Theorem. *Let* A *be a closed set in a metric space* (X,d) *and* $A \in LW2(Fr\ A)$ *for a* Q. *Then* A *is a retract of its neighbourhood* $W \cup A$, *where*

$$(6.7)\quad W = \bigcup_{q>0} \{x \in X: (B(x,d(x,Fr\ A)+q) \cap Fr\ A,\ Q) \text{ is a weed in the set } A\}.$$

Proof. By (6.6) W is a neighbourhood of $Fr\ A$ and, therefore, $A \cup W$ is a neighbourhood of A. Let us define $l: W\setminus A \rightarrow]0,\infty[$ as follows

$$(6.8)\quad l(x) = \sup \{q: B(x,d(x,Fr\ A)+q) \cap Fr\ A,\ Q) \text{ is a weed in the set } A\}.$$

We will prove that mapping l is sci. Assume that $a < l(x)$. Then there exists a $q > 0$ such that $a < l(x) - q$. For $z \in B(x,q/2)$ the inequality $d(y,z) < d(z,Fr\ A) + l(x) - q$ means that $d(x,y) \leq d(x,z) + d(y,z) < q/2 + d(z,Fr\ A) + l(x) - q \leq q/2 + d(x,z) + d(x,Fr\ A) + l(x) - q < d(x,Fr\ A) + l(x)$. It means that we have $B(z,d(z,Fr\ A)+l(x)-q) \subset B(x,d(x,Fr\ A)+l(x))$ and, therefore, $a < l(x) - q \leq l(z)$ for $z \in B(x,q/2)$. Hence, l is sci. Now, by considering $p(x) = \min \{l(x),d(x,Fr\ A)\}$ we can see that r defined by (6.4) is continuous as $\{y_{j_1}, \ldots, y_{j_m}\} \subset B(x,d(x,Fr\ A)+l(x))$. What is more, the continuity of r on $Fr\ A$ follows from the fact that $p(x) \rightarrow 0$ if $x \rightarrow z \in Fr\ A$ $(x \in X\setminus A)$. □

Similarly to Th. 6.1.4, from the above we obtain

6.1.7. Theorem. *Let* A *be a closed set in a metric space* (X,d) *and* $g: A \rightarrow Y$ *a map. If* $Y \in LW2(g(Fr\ A))$, *then* g *can be extended to a map* $f: A \cup W' \rightarrow Y$, *where*

$$(6.7')\quad W' = \bigcup_{q>0} \{x \in X: (g(B(x,d(x,Fr\ A)+q) \cap Fr\ A),Q) \text{ is a weed in } Y\},$$

and in such a way that $f(A \cup W') \subset g(A) \cup cQ\ g(Fr\ A)$ (Q *as in* 6.1.5). *In particular, if* $g(A) \in LW2$, *then* $g(A) \cup cQ\ g(Fr\ A) \subset cQ\ g(A)$.

Bielawski has proved [3, Corol. (2.2) p. 162] that for metric spaces we have $X \in AR(\mathcal{M})$ if and only if $X \in W2$. Now we will obtain a similar result for $LW2$.

6.1.8. Theorem. *Let* $f: X \to Y$ *be an r-map with the right inverse* $g: Y \to X$. *If* $X \in Wi(g(E))$ $(X \in LWi(g(E)))$, *then* $Y \in Wi(E)$ $(Y \in LWi(E))$, $i = 1,2$. *In place of* i *one may write* ki *(see* Def. 6.2.2, 6.2.4, 6.2.9).

Proof. For example, let $X \in W2$. Let us consider Q^* as in 1.2.10. Assume that Q satisfies (6.3) for $z = g(x)$ $(x \in E)$. If W is a neighbourhood of x in Y, then we may require $f(V_z) \subset W$ to be satisfied. What is more, $T_x = g^{-}(U_z)$ is a neighbourhood of x. From (1.9) and (6.3) it follows that $Q_n^*(T_x) \subset W$, $n \in N$, i.e. (6.3) holds for Q^* and E. □

6.1.9. Remark. *As it is known* [4, Th. (2.1) p. 85], *if* $X \in AR(\mathcal{M})$, *then* X *is an r-image of a convex subset of a normed space and in view of* 6.1.8 $X \in W2$. *Similarly, as* $X \in ANR(\mathcal{M})$ *means that* X *is an r-image of an open subset of a convex set in a normed space* [4, Th. (3.1) p. 86], *then by* 6.1.8 $X \in LW2$.

From the above facts we obtain ($\in$ replaces is)

6.1.10. Theorem. *If* X *is a metrisable space, then* $X \in AR(\mathcal{M})$ $(X \in ANR(\mathcal{M}))$ *if and only if* $X \in W2$ $(X \in LW2)$.

6.1.11. Corollary. *Let* A *be a closed set in a metric space* (X,d) *and* $A \in LW2(\mathrm{Fr}\ A)$. *If* $\inf \{l(x): x \in \mathrm{Fr}\ A\} = a > 0$ *(see* (6.8)), *then* A *is a retract of its neighbourhood* $A \cup B(\mathrm{Fr}\ A, a/2)$.

