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FIXED POINT THEOREMS FOR UHC MAPPINGS

First let us present the notations applied in this paper. The family of all subsets of a set X is written as 2^X. The notation $F: 2^X \to 2^Y$ means that for each $A \subset X$, $F(A)$ is a subset of Y; for $G: 2^Y \to 2^Z$, $G \circ F: 2^X \to 2^Z$ is defined by $(G \circ F)(A) = G(F(A))$, $A \subset X$. We require $X \cap 2^X = \varnothing$ in order to replace $F(\{x\})$ by $F(x)$, $x \in X$ which is more convenient. If $F: 2^X \to 2^Y$ and $F(A) = \bigcup \{F(x): x \in A\}$, $A \subset X$ then we write $F: X \to 2^Y$; in such a case F is a mapping if $F(x) \neq \varnothing$, $x \in X$. Any uniformity $\mathcal{U}$ for a set X consists of mappings as $U(x) = \{y \in X: (x,y) \in U\}$, $U \in \mathcal{U}$, $x \in X$. If $F: 2^X \to 2^Y$, then $F^-: Y \to 2^X$ is defined as follows: $x \in F^-(y)$ if $y \in F(x)$; in particular, for $F: X \to 2^Y$, $G: Y \to 2^Z$ we have $(G \circ F)^- = F^- \circ G^-$ [1, p. 119]. As regards topological notions, we do not require compact, regular nor locally convex spaces to be T_2.

The two subsequent lemmas are useful tools in proving fixed point theorems.

Lemma 1. Let E be a finite subset of a set Y in a space M. If $F: 2^X \to 2^Y$ is such that $\operatorname{conv} E \subset F^-(E)$ and $F^-(e) \cap \operatorname{conv} E$ is open in $\operatorname{conv} E$, then there exists a set $Z \subset E$ and a point y such that $y \in \operatorname{conv} Z \in \operatorname{conv} F(y)$.

Proof. Assume that $E = \{e_0, ..., e_n\}$ and let P be n-simplex ($P = \{t \in R^{n+1}: 0 \le t_i,\ i = 0, \dots, n \text{ and } \Sigma t_i = 1\}$). Mapping $g: P \to \operatorname{conv} E$ defined by $g(t) = \Sigma t_i e_i$, $t \in P$ is continuous and therefore $M_i = P \setminus g^-(F^-(e_i)) = P \setminus g^-(F^-(e_i) \cap \operatorname{conv} E)$ is closed, $i = 0, \dots, n$. On the other hand we have

$\bigcap M_i = P \setminus \bigcup g^{-}(F^{-}(e_i)) = P \setminus g^{-}(F^{-}(E)) = P \setminus g^{-}(\text{conv } E) = \varnothing$. By referring to the Knaster-Kuratowski-Mazurkiewicz theorem (see e.g. [2, (13.3) p. 102]) we note that the condition $t \in \bigcup \{M_i: t_i \neq 0\}, t \in P$ cannot be satisfied. Thus there exists a $t \in P$, for which we have $t \notin \bigcup \{M_i: t_i \neq 0\} = P \setminus \bigcap \{g^{-}(F^{-}(e_i)): t_i \neq 0\}$ and consequently $t \in g^{-} \bigcap \{F^{-}(e_i): t_i \neq 0\}$, $g(t) \in \bigcap \{F^{-}(e_i): t_i \neq 0\}$. For $y = \Sigma\, t_i e_i = g(t)$ we obtain $y \in F^{-}(e_i), t_i \neq 0$ and $Z = \{e_i \in E: t_i \neq 0\} \subset F(y)$. Now it is clear that $y \in \text{conv } Z \subset \text{conv } F(y)$ holds.

Definition 1. [7, Def. 2.1.1]. We say that $F: 2^Y \to 2^M$ is increasing if for each $A, B \subset Y$ from $A \subset B$ follows $F(A) \subset F(B)$.

It is clear that any $F: Y \to 2^M$ is increasing.

Lemma 2. Let E be a finite subset of a set Y in a linear topological space M. Assume that $W: 2^Y \to 2^M$ is increasing, $\text{conv } E \subset W^{-}(E)$ and let $V: 2^Y \to 2^M$ satisfy

for each $y \in \text{conv } E$ there exists a neighbourhood U_y of y in conv E for which $W(U_y) \subset V(y)$ holds. (1)

Then we have $y \in \text{conv } E \cap \overline{\text{conv}}\ V(y)$ for a $y \in Y$.

