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On the KKM Theorem

by

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Summary. The paper is devoted to a more thorough investigation of the theorem of Knaster-Kuratowski-Mazurkiewicz (Knaster and Kuratowski 1929, Satz p. 134) concerning the nonempty intersection property of a cover of a simplex. A simple modification of the KKM theorem (Theorem 2) leads in consequence to an extension of a fixed point theorem of Himmelberg (Theorem 7). On the other hand, a more "geometrical" interpretation of KKM theorem enables us to obtain some theorems on nonempty intersection $(10, \ldots, 14)$. In conclusion we give simple proofs of two fixed point theorems (16, 17) for mappings satisfying a "boundary" condition.

Let us adopt $I = \langle 0, 1\rangle \subset R$ and $P^n = \{(t_0, \ldots, t_n) \in I^{n+1} : \sum t_i = 1\}$ (standard n-simplex in R^{n+1}).

The "open" case of the theorem to follow is a simplified version of ([7], Thm 1 p. 573) and the "closed" one was given in [6].

THEOREM 1. *Let $\mathcal{A} = \{A_0, \ldots, A_n\}$ be a closed or open cover of P^n. If for each $t \in P^n$ we have $t \in \bigcup\{A_i : t_i \neq 0\}$ then $\bigcap \mathcal{A}$ is nonempty.*

This theorem leads to fixed point theorems and related results. Let us present a simple proof of ([8], Thm 2.2.12 p. 34) as the original one was complicated. The final reasoning is based on Thm 4 and Lemma 5.

First let us note that if Y is a finite dimensional linear topological space then any closed (open) set $A \subset Y$ is closed (open) in Y equipped with the natural topology for which Y is homeomorphic with R^n or C^n.

This comment enables us to refolmulate Theorem 1 as follows.

THEOREM 2. *Let $\mathcal{A} = \{A_0, \ldots, A_n\}$ be a closed or open cover of a n-simplex $\Delta = \operatorname{conv}\{v_0, \ldots, v_n\}$ in a linear topological space Y. If for each*

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$t = (t_0, \ldots, t_n) \in \operatorname{Fr} P^n$ we have $\sum t_i v_i \in \bigcup\{A_i : t_i > 0\}$ then $\bigcap \mathcal{A}$ is nonempty.

The methods based on the Knaster-Kuratowski-Mazurkiewicz (KKM) theorem were investigated by Granas and his school (see e.g. [2] p. 72–78).

Let us recall some notions concerning multivalues mappings. For arbitrary sets X, Y the notation $F : X \to 2^Y$ means that $F(x) \subset Y$, $x \in X$ and $F(A) = \bigcup\{F(x) : x \in A\}$, $A \subset X$; F is a mapping if $F(x) \neq \emptyset$, $x \in X \neq \emptyset$. For $F : X \to 2^Y$ and any $B \subset Y$ we write $F^-(B) = \{x \in X : F(x) \cap B \neq \emptyset\}$; it can be easily checked that $F^- : Y \to 2^X$. If Y is a topological space then $F : X \to 2^Y$ is closed (open) if $F(x)$ is closed (open) for each $x \in X$. If $F : X \to 2^Y$ and B is a subset of Y, then $B \cap F$ is defined by $(B \cap F)(x) = B \cap F(x)$, $x \in X$.

THEOREM 3. *Let Z be a finite set in a linear topological space Y and $H : Z \to 2^Y$ such that $\operatorname{conv} K \subset H(K)$, $K \subset Z$. If $(\operatorname{conv} Z) \cap H$ is closed or open (in $\operatorname{conv} Z$) then $\bigcap\{H(z) : z \in Z\} \neq \emptyset$.*

Proof. Assume $Z = \{z_0, \ldots, z_n\}$ and let $h : P^n \to \operatorname{conv} Z$ be defined by $h(t) = \sum t_i z_i$. Let us consider $A_i = (h^{-1} \circ H)(z_i)$, $i = 0, \ldots, n$ and $\mathcal{A} = \{A_0, \ldots, A_n\}$. Clearly all members of $\mathcal{A}$ are closed (open) in $\operatorname{conv} Z$ and the other assumptions of Theorem 2 are satisfied. Therefore $\bigcap \mathcal{A} \neq \emptyset$ and we have $h(\bigcap \mathcal{A}) \subset \bigcap h(\mathcal{A}) = \bigcap\{H(z) : z \in Z\}$. □

The result to follow is a modification of ([7], Thm 3 p. 574).