Proof. If $z \in \mathrm{Fr}\ A$ and $d(x,z) < a/2$, then $B(x,a/2) \subset B(z,a)$ and hence, $B(\mathrm{Fr}\ A, a/2) \subset W$ (see (6.7)). □

6.1.12. Theorem. *Let* (X,d) *be a metric space, and* $A \in W2$ *a closed subset of* X. *If a compact map* $f: A \to X$ *satisfies*

(i) *there exists an sci mapping* $p: C = f(A)\setminus A \to]0,\infty[$ *that* $0 < d(f^{-}(z), cQ[B(z,d(z,A)+p(z)) \cap A])$, $z \in C$,

then f *has a fixed point.*

Proof. Let $r: C \to A$ be as in (6.4). In view of Th. 2.2.20 it follows that $r\circ f\colon A \to A$ has a fixed point, say $x \in (r\circ f)(x)$. If $z = f(x) \notin A$, then we have $r(z) \in cQ[B(z,d(z,A)+p(z)) \cap A]$ (see the proof of Th. 6.1.6), i.e. $x \neq r(z)$ (see (i)). Hence, $f(x) \in A$ and $x = (r\circ f)(x) = f(x)$. □

The above theorem extends [21, Th. 2 p. 235] as in the case of A being a compact convex set in a normed space, condition (i) of 6.1.2 is equivalent to $f^{-}(z) \cap \bar{B}(z,d(z,A)) = \emptyset$, $z\in C$ [P4, Lemma].

6.2. FINITE DIMENSIONAL CASE

Some theorems of the previous section have their finite dimensional versions with relaxed assumptions on the structure of weeds under consideration. The final part of this section is devoted to comparing various definitions of special weeds.

We adopt the following definition of the covering dimension dim (see [15, 7.2.4 p. 484])

6.2.1. Definition. *Let* X *be a metrizable space. Then we write* $\dim X \le n$ *if every locally finite open cover of* X *has an open refinement of order* $\le n$ *(at most* $n+1$ *elements of such a refinement intersect).*

6.2.2. Definition (cp. [P6, Def. 1.7]). *We write* $X \in Wk1(E)$, *if there exists a* Q *such that* (E,Q) *is a weed in the set* X, X *is a space,* Q_k *satisfies the continuity assumptions of* Def. 1.2.7 *and the following is satisfied*

(6.9) *for each* $z\in E$ *and each* $V_z \subset X$ *there exists a* $U_z \subset X$ *such that* $Q_k(U_z) \subset V_z$.

In place of $X \in Wk1(X)$ *we write* $X \in Wk1$.

Clearly, if $X \in Wk1(E)$, then for $m \le k$ $X \in Wm1(E)$.

6.2.3. Theorem. *Let* A *be a closed set in a metric space* (X,d) *and* $g\colon A \to Y$ *a map. If* $\dim(X\setminus A) \le k-1$ *and* $Y \in Wk1(g(\mathrm{Fr}\, A))$, *then* g *can be extended to a map* $f\colon X \to Y$ *such that* $f(X) \subset g(A) \cup Q_k(g(\mathrm{Fr}\, A))$ *(*Q *as in* 6.2.2). *In particular, for* $Y \in Wk1$ *we obtain* $f(X) \subset Q_k(g(A))$.

Proof. In view of $\dim(X\setminus A) \le k-1$ we may require $\mathcal{U}$ to be of order $\le k-1$ (see the proof of Th. 6.1.3) [15, Lemma 5.1.15 p.

377]. Hence, $t(x)$ has at most k coordinates different from zero, and f obtained by (6.5) is continuous. □

6.2.4. Definition (cp. [P8, Def. 1.7]). *We write* $X \in LWk1(E)$ *if there exists a* Q *such that* (E,Q) *is a weed in the set* X, X *is a space and* Q_k *satisfies* (6.9) *and* (6.6) *for* $m = k$. *In place of* $X \in LWk1(X)$ *we write* $X \in LWk1$.

6.2.5. Theorem. *Let* A *be a closed set in a metric space* (X,d) *and* $g\colon A \to Y$ *a map. If* $\dim(X\setminus A) \le k-1$ *and* $Y \in LWk1(g(\mathrm{Fr}\,A))$, *then* g *can be extended to a map* $f\colon A \cup W \to Y$ *such that* $f(A \cup W) \subset g(A) \cup Q_k(g(\mathrm{Fr}\,A))$ (Q *as in* 6.2.4) *for* W *defined as follows*

$$(6.10)\quad W = \bigcup_{q>0} \{x\in X\colon Q_k(y,\cdot)\colon P^{k-1} \to A \text{ is continuous for } y \in [g(B(x,d(x,\mathrm{Fr}\,A)+q) \cap \mathrm{Fr}\,A]^k\}.$$

In particular, if $Y \in LWk1$, *then* $f(A \cup W) \subset Q_k(g(A))$.