Proof. We can treat conv E as a uniform space (conv E, $\mathcal{U}$) with uniformity $\mathcal{U}$ consisting of open symmetric entourages [3, Th. 6 p. 179]. Then for $U \in \mathcal{U}, e \in E$ the set $(U \circ W^{-})(e)$ is open and we have $(U \circ W^{-})(e) = (U^{-} \circ W^{-})(e) \subset (W \circ U)^{-}(e)$, W being increasing. Thus we note that $\text{conv } E \subset W^{-}(E) \in (U \cdot W^{-})(E) \subset (W \cdot U)^{-}(E)$. In view of Lemma 1 there exists a point $y \in \text{conv}(W \circ U)(y) \cap \text{conv} E$. Let us adopt $A(U) = \{y \in \text{conv} E: y \in \text{conv } (W \circ U)(y)\}, U \in \mathcal{U}$. Clearly, $U_1 \subset U_2$ means that $A(U_1) \subset A(U_2)$ (W is increasing) and therefore the family $\{A(U): U \in \mathcal{U}\}$ has the finite intersection property ($\mathcal{U}$ is a uniformity). Now it is clear that $A = \bigcap \{\overline{A(U)} \cap \text{conv} E: U \in \mathcal{U}\}$ is nonempty, conv E being compact. If y is a point of A then $U(y) \cap \text{conv}(W \circ U)(y) \neq \varnothing, U \in \mathcal{U}$ and in view of (1) $U(y) \cap \text{conv} V(y) \neq \varnothing, U \in \mathcal{U}$. Hence follows $y \in \overline{\text{conv}} V(y)$ [3, Th. 7 p. 179].

Definition 2. (cp. [5, p. 1435]). Let Y be a topological space and M a linear topological space. We say that a mapping $F: Y \to 2^M$ is **uhc** if for each continuous linear functional x on M (i.e. $x \in M^*$) sup $rx \circ F: Y \to \overline{R}$ is upper semicontinuous, where rx is the real part of x.

As regards the convexity of a set it has no meaning whether the space under consideration is real or complex. On the other hand every linear functional x on a complex space M is defined by its real part rx which is a linear functional on the real restriction of M [4, 1.4 p. 9]; and what is more, the continuity of x is equivalent to the continuity of rx [4, 5.4 p. 37]. Therefore, if we restrict our considerations to the real spaces (the notations are simpler) then the results do not lose their generality.

Lemma 3 enables us to prove the following in a simple way

Theorem 1. Let Y be a compact convex set in a locally convex space M and let $F\colon Y \to 2^M$ be **uhc** with $F(y) \cap Y \neq \emptyset$, $y \in Y$. Then $\overline{\mathrm{conv}}\, F$ has a fixed point.

Proof. We may assume that M is real. Let $x_1, \dots, x_m \in M^*$, $q > 0$ be fixed and let $V_q(B) = \{z \in Y\colon x_i(z) < q + \sup x_i(B),\ i = 1, \dots, m\}$, $B \subset M$. We consider the increasing mappings $W = V_p \circ F$ and $V = V_{2p} \circ F$. From the compactness of Y it follows that $W^-(E) = Y$ for a finite set $E \subset Y$. On the other hand (1) holds for W and V, F being **uhc.** In view of Lemma 2 the family $\{A(x, q)\colon x \in M^*, q > 0\}$, where $A(x, q) = \{y \in Y\colon x(y) < q + \sup x(F(y))\}$ has the finite intersection property. The compactness of Y guarantees that $A = \bigcap \{Y \cap \overline{A(x, q)}\colon x \in M^*, q > 0\}$ is nonempty. For each $y \in A$ we have $y \in \overline{\mathrm{conv}} F(y)$ (see [4, Corol. 14.4 p. 119]).

The above theorem is a simplified version of Th. 2, nevertheless it is worth of being noted as it extends a classical result of Fan and the proof is short.

Definition 2 is general, but it is convenient to consider weakly upper semicontinuous (**wusc**) mappings which we introduce below

Definition 3. [7, Def. 2.1.3 p. 26]. An increasing $F\colon 2^Y \to 2^M$ is **usc** if Y, M are topological spaces and for each $y \in Y$ and any neighbourhood W of $F(y)$ there exists a neighbourhood U of y such that $F(U) \subset W$.

The **usc** mappings in the above sense have nice properties (see [7, 2.1]).