THEOREM 4. *Let Z be a finite set in a linear topological space Y and $G : Y \to 2^Y$ such that $\operatorname{conv} Z \subset G^-(Z)$ and $(\operatorname{conv} Z) \cap G^-$ is closed (open) in $\operatorname{conv} Z$. Then there exist a set $K \subset Z$ and a point $x \in \operatorname{conv} K$ such that $K \subset G(x)$; consequently $x \in \operatorname{conv} K \subset \operatorname{conv} G(x)$.*

Proof. Let us consider $H : Z \to 2^Y$, $H(z) = (\operatorname{conv} Z) \setminus G^-(z)$, $z \in Z$. Suppose $\operatorname{conv} K \subset H(K)$, $K \subset Z$. In view of Theorem 3 we have $\emptyset \neq \bigcap\{H(z) : z \in Z\} = (\operatorname{conv} Z) \setminus G^-(Z) = \emptyset$. Thus there exist a set $K \subset Z$ and a point $x \in K$ such that $x \in \operatorname{conv} K$ and $x \notin H(K)$. It means $x \in G^-(z)$, $z \in K$ and $K \subset G(x)$. □

Each uniform space is regular ([4], p. 180) and therefore the lemma to follow is a consequence of ([4], Thm 27 p. 155, Corol. 30 p. 198). Nevertheless, the direct proof is short and therefore it is worth of being presented.

LEMMA 5. *Let $\mathcal{B}$ be an open cover of a compact set Y in a uniform space $(X, \mathcal{V})$. Then there exists a symmetric $V \in \mathcal{V}$ such that for each $y \in Y$ there is a $B \in \mathcal{B}$ with $V(y) \subset B$.*

P r o o f . For any point $y \in Y$ let a symmetric entourage $W_y \in \mathcal{V}$ be such that $2W_y(y) := (W_y \circ W_y)(y) \subset B$ for a $B \in \mathcal{B}$. There exists a set $\{y_1, \dots, y_n\} \subset Y$ such that $\bigcup\{W_{y_i}(y_i) : i = 1, \dots, n\}$ covers Y. Let us adopt $V = \bigcap W_{y_i}$. For any $y \in Y$ there is a point y_i with $y \in W_{y_i}(y_i)$ which implies $V(y) \subset 2W_{y_i}(y_i)$ and consequently $V(y) \subset B$ holds for a $B \in \mathcal{B}$. □

Let us recall that for topological spaces $X, Y, F : X \to 2^Y$ is usc (upper semicontinuous) if for each closed set $B \subset Y$, $F^-(B)$ is closed; $F : X \to 2^Y$ is compact (relatively compact) if it is usc and $\overline{F(X)}$ is compact ($F(X)$ is contained in a compact set). If $F(X)$ is uniformizable (in particular, if Y is a linear topological space) then each relatively compact $F : X \to 2^Y$ is compact ([4], p. 180, B(b) p. 161).

The fact to follow extends ([7], Lemma p. 576). It will enable us to prove Theorem 7 in an elegant way.

LEMMA 6. *Let $(X, \mathcal{V})$ be a uniform space and $F : X \to 2^X$ compact with closed values. If F has no fixed point then there exists a closed and symmetric $V \in \mathcal{V}$ such that $V \circ F \circ V$ has no fixed point.*

P r o o f . For each $y \in F(X)$ there exists a closed entourage $W_y \in \mathcal{V}$ such that $y \notin (F^- \circ W_y)(y)$. In consequence, there exists a $V_y \in \mathcal{V}$, $(V_y \subset W_y)$ for which $V_y(y) \cap (F^- \circ V_y)(y) = \emptyset$ holds as $(X, \mathcal{V})$ is regular. The family $\{V_y(y) : y \in \overline{F(X)}\}$ covers $Y = \overline{F(X)}$ (the space is regular) and for V as in Lemma 5 we have $V(y) \cap (F^- \circ V)(y) \subset V_{z(y)}(z(y)) \cap (F^- \circ V_{z(y)})(z(y)) = \emptyset$, $y \in Y$. Assume $(2V)(y) \cap (F^- \circ (2V))(y) = \emptyset$, $y \in Y$ holds. Then for $x \in X \backslash (V \circ F)(X)$ we have $x \notin (V \circ F \circ V)(x) \subset (V \circ F)(X)$. If $x \in (V \circ F)(X)$ then $V(x) \subset (2V)(y)$ for a $y \in F(X)$ holds and hence $V(x) \cap (F \circ V)(x) \subset (2V)(y) \cap (F \circ 2V)(y) = \emptyset$. □

Now we are ready to give a simple direct proof of "locally convex" version of ([8], Thm 2.2.12 p. 34), extending a theorem of Himmelberg [3].