Proof. It suffices to follow the proofs of Theorems 6.1.6, 6.1.7 for $l(x)$ depending on Q_k and the cover $\mathcal{U}$ of order $\le k-1$ (the proof of Th. 6.1.3). □

6.2.6. Remark. *In particular, if* $Y = A$, $g = \mathrm{id}_A$, *then* 6.2.3, 6.2.5 *become theorems on retractions.*

6.2.7. Theorem. *Let* Y *be a metrizable space. If* $Y \in W(k+2)1$ ($Y \in LW(k+2)1$), *then* $Y \in LC^k$ *and* $Y \in C^k$ ($Y \in LC^k$). *If in addition* $\dim Y \le k$, *then* $Y \in AR(\mathcal{M})$ ($Y \in ANR(\mathcal{M})$).

Proof. From 6.2.3 it follows that if A is a closed subset of a metric space X and $\dim(X\setminus A) \le k+1$, then each map $g\colon A \to Y$ can be extended on X if $Y \in W(k+2)1$. Hence, in view of [11, Th. 3.2 p. 232] $Y \in LC^k$ and by [11, Th. 1 p. 238] $Y \in C^k$. This fact and [11, Th. 15.2 p. 244] mean that $Y \in AR(\mathcal{M})$. Similarly, for $Y \in LW(k+2)1$, from Th. 6.2.5, [11, Th. 3.2 p. 232] and [11, Th. 15.1 p. 244] we obtain $Y \in ANR(\mathcal{M})$. □

For any finite dimensional metrizable space Y, we have $Y \in ANR(\mathcal{M})$ if and only if $Y \in LC$ [11, Th. 15.1 p. 244] and $Y \in AR(\mathcal{M})$ if and only if $Y \in LC$ and $Y \in C$ [11, Th. 15.2 p. 244], then by the above and 6.1.10 (see [3, Corol. (2.3) p. 163]) we obtain

6.2.8. Theorem. *Let* X *be a metrizable space and* $\dim X \leq k$. *Then we obtain* $X \in LC^k \Leftrightarrow X \in LC \Leftrightarrow X \in LW(k+2)1 \Leftrightarrow X \in LW1 \Leftrightarrow X \in LW2 \Leftrightarrow$ $\Leftrightarrow X \in ANR(\mathcal{M})$ *and* $(X \in LC^k$ *and* $X \in C^k) \Leftrightarrow (X \in LC$ *and* $X \in C) \Leftrightarrow$ $\Leftrightarrow X \in W(k+2)1 \Leftrightarrow X \in W1 \Leftrightarrow X \in W2 \Leftrightarrow X \in AR(\mathcal{M})$.

6.2.9. Definition (cp. [P6, Def. 1.7]). *We write* $X \in W1(E)$ $(X \in$ $\in LW1(E))$ *if for a fixed* Q *and all* $k \in N$ $X \in Wk1(E)$ $(X \in$ $\in LWk1(E))$. *In place of* $X \in W1(X)$ $(X \in LW1(X))$ *we write* $X \in W1$ $(X \in LW1)$.

From the very definitions we obtain

6.2.10. Corollary. $X \in W2(E) \Rightarrow X \in W1(E) \Rightarrow X \in Wk1(E)$, $k \in N$, *and* $X \in LW2(E) \Rightarrow X \in LW1(E) \Rightarrow X \in LWk1(E)$, $k \in N$.

6.2.11. Lemma. *If* $X \in LW1(E)$ *and* E *is a* T_1*-space, then* $cQ\{x\} = \{x\}$, $x \in E$ *and each subset of* E *is an underhull (cp.* Remark 1.2.5).

Proof. It follows from (6.9) that $Q_k(z) \subset V_z$ for each V_z in X, $k \in N$. Hence, $cQ\{z\} \subset \{z\}$ and from the fact that $cQ\{z\} \neq \emptyset$ we obtain $cQ\{z\} = \{z\}$, $z \in E$. □

The subsequent lemma will be applied in Section 3.

6.2.12. Lemma. *Let* X *be a space and* $S\colon X \times I \times X \longrightarrow X$ *a mapping such that for each* $z \in X$, $V = V_z$ *there exists a* $U = U_z$ *for which* $S(U,I,U) \subset V$. *Then for* Q *defined in* (1.13), p. 21 *and each* $k \in N$ (6.9) *holds.*

Proof. If W is a neighbourhood of z, then there exists a $V =$ $= V_z$ such that $S(V,I,V) \subset W$. In turn, there exists a $U = U_z \subset$ $\subset V$ such that $S(U,I,U) \subset V$ and, therefore, $S(U,I,S(U,I,U)) \subset$ $\subset S(U,I,V) \subset S(V,I,V) \subset W$ holds. Consequently, our lemma can be proved by induction. □

6.3. WEEDS AND OTHER STRUCTURES

In this section we compare weeds with some well known classical notions of similar type (Def. 6.3.1, 6.3.2, 6.3.3).

Fox [24, p. 733] has introduced the notion of the local equiconnecting function which was applied to defining LEC and EC space.

6.3.1. Definition (cp. [7, pp. 101, 102]). *A space* X *is* LEC (X ∈ LEC) *if there exists a local equiconnecting function, i.e. a map* $\lambda: X \times X \times I \to X$ *which is continuous on* $V \times I$, *where* V *is a neighbourhood of the diagonal in* $X \times X$ *and* $\lambda(x_0, x_1, i) = x_i$, $i = 0,1$ *and* $\lambda(x,x,t) = x$, $(x_0,x_1) \in V$, $x \in X$, $t \in I$. *If in addition* $V = X \times X$, *then* X *is* EC (X ∈ EC).