Definition 4. Let Y be a topological space and Z a set in a linear topological space M. We say that an increasing $F\colon 2^Y \to 2^Z$ is **wusc** (weakly upper semicontinuous) if it is **usc** for Z equipped with the topology induced by the weak topology of M.

It is clear that every **wusc** mapping $F\colon Y \to 2^Z$ is **uhc** as $\{z \in Z\colon rx(z) \sup rx(F(y)) + p\}$ $(p > 0)$ is a neighbourhood of $F(y)$ in the weak topology in Z and, consequently, we have $\sup rx(F(U_y)) < \sup rx(F(y) + p$ for a neighbourhood U_y of y.

Lemma 3. Let M be a linear topological space and let $F\colon Y \to 2^M$ be an **uhc** mapping. If $Z \subset M$ is such that all values of $Z \cap F$ are convex and weakly compact, then $Z \cap F\colon Y \to 2^Z$ is **wusc.**

Proof. For simplicity let us assume that M is real. For $A = F(y)$ any neighbourhood of the set $Z \cap A$ contains a uniform neighbourhood $W(Z \cap A)$ in Z [3, Th. 33 p. 199] (both of them in the weak topology). We may assume that $W(z) = \{s \in Z\colon |x_i(s - z)| < q,\ i = 1, \dots, m\}$, where $x_1, \dots, x_m \in M^*$ are arbitrary (the case of $M^* = \{0\}$ is trivial). We will show that there exists a $n \in \mathcal{N}$ and $t_1, \dots, t_n \in M^*$ such that for

$$V = \{s \in M: \inf t_j(A) - p < t_j(s) < \sup t_j(A) + p, j = 1, \dots, n\}$$

we have $Z \cap V \subset W(A)$. For $Q = \{z \in M: x_i(z) = 0, i = 1, \dots, m\}$ the space M/Q is at most m-dimensional [4, B (a) p. 11], locally convex (weak topology) and T_2, and in view of [4, 6.1 p. 44] it can be identified with R^k (Euclidean space). For simplicity of notations we identify M/Q with M (for the topology induced by $x_1, \dots, x_m$). Clearly, $2\overline{W}(Z \cap A) = \overline{(W \circ W)}(Z \cap A)$ is compact as being closed and bounded. Let T consist of linear combinations of $x_1, \dots, x_m$ and let $V_{t,p} = \{z \in M: \inf t(A) - p < t(z) < \sup t(A) + p\}$, $t \in T$, $p > 0$. Suppose that the family of sets $\{\overline{V_{t,p}} \cap 2\overline{W}(Z \cap A) \setminus W(Z \cap A): p > 0, t \in T\}$ has the finite intersection property. Then the intersection of this family consisted of closed compact sets must be nonempty. If $z \in \bigcap \{\overline{V_{t,p}} \cap \overline{2W}(Z \cap A) \setminus W(Z \cap A): p > 0, t \in T\}$, then we have $z \notin W(Z \cap A)$ and z cannot be strongly separated from $Z \cap A$ by a linear functional. This fact contradicts [5, Corol. 14.4 p. 119] and consequently there exist $t_1, \dots, t_n \in T$ and a $p > 0$ such that for $V = V_{t_j p}$ we have $V \cap Z \subset W(A)$.

The upper hemicontinuity of F means that $\inf x \circ F = -\sup(-x) \circ F$ is lower semicontinuous and hence

$$U = \{z \in Y: [\inf t_j(F(z)), \sup t_j(F(z))] \subset [\inf t_j(A) - p, \sup t_j(A) + p], j = 1, \dots, n\}$$

is a neighbourhood of y and, in view of the way V was defined we have $U \subset \{z \in Y: F(z) \subset W(A)\}$. Thus F is **wusc**.

Clearly, if M is a linear topological space then any $F: Y \to 2^M$ which is **usc** is **uhc**. Therefore the theorem to follow is fairly general. Its proof is based on the proof of [7, Th. 2.2.11 p. 33].

Theorem 2. Let Y be a convex set in a locally convex space and $F: Y \to 2^M$ a **uhc** mapping with closed convex values. Assume that $Y \cap F(Y)$ is relatively compact in Y and $Y \cap F(y) \neq 0$, $y \in Y$. Then F has a fixed point.