THEOREM 7. *Let X be a convex set in a locally convex space Y (not necessarily Hausdorff) and $F : X \to 2^X$ a compact mapping with closed convex values. Then F has a fixed point.*

P r o o f . Suppose F has no fixed point. We can treat X as being a uniform space $(X, \mathcal{V})$. In view of Lemma 6 $V \circ F$ has no fixed point for a closed, symmetric $V \in \mathcal{V}$. On the other hand, there exists a finite set $Z \subset F(X)$ such that $F(X) \subset V(Z)$. Let us consider $G = V \circ F$. In view of Theorem 4, G has a fixed point (we may require the values of $V \circ F$ to be convex, Y being locally convex) which is a contradiction. □

In the sequel we derive a fixed point theorem with "sector" condition. The proof is based in a more direct way on Theorem 2.

Proposition 8. *Let $v_0, \ldots, v_n$ be vectors in R^n. Then the following conditions are equivalent*

(1) *$v_1, \ldots, v_n$ are linearly independent and v_0 is their linear combination with negative coefficients*

(2) *for each $x \in R^n$ there exist $t_0, \ldots, t_n \geqslant 0$ such that $x = \sum t_j v_j$, and a t_i equals to zero. This representation is unique*

(3) *for each $x \in R^n$, $x \neq 0$ there exists an index i such that $(v_i, x) > 0$ ($(\cdot,\cdot)$ is inner product).*

Proof. Let us assume (1). There exists a linear homeomorphism $h: R^n \to R^n$ such that $w_1 = h(v_1) = (1, 0, \ldots, 0)$, $w_2 = h(v_2) = (0, 1, 0, \ldots, 0)$, $\ldots$, $w_n = h(v_n) = (0, \ldots, 0, 1)$ and then for $v_0 = \sum t_j v_j$, $t_1, \ldots, t_n < 0$ we have $w_0 = h(v_0) = (t_1, \ldots, t_n)$. Now it is clear that (1) implies (2). Let us show that from (2) follows (3). If $0 \neq x = \sum t_j v_j$ and $t_j \geqslant 0$, $j = 0, \ldots, n$ then we have $\sum t_j (v_j, x) = (x, x) > 0$ and consequently $(v_i, x) > 0$ for an index i.

Assume (3) is satisfied. Suppose that $v_1, \ldots, v_n$ are dependent. Then there exists an $x \in R^n$ such that $(v_i, x) = 0$, $i = 1, \ldots, n$ and therefore (v_0, x) or $(v_0, -x)$ is not positive which contradicts (3). Now let $v_0 = \sum\{t_j v_j : j = 1, \ldots, n\}$ and $t_1 \geqslant 0$. Then $v_2, \ldots, v_n$ are contained in a subspace $V = \{v \in R^n : (v, x) = 0\}$ and v_0, v_1 are on the same side of V. Therefore we have $(v_0, x) \leqslant 0$ and $(v_1, x) \leqslant 0$ or $(v_0, -x) \leqslant 0$ and $(v_1, -x) \leqslant 0$, and clearly $(v_2, x) = 0, \ldots, (v_n, x) = 0$. This contradicts (3) and therefore $t_1, \ldots, t_n$ must be negative. □

It can be noted that (1) is equivalent to

(4) each n vectors among $v_0, \ldots, v_n$ are linearly independent and the remaining one is their combination with negative coefficients.

Definition 9. Let Y be a linear n-dimensional space. If $v_0, \ldots, v_n \in Y$ are such that (2) is satisfied then we say that $(v_0, \ldots, v_n)$ is a sector representation of Y; for any $x = \sum t_i vi$ with $t_i \geqslant 0$, $i = 0, \ldots, n$ and a $t_i = 0$, x is sector combination of $v_0, \ldots, v_n$ while $t_0, \ldots, t_n$ are sector coefficients of x (for $(v_0, \ldots, v_n)$). □

The initial part of the proof of Proposition 8 shows that for $v_0, \ldots, v_n$ as in Definition 9 $\operatorname{conv}\{v_0, \ldots, v_n\}$ is a n-simplex in Y. Therefore Theorem 2 implies the following:

Theorem 10. *Let Y be a n-dimensional linear topological space, $(v_0, \ldots, v_n)$ — a sector representation of Y and $A = \operatorname{conv}\{v_0, \ldots, v_n\}$. Assume that $\mathcal{A} = \{A_0, \ldots, A_n\}$ is a closed or open cover of A (in the induced topology) satisfying*

(5) *for each $x \in \operatorname{Fr} A$ in the Euclidean topology of Y if $t_0, \ldots, t_n$ are sector coefficients of x then $x \in \bigcup\{A_i : t_i > 0\}$.*
Then $\bigcap \mathcal{A}$ is nonempty.