As it is seen, for $S_x(t,y) = \lambda(x,y,1-t)$, where λ is as in 6.3.1, condition (1.11) is satisfied. This fact and Remark 1.3.2 mean that from X ∈ EC it follows that X is S-contractible (Def. 1.3.1). From the continuity of $\lambda : V \times I \to X$ it follows that for each V_z the inclusion $\lambda(z,z,I) \subset V_z$ holds, and by the theorem of Wallace [31, Th. 12 p. 142] we have $\lambda(U_z,U_z,I) \subset V_z$ for a $U_z \subset X$. Hence, X ∈ EC (X ∈ LEC) implies (see 6.2.12) that X is (locally) of type II [P1, Def. 4 p. 597] ([P1, Def. 6 p. 598]) and, in turn, X ∈ W1 (X ∈ LW1).

6.3.2. Definition [12, p. 191]. *A space* X *is stably* EC (LEC) *if* X ∈ EC (X ∈ LEC) *for* λ *such that for each* W_z, $z \in X$ *there exists a* $U = U_z$ *such that* $U^1 = \lambda(U,U,I)$, ..., $U^{n+1} = \lambda(U,U^n,I) \subset W_z$, $n \in \mathbb{N}$.

If in place of λ, as in 6.3.1, we apply S from 1.3.1, then the above becomes the definition of a space (locally) of type I [P1]. In addition, the inclusions of 6.3.2 mean that $cQ\, U \subset W_z$ for Q as in (1.13). Hence, we have (X is stably EC) ⇒ (X is of type I) ⇒ X ∈ W2, and similarly for the local case.

Himmelberg [28, Th. 3 p. 45] has defined the space of class CS (LCS). The respective definition can be formulated as follows (cp. [7, p. 102]

6.3.3. Definition. *A space* X *is* LCS (X ∈ LCS) *if there exists an open cover* $\mathcal{U}$ *of* X *and a sequence of maps* $\lambda^n: (\bigcup_{U \in \mathcal{U}} U^n) \times P^{n-1} \to X$ *such that for each* $x \in X^n$, $t \in P^{n-1}$ *we have* $\lambda^n(x,t) = \lambda^{n-1}(\partial_i x, \partial_i t)$ *whenever* $t_i = 0$, *and for each* V_z, $z \in X$ *there exists a* U_z *such that* $\lambda^n(U_z^n \times P^{n-1}) \subset V_z$, $n \in \mathbb{N}$. *If, in addition,* $\mathcal{U} = X$, *then we have* X ∈ CS.

By considering the composition of λ as in Def. 6.3.2 (cp. (1.12), (1.13)) we can see that if X is stably EC (LEC), then X ∈ CS (LCS). What is more, it follows from 6.3.3 that if X ∈ CS (LCS), then $X \in W2$ $(LW2)$.

By putting the above information together we obtain

6.3.4. <u>Corollary</u>. (X *is stably* EC) $\Rightarrow$ $X \in EC$ $\Leftrightarrow$ (X *is of type* II) $\Leftrightarrow$ ($X \in C$ *and* $X \in LC$); (X *is stably* EC) $\Rightarrow$ $X \in CS$ $\Rightarrow$ $X \in W2$ $\Rightarrow$ $X \in W1$; (X *is stably* EC) $\Rightarrow$ (X *is of type* I) $\Rightarrow$ (X *is of type* II) $\Rightarrow$ $X \in W1$. *For the local case we obtain similar implications. Let us write the first of them* (X *is stably* LEC) $\Rightarrow$ $X \in LEC$ $\Leftrightarrow$ (X *is locally of type* II) $\Leftrightarrow$ $X \in LC$.

From 6.3.4, [28, Th. 3 p. 45] and 6.1.10 follows

6.3.5. <u>Corollary</u>. *If* X *is a metrisable space, then we have* (X *is stably* EC) $\Leftrightarrow$ (X *is of type* I) $\Leftrightarrow$ $X \in CS$ $\Leftrightarrow$ $X \in W2$ $\Leftrightarrow$ $X \in AR(\mathcal{M})$; (X *is stably* LEC) $\Leftrightarrow$ (X *is locally of type* I) $\Leftrightarrow$ $X \in LCS$ $\Leftrightarrow$ $X \in LW2$ $\Leftrightarrow$ $X \in ANR(\mathcal{M})$.

What is more, it follows from 6.1.9 that if $X \in AR(\mathcal{M})$ ($X \in ANR(\mathcal{M})$), then $X \in EC$ ($X \in LEC$).

In view of Th. 6.2.8 we may say that for a finite dimensional metrizable space X all conditions of 6.3.4 except of stably (LEC) EC and (locally) of type I are equivalent to ($X \in ANR(\mathcal{M})$) $X \in AR(\mathcal{M})$.

It should be noted that Mazurkiewicz's example [4, pp. 152 - 155] presents a finite dimensional space $X \in CS$ which is not of type I.