Proof. In view of Lemma 3 $Y \cap F$ is **wusc**. The set Y with the induced topology and the weak topology can be treated as two uniform spaces with uniformities $\mathcal{U}$, $\mathcal{V}$ respectively. In addition, we may assume that $U(A)$, $V(A)$ are open and $V(A)$ convex for $U \in \mathcal{U}$, $V \in \mathcal{V}$, $A = \operatorname{conv} A \subset Y$. For each $W_1 \in \mathcal{V}$ there exists a finite set $E \subset Y \cap F(Y)$ such that $(W_1 \circ F)^-(E) = (F^- \circ W_1^-)(E) = Y$ as $Y \cap F(Y)$ is precompact in the weak topology and $Y \cap F(y) \neq \varnothing$, $y \in Y$. On the other hand, if $V_1, W_1 \in \mathcal{V}$ are such that $2W_1 \subset V_1$ then for $W = W_1 \circ F$, $V = V_1 \circ F$ (1) holds F being **wusc**. Now by Lemma 2 the set $A(V_1) = \{y \in Y: y \in (V_1 \circ F)(y)\}$ is nonempty. The family $\{A(V): V \in \mathcal{V}\}$ has the finite intersection property as $V_1 \subset V_2$, $V_1, V_2 \in \mathcal{V}$ imply $A(V_1) \subset A(V_2)$ and for $V_1, \dots, V_m \in \mathcal{V}$ there exists a $V \in \mathcal{V}$ such that $V \subset V_1 \cap \dots \cap V_n$. Clearly, for $C = Y \cap \overline{F}(Y)$ there exists a $y \in C \cap \overline{V(A(V))}$, $V \in \mathcal{V}$ (closures in $(Y, \mathcal{U})$) as C is compact. Hence it follows that

$\varnothing \neq U(y) \cap (V \circ F \circ U)(y)$, $U \in \mathcal{U}$, $V \in \mathcal{V}$ and consequently, $U(y) \cap (V \circ F)(y) \neq \varnothing$, $U \in \mathcal{U}$, $V \in \mathcal{V}$ F being **wusc**. Therefore we have $V(y) \cap (V \circ F)(y) \neq \varnothing$, $V \in \mathcal{V}$ which means that y, $F(y)$ can not be strongly separated by any $x \in M^*$. Now, in view of [4, Corol. 14.4 p. 119] we obtain $y \in F(y)$.

The above theorem can be extended to the form of [7, Th. 2.2.13] with **uhc** in place of **usc** (see the respective proof).

The theorem to follow is a natural extension of Th. 2 and the method of proving it is adopted from [7, 3.1].

Theorem 3. Let $Y_1, \dots, Y_m$ be convex sets in locally convex spaces $M_1, \dots, M_m$, respectively, and $F_1: Y_1 \to 2^{M_2}, \dots, F_{m-1}: Y_{m-1} \to 2^{M_m}, F_m: Y_m \to 2^{M_1}$ **uhc** mappings with closed convex values and such that $Y_1 \cap F_m(Y_m), Y_2 \cap F_1(Y_1), \dots, Y_m \cap F_{m-1}(Y_{m-1})$ are relatively compact and $Y_1 \cap F_m(y_m), Y_2 \cap F_1(y_1), \dots, Y_m \cap F_{m-1}(y_{m-1})$ are non void $y_1 \in Y_1, \dots, y_m \in Y_m$. Then $F_1 \circ \dots \circ F_m$ has a fixed point.

Proof. The weak topologies of a complex space and of its real restriction coincide [4, 5.4 p. 37] and therefore we may require all spaces under consideration to be real. Let us adopt $M = M_1 \times \dots \times M_m$, $Y = Y_1 \times \dots \times Y_m$ and let $F: Y \to 2^M$ be defined as follows $F(y_1, \dots, y_m) = F_m(y_m) \times F_1(y_1) \times \dots \times F_{m-1}(y_{m-1})$. It can be easily verified that $F_1 \circ \dots \circ F_m$ has a fixed point iff F as above has a fixed point. In view of [4, 17.13 p. 160] the weak topology for a product of linear topological spaces is the product of the weak topologies and by [3, Th. 12 p. 142] any neighbourhood of $Y \cap F(y)$ in the weak topology contains the product of neighbourhoods of the projections of $Y \cap F(y)$. Therefore from the assumption of $F_i: Y_i \to 2^{M_j}$ (the respective i,j) being **uhc** and Lemma 3 it follows that F is **uhc** and (see Theorem 2) has a fixed point.