7. Selections

In the present chapter we prove generalizations of some known theorems on selections (Th. 7.11 is the most important one here). The convex structure presented in Def. 7.8 extends the respective ideas of Michael and Curtis (see considerations following Remark 7.12) and it is related to a concept of Bielawski. The results of this chapter were obtained in [P8] for the different definitions.

If X,Y are spaces and $F\colon X \to 2^Y$ is a mapping, then each map $f\colon X \to Y$ such that $f(x) \in F(x)$, $x \in X$ will be called a *selection* (for F).

Let us recall that a space is *paracompact* if it is regular and each open cover has an open locally finite refinement [31, p. 156]. *A partition of unity on a space* X is a family $\mathcal{T} = \{t_j\colon j \in J\}$ of maps on X to I such that for each $x \in X$ $t_j(x) \neq 0$ holds only for a finite number of $j \in J$ and we have $\sum \{t_j(x)\colon j \in J,\ t_j(x) \neq 0\} = 1$ [31, p. 171]. A partition of unity $\mathcal{T}$ on X defines an open cover $\mathcal{U}_{\mathcal{T}} = \{U_j = t_j^{-}(]0,1])\colon j \in J\}$. By saying that a partition of unity $\mathcal{T}$ has a property which is characteristic of covers, we understand that it concerns $\mathcal{U}_{\mathcal{T}}$. In addition we assume that J is linearly ordered by a relation $<$.

Paracompact spaces have the following property (cp [15, Lemma 5.1.13 p. 377, Th. 5.1.9 p. 375]) which is important for the proofs of this chapter

7.1. <u>Theorem</u>. *Every open cover $\mathcal{W}$ of a paracompact space has a partition of unity being a locally finite star refinement of $\mathcal{W}$.*

Proof. Every open cover $\mathcal{W}$ of a paracompact space has a locally finite closed refinement [31, Lemma 29 p. 157]. Hence, $\mathcal{W}$ has an open star refinement $\mathcal{V}$ [15, Lemmas 5.1.13, 5.1.15 p. 377]. The cover $\mathcal{V}$ has a locally finite open refinement and a locally finite partition of unity $\mathcal{T}$ subordinate to $\mathcal{V}$ [31, V p. 171] as X is a normal space [31, Corol. 32 p. 159]. Hence, $\mathcal{T}$ is a locally finite partition of unity which is a star refinement of $\mathcal{W}$. □

7.2. Theorem (cp. [3, Prop. (3.9) p. 166]). *Assume that* X *is a paracompact space,* $F: X \to 2^Y$ *a mapping such that all values of* F^- *are open and* $(F(X),Q)$ *is a weed in* Y. *Then* $cQ \circ F$ *admits selection. If, in addition, all values of* F *are overhulls, then* F *has a selection.*

Proof. The family $\mathcal{V} = \{F^-(y): y \in Y\}$ is an open cover of X. Let $\mathcal{T}$ be a locally finite partition of unity subordinate to $\mathcal{V}$. For each $x \in X$ there exists a W_x such that $J(x) = \{j \in J: t_j(W_x) \neq 0\} = \{j_1, \ldots, j_m\}$ $(m = m(x))$. We may assume that $j_1 < \ldots < j_m$. Let $y_j \in Y$ be such that $U_j \subset F^-(y_j)$, $j \in J$ and $y = (y_{j_1}, \ldots, y_{j_m})$, $x \in X$. Then for $t(x) = (t_{j_1}(x), \ldots, t_{j_m}(x))$ we adopt

$$(7.1)\quad f(x) = Q_m(y, t(x)), \quad x \in X \quad (m = m(x)).$$

By (1.1) $f: X \to Y$ is a mapping and by the continuity assumptions of Def. 1.2.7 $f|W_x$ is continuous, $x \in X$. Thus, f is continuous. If $t_j(x) \neq 0$, then $x \in U_j$ and, hence, $x \in F^-(y_j)$ which means that $y_j \in F(x)$. Therefore, we have $f(x) \in Q_m(F(x)) \subset cQ\, F(x)$, $x \in X$. □

Each regular compact space is paracompact. Consequently, every compact subset of a linear space is paracompact. Hence, from Th. 7.2 we obtain (cp. [5, proof of Th. 1. p. 285])

7.3. Theorem. *Let* X *be a compact set in a linear space and* Y *a convex set in a linear space. If* $F: X \to 2^Y$ *is a mapping such that all values of* F^- *are open, then* $\mathrm{conv} \circ F$ *has a selection. If in addition all values of* F *are convex, then* F *has a selection.*

Mapping F as in Th. 7.3 is, in paritcular, lsc (Def. 2.2.15 p. 36). Let us recall some of the properties of lsc mappings.

7.4. Lemma [37, Ex. 1.3* p. 362]. *If* $F: X \to 2^Y$ *is lsc* $C = \overline{C} \subset X$ *and* g *is a selection for* $F|C$, *then* $F: X \to 2^Y$ *defined by*

$$(7.2)\quad G(x) = \begin{cases} g(x), & x \in C, \\ F(x), & x \in X \setminus C, \end{cases}$$

is lsc.

Proof. Let $W \subset Y$ be an open set. If $x \in G^{-}(W)$, then $x \in V = F^{-}(W)$. What is more, if $x \in G^{-}(W) \cap C$, then there exists a neighbourhood U of x such that $g(U \cap C) \subset W$. Hence, for $V_1 = U \cap V$ we have $g(V_1 \cap C) \subset W$ and $V_1 \cap (X \setminus C) \subset F^{-}(W)$, i.e $V_1 \subset G^{-}(W)$. If $x \in X \setminus C$, then $V_1 = V \cap (X \setminus C)$ is the respective neighbourhood of x. □

7.5. <u>Lemma</u> [37, Prop. 2.3 p.366]. *A mapping* $F: 2^X \to 2^Y$ *is lsc if and only if* $\bar{F}$ *is lsc.*

Proof. For each $A \subset Y$ and any open $W \subset Y$ we have $\bar{A} \cap W \neq \emptyset$ if and only if $A \cap W \neq \emptyset$. Hence, $(\bar{F})^{-}(W) = F^{-}(W)$ is open. □

7.6. <u>Lemma</u>. *Let* $F: X \to 2^Y$ *be a mapping,* X *a space and* (Y,d) *a metric space. If the following condition is satisfied*

(7.3) *for each* $p > 0$, $x \in X$ *there exists a* U_x *such that* $F(x) \subset B(F(z),p)$, $z \in U_x$,

then F *is lsc. If all values of* F *are compact and* F *is lsc, then* (7.3) *holds.*

Proof. Let V be an open subset of Y and let $x \in F^{-}(V)$. Thus there exists a $y \in F(x) \cap V$ and for a $p > 0$ we have $B(y,p) \subset V$. This proves that for each $z \in X$ such that $F(x) \subset B(F(z),p)$ we have $F(z) \cap V \neq \emptyset$ and in view of (7.3) the set of all such points is a neighbourhood of x, i.e. $F^{-}(V)$ is an open set. The remaining part of our lemma follows from [38, Lemma 11.3 p. 578]. □

7.7. <u>Lemma</u> (cp. [2, Th. 1 p. 133]). *If* (Y,d) *is a metric space and* $F: X \to (C(Y),D)$ *(see (6.1)) is a map, then* F *is lsc.*

Proof. The set $\{z \in X: F(x) \subset B(F(z),p)\}$ contains a neighbourhood $F^{-}(B_D(F(x),p))$ of x and (7.3) holds. □

7.8. <u>Definition</u> (cp. [P6, Def. 4.6]). *Let* (Y,d) *be a metric space and* $\emptyset \neq \mathcal{A}$ *a family of nonempty subsets of* Y. *We write* $Y \in W4(\mathcal{A})$ *if* $Y \in W2(\bigcup \mathcal{A})$ (Def. 6.1.2 p. 78) *for a* Q *satisfying*

(7.4) *for each* $q > 0$ *there exists a* $p > 0$ *such that* $cQ\ B(A,p) \subset B(cQ\ A,q)$, $A \in \mathcal{A}$.

If $\mathcal{A} = 2^Y \setminus \{\emptyset\}$, *then we write* $Y \in W4$. *By replacing* $W2(\bigcup\mathcal{A})$ *by* $LW2(\bigcup\mathcal{A})$ *we define* $LW4(\mathcal{A})$.

Every overhull which is contained in $\bigcup\mathcal{A}$ as in the above definition, is a convex set (see Lemma 6.2.11, Remark 1.2.5).

Condition (7.4) resembles [3, (3) p. 164] and is an extension of [P5, (4)].

7.9. Lemma. *Let* X *be a space,* (Y,d) *a metric space,* $Y \in W4(\{F(x): x \in X\})$. *If* $F: X \to 2^Y$ *satisfies* (7.3), *then* $cQ \circ F$ *is lsc.*

Proof. From $cQ\, B(F(z)),p) \subset B(cQ\, F(z),q)$ (see (7.4)) it follows that $cQ \circ F$ satisfies (7.3) and in view of Lemma 7.6 $cQ \circ F$ is lsc. □

7.10. Corollary. *Let* X *be a space and* (Y,d) *a metric space,* $Y \in W4(\{F(x): x \in X\})$. *If all values of* $F: X \to 2^Y$ *are compact and* F *is lsc, then* $\overline{cQ} \circ F$ *is lsc.*

Proof. In view of Lemma 7.6 F satisfies (7.3) and by Lemmas 7.9, 7.5 the proof is completed. □

For a metric space (Y,d), let $Z(Y)$ denote the family of all complete nonempty subsets of Y.

Now let us present a generalization of [3, Th. (3.5), p. 164]. The proof is a modification of the proof of Th. 1 [P5] and it resembles the proof of Th. 3.2'' [37, p. 367].