An almost immediate consequence of Theorem 2 is the following

Theorem 4. Let Y be a closed convex set in a locally convex space M and $F: Y \to 2^M$ an **uhc** mapping with closed compact convex values and such that $Y \cap F(Y)$ is relatively compact. If the following is satisfied

$$\inf rx(Y) \leq \sup rx(F(y)), x \in M^*, y \in Y, \tag{2}$$

then F has a fixed point.

Proof. The only fact to verify (cp. Th. 2) is that $Y \cap F(y) \neq \varnothing$, $y \in F$. This dependence follows from (2) and [5, Corol. 14.4 p. 119].

In Theorem 5 we apply a little modified (see [3, B (b) p. 161] notion of c-compactness.

Definition 5. (cp. [6, Def. 3 p. 154]). Let X be a linear topological space. A nonempty set $K \subset X$ is c-compact if for each finite set $Z \subset X$, $\overline{\mathrm{conv}}(K \cup Z)$ is compact.

The theorem we present below extends a little [7, Th. 4.3.2 p. 62] and its proof is more elegant than the original one. It should be noted that a set in a locally convex space is bounded if it is weakly bounded [4, 17.5 p. 155].

Theorem 5. Let Y be a bounded convex set in a locally convex space M and $F: Y \to 2^M$ a mapping with closed convex values which is **uhc** on compacta. If (2) and the following are satisfied

$$B = \{y \in F: rx(y) \le \sup rx(F(y)) + q, x \in K\} \text{ is compact for a set } K \subset X = M^* \tag{3}$$
which is c-compact in the finite topology, and a $q > 0$,

then F has a fixed point in B.

Proof. We assume that M is real. Let us adopt $T(x) = \{y \in Y: x(y) - p < \inf x(Y)\}$, $x \in X$ and $J(y) = \{x \in X: \sup x(F(y)) < x(y) - p\}$, $y \in Y$ for a $p > 0, p < q$. Clearly, the values of $T: X \to 2^Y$ are nonempty. Suppose J is a mapping too. Now let us verify the remaining assumptions of [7, Th. 3.2.1 p. 43]. Clearly, all values of T are convex. On the other hand if $x, z \in J(y)$ then $(tx + [1-t]z)(u) \le tx(u) + (1-t)\sup z(F(y))$ holds for each $u \in F(y)$ and, consequently, we obtain $\sup (tx + [1-t]z)(F(y)) \le t \sup x(F(y)) + (1-t)\sup z(F(y))$. Hence we obtain $\sup (tx + [1-t]z)(F(y))\, tx(y) + (1-t)z(y) - p$ which means that all values of J are convex. Now we are going to show that all values of T^-, J^- are compactly open. As it is seen $J^-(x) = \{y \in Y: \sup x(F(y)) < x(y) - p\}$ and the intersection of $J^-(x)$ with any compact set C in Y is open in C as F is **uhc** on compacta. As regards the finite topology in X, it is such that every simplex in X is topologically isomorphic with the standard simplex in Euclidean space (of the respective dimension). Mapping $x \to x(y) - \inf x(Y)$ is convex for fixed y [1, Th. 5 p. 199] and therefore it is continuous in each simplex [1, Th. 7 p. 201]. Thus $T^-(y) = \{x \in X: x(y) - \inf x(Y) < p\}$ is an open set. As regards condition (iii) of [7, Th. 3.2.1], we have

$$Y \setminus J^-(K) = \{y \in Y: x(y) - p \le \sup x(F(y)), x \in K\} \subset \{y \in Y: x(y) \le \sup x(F(y)) + q, x \in K\}$$

and in view of (3) $Y \setminus J^-(K)$ is compact. Now by [7, Th. 3.2.1] there exists a point $(x, y) \in X \times Y$ such that $(x, y) \in J(y) \times T(x)$, i.e. $\sup x(F(y)) < x(y) - p < \inf x(Y)$. This contradiction (see (2)) shows that J cannot be a mapping and consequently there exists a $y \in Y$ such that $x(y) - p \le \sup x(F(y))$ holds for all $x \in X$. By considering nx in place of x we obtain $x(y) - p/n \le \sup x(F(y))$ for all $n \in N$ which means that $x(y) \le \sup x(F(y))$, $x \in X$ and consequently (see [4, Corol. 14.4 p. 119]) we have $y \in F(y)$.

Remark 1. In the proofs of all theorems in this paper we applied [4, Corol. 14.4 p. 119]. Instead of M being locally convex it suffices to assume in our theorems that $y \in F(y)$ holds if none of $x \in M^*$ separates strongly y and $F(y)$.

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