7.11. Theorem. *Let* X *be a paracompact space,* (Y,d) *a metric space,* $Y \in W4(\{F(x): x \in X\} \cup \{\{y\}: y \in F(X)\})$ *for* Q *while* $F: X \to Z(Y)$ *is an lsc mapping with Q-convex values. Then* F *admits a selection.*

Proof. Let $p = p_1$, $q = q_1$ be as in condition (7.4). By considering $H = B(F(\cdot),p)$ in place of F $(H^- = F^-(B(\cdot,p)))$ in the proof of Th. 7.2, we find a selection f_1 for $cQ \circ H$. By taking (7.4) into account we obtain $f_1(x) \in cQ\, B(F(x),p_1) \subset B(cQ\, F(x),q_1) \subset B(F(x),q_1)$, $x \in X$ as the values of F are Q-convex. Let us consider $F_1(x) = F(x) \cap B(f_1(x),q_1)$, $x \in X$. Clearly, $F_1: X \to 2^Y$ is a mapping and $F_1^-(V) = V^-(V) \cap f_1^-(B(V,q_1))$ is an open set if V is open. Hence, F_1 is lsc. Now we consider $H = B(F_1,p_2)$ in order to obtain a selection f_2 for $cQ \circ H$, i.e. $f_2(x) \in cQ\, B(F_1(x),p_2) \subset cQ\, B(Fx),p_2) \subset B(cQ\, F(x),q_2) \subset B(F(x),q_2)$ and, moreover, $f_2(x) \in$

$\in$ cQ $B(F_1(x),p_2) \subset$ cQ $B(f_1(x),q_1+p_2) \subset B($cQ $f_1(x),r_2) = B(f_1(x),r_2)$. In turn, we define $F_2(x) = F(x) \cap B(f_2(x),q_2)$, $x\in X$. By induction we obtain a sequence of maps f_n: $X\rightarrow Y$ and a sequence of lsc mappings F_n: $X\rightarrow 2^Y$ such that

(7.5) $f_n(x) \in$ cQ $B(F_{n-1}(x),p_n) \subset$ cQ $B(F(x),p_n) \subset B(F(x),q_n)$, $x\in X$,

(7.6) $f_n(x) \in$ cQ $B(f_{n-1}(x),q_{n-1}+p_n) \subset B(f_{n-1}(x),r_n)$, $x\in X$,

where p_n, q_n are as in (7.4) and $p = q_{n-1}+p_n$, $q = r_n$ satisfy (7.4). We may require $\sum r_n < \infty$ (i.e. $q_n \rightarrow 0$, see Lemma 6.2.11). Then for each $x \in X$, $k \le n$, $k,n \in N$ we obtain $d(f_k,f_n(x)) \le$ $\le d(f_k(x),f_{k+1}(x))+ \dots +d(f_{n-1}(x),d_n(x)) \le r_{k+1}+ \dots +r_n$ (see (7.6)) which means that the sequence (f_n) of maps satisfies uniformly the Cauchy condition. By (7.5) and the completeness of $F(x)$, $x\in X$ the sequence $(f_n(x))$ converges to an element of $F(x)$. Hence, (f_n) is a sequence of maps which converges uniformly to a selection f for mapping F. □

7.12. Remark. *If* $Y \in W2(\bar{F}(X))$ *and* $\bar{F}(X)$ *is compact, then in view of* Lemma 2.2.22, *in* 7.11 *it suffices to assume* $Y \in W4(\{F(x): x\in X\})$ *in place of the respective assumption* $Y \in W4(\dots)$.

We will show that the convex structure presented in Def. 7.8 is more general than those of Michael [40, Def. 1.1 p. 559] and Curtis [8, Def. 2.1 p. 748]. Both of them assume that (Y,d) is a metric space, for each $n\in N$ M_n is a subset of Y_n and k_n: $M_n \times P^{n-1} \rightarrow Y$ is a mapping such that for $e \in M_n$ (1.1) holds and $k_1(x,I) = x$, $x \in$ $\in M_1$. Hence, we have $(M_1)^n \subset M_n$, $n\in N$. What is more, mappings k_n are such that for each $q > 0$ there exists a $p > 0$ such that for each $y,y' \in M_n$, $n \in N$ if $d(y_i,y_i') < p$, $i = 1, \dots, n$, then $d(k_n(y,t),k_n(y',t)) < q$. From this condition it follows that for $A \subset$ $\subset M_1$ and $Q_n = k_n$ we obtain $Q_n(B(A,p)) \subset B(Q_n(A),q)$ and, clearly, cQ $B(A,p) \subset B($cQ $A,q)$, i.e. (7.4) holds. Naturally, (7.4) does not imply the uniform continuity of Q on $x \in E^n$. What is more, Michael assumes $k_n(e,\cdot)$: $P^{n-1} \rightarrow Y$ to be continuous which together with the uniform continuity on x variable means that k_n: $M_n \times P^{n-1} \rightarrow Y$ is continuous, $n \in N$ (in the paper of Curtis this requirement is a part of definition). Thus, the continuity requirement of Def. 7.8 is less restrictive (see Def. 1.2.7). In Def. 7.8 we assume that each singleton in $\mathcal{U}$ is Q-convex. This requirement in Michael's paper follows from conditions (a) and (c) (Curtis assumes it explicitely). What is more, Michael's definition contains an additional condition (c) concerning t.

Let us present further theorems on selections.

7.13. Theorem (cp. [40, Th. 1.5 p. 559]). *Assume that* X *is a paracompact space and* (Y,d) *a metric space. If* $F: X \to Z(Y)$ *is an lsc mapping,* $Y \in W4(\{F(x): x \in X\} \cup \{\{y\}: y \in F(X)\})$ *for* Q, *all values of* F *are Q-convex, and* g *is a selection for* $F|C$, *where* $C = \bar{C} \subset X$, *then* g *can be extended to a selection* f *for* F.

Proof. Let us consider G as in (7.2). Then, $G(X) \subset F(X)$, and $G: X \to Z(Y)$, all values of G are Q-convex (Lemma 6.2.11) and G is lsc (Lemma 7.4). Now by Th. 7.11 G has a selection. □

From the above and Lemma 7.7 we obtain

7.14. Theorem (cp. [39, Th. 8.1 p. 388]). *Assume that* X *is a paracompact space,* (Y,d) *a metric space,* $F: X \to (Z(Y),D)$ *a map and* $Y \in W4(\{F(x): x \in X\} \cup \{\{y\}: y \in F(X)\})$ *for* Q. *If all values of* F *are Q-convex and* g *is a selection for* $F|C$, *where* $C = \bar{C} \subset X$, *then* g *can be extended to a selection* f *for* F.

The subsequent theorem is a generalization of the Bartle-Graves theorem (cp. [37, p. 369])

7.15. Theorem. *Let* X *be a paracompact space,* (Y,d) *a metric space,* $Y \in W4$ *for* Q *and let* $f: Y \to X$ *be an open mapping such that all values of* f^- *are complete and Q-convex. Then* f *has a right inverse map* g, *i.e. such that* $f \circ g = id_X$.

Proof. For simplicity of notations let us assume that $f(Y) = X$. In view of Theorem 7.11 it suffices to verify that $F = f^-$ is lsc. If $V \subset Y$ is open, then $F^-(V) = f(V)$ is open. □

Let us prove a fixed point theorem related to selections

7.16. Theorem. *If* (X,d) *is a metric space,* $F: X \to Z(X)$ *is lsc,* $\bar{F}(X)$ *is compact,* $X \in W4(\{F(x): x \in X\})$ *and* $X \in W2(\bar{F}(X))$ *for* Q *and all values of* F *are Q-convex, then* F *has a fixed point.*

Proof. By Remark 7.12 and Th. 7.11 F has a selection, which in view of Th. 2.2.20 has a fixed point. □

Now let us consider the case of X finite dimensional. In view of Th. 7.1 and [31, W p. 171] we may assume that Def. 6.2.1 concerns paracompact spaces. If X is a T_1-space, then we obtain a condition equivalent to the classical definition [15, Th. 7.2.4 p. 484].

7.17. Definition (cp. [P6, Def. 4.15]). *Let* (Y,d) *be a metric space. We write* $Y \in Wk3(\mathcal{A})$ *if* $\emptyset \neq \mathcal{A}$ *is the family of nonempty subsets of* Y *and* $Y \in Wk1(\bigcup\mathcal{A})$ (Def. 6.2.2) *for a* Q *such that*

(7.7) *for each* $q > 0$ *there exists a* $p > 0$ *such that for each* $A \in \mathcal{A}$ $Q_k(B(A,p)) \subset B(Q_k(A),q)$.

If $\mathcal{A} = 2^Y \setminus \{\emptyset\}$, *then we write* $X \in Wk3$.

One can define class $LWk3(\mathcal{A})$ by considering $LWk1(\bigcup\mathcal{A})$ (Def. 6.2.4) in place of $Wk1(\bigcup\mathcal{A})$ in the above.

Now we present a finite dimensional version of Th. 7.11.

7.18. Theorem. *Let* X *be a paracompact space,* $\dim X \leq k-1$ *and* (Y,d) *a metric space,* $Y \in Wk3(\{F(x)\colon x \in X\} \cup \{\{y\}\colon y \in F(X)\})$ *for* $F\colon X \longrightarrow Z(Y)$ *being lsc and such that* $Q_k(F(x)) = F(x)$, $x \in X$. *Then* F *has a selection.*

Proof. In the proof of Th. 7.11 we assume that the respective partitions of unity are of order $\leq k$ (see the proof of Th. 7.2). By writing Q_k in place of cQ we define a sequence of maps which converges to a selection f. □

In a similar way one can formulate the finite dimensional version of Th. 7.13.

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S t r e s z c z e n i e

Teoria punktów stałych i pewne inne zastosowania chwastów

Praca ta jest poświęcona badaniu struktury wypukłej zwanej chwastem (ang. weed). Podstawową koncepcją jest zastąpienie kombinacji wypukłych, właściwych przestrzeniom liniowym, przez ciąg odwzorowań dla przestrzeni topologicznych (zob. Def. 1.2.7, str. 18). Rolę zbiorów wypukłych przejmują nadotoczki (ang. overhulls) (Def. 1.2.2, p. 16).
Jesteśmy tu głownie zainteresowani teorią punktów stałych, a otrzymane wyniki uogólniają dobrze znane twierdzenia, nawet w przypadku przestrzeni lokalnie wypukłych. Dwie końcowe części, które są poświęcone chwastom o bardziej skomplikowanej strukturze, dotyczą odpowiednio, retrakcji oraz ciągłych selekcji.