The previous theorem can be presented in the following equivalent form:

THEOREM 11. *Let A be a bounded closed convex set in a n-dimensional separated linear topological space Y and let $0 \in \operatorname{Int} A$. Assume that $\mathcal{A} = \{A_0, \ldots, A_n\}$ is a closed or open cover of A (in the induced topology). If (5) is satisfied then $\bigcap \mathcal{A}$ is nonempty.*

Proof. Let $k_0, \ldots, k_n > 0$ be such that $A \subset \triangle = \operatorname{conv}\{k_0 v_0, \ldots, k_n v_n\}$ holds. There exists a homeomorphism $h : A \to \triangle$ such that $h(x) = g(x)x$, $x \in A$, where $g : A \to (0, \infty)$ is continuous. Clearly (5) holds for $\triangle$, $h(\mathcal{A})$ in place of A, $\mathcal{A}$. □

Remark 12. By using $-v_0, \ldots, -v_n$ in (5) we obtain
(6) for each $x \in \operatorname{Fr} A$ in the Euclidean topology of Y if $t_0, \ldots, t_n$ are sector coefficients of $-x$ for $(v_0, \ldots, v_n)$ then $x \in \bigcup\{A_i : t_i > 0\}$.

As a consequence of Theorem 11 and Remark 12 we obtain the following version of ([1], Lemma 3 p. 178):

THEOREM 13. *Let A be a bounded closed convex set in a n-dimensional separated linear topological space Y and $0 \in \operatorname{Int} A$. Let $(v_0, \ldots, v_n)$ be a sector representation of Y. Assume that $\mathcal{A} = \{A_0, \ldots, A_n\}$ is a closed or open cover of A such that A_i contains all points of $\operatorname{Fr} A$ for which $t_i = 0$ (in the sector combination), $i = 0, \ldots, n$. Then $\bigcap \mathcal{A}$ is nonempty.*

Proof. If $-x = \sum t_j v_j$ (sector combination) for a $x \in \operatorname{Fr} A$ then there exists an index i such that $x = \sum s_j v_j$ (sector combination) with $s_i = 0$ and $t_i > 0$ (on the contrary (1) would not hold). Now it is clear that (6) is satisfied and we apply Theorem 11. □

The proofs of Theorems 11 and 13 suggest that one may assume A to be a retract of a special kind of a simplex.

Theorem 11 has an obvious geometrical interpretation, nevertheless condition (5) is not too elegant. The theorem to follow is weaker than it, but has a more convenient form.

THEOREM 14. *Let A be a bounded closed convex set in R^n, $0 \in \operatorname{Int} A$. Assume that $\mathcal{A} = \{A_0, \ldots, A_n\}$ is a closed or open cover of A and $\varphi_0, \ldots, \varphi_n$ are linear functionals on R^n such that*
(7) *$\varphi_i(x) > 0$ for a i, and x belongs to all A_i with $\varphi_i(x) > 0$, $x \in \operatorname{Fr} A$ holds. Then $\bigcap \mathcal{A}$ is nonempty.*

P r o o f . There exist vectors $v_0, \ldots, v_n$ such that $\varphi_i(x) = (v_i, x)$, $i = 0, \ldots, n$, $x \in R^n$. From (7) and Proposition 8 it follows that $(v_0, \ldots, v_n)$ is a sector representation of R^n (A is bounded). If $x \in \operatorname{Fr} A$ and $x = \sum t_i v_i$ (sector combination) then $\sum t_i(v_i, x) = (x, x) > 0$ means that $(v_i, x) > 0$ for a $t_i > 0$. In view of (7) $x \in \bigcup\{A_i : t_i > 0\}$ and consequently we obtain (5). □

Clearly we may assume in (7) (see (6)) that x belongs to all A_i with $\varphi_i(x) < 0$.

In the theorem to follow we require more as regards the family of functionals. The respective boundary condition becomes more elegant.

THEOREM 15. *Let A be a bounded closed convex set in R^n, $0 \in \operatorname{Int} A$. Assume that $\mathcal{A} = \{A_0, \ldots, A_n\}$ is a closed or open cover of A, and $v_0, \ldots, v_n \in R^n$ are such that $(v_i, v_j) \leqslant 0$, $i \neq j$, $i, j = 0, \ldots, n$ and*

$$x \in \bigcup\{A_i : (v_i, x) > 0\}, \quad x \in \operatorname{Fr} A \tag{8}$$

holds. Then $\bigcap \mathcal{A}$ is nonempty.

P r o o f . We will show that under the assumption $(v_i, v_j) \leqslant 0$, $i \neq j$ condition (8) implies (5). If $t_0, \ldots, t_n$ are sector coefficients of a point $x \in \operatorname{Fr} A$ and $t_i = 0$ then we have $(v_i, x) = \sum t_j(v_i, v_j) \leqslant 0$ as for $t_j > 0$, $i \neq j$ holds. Thus $x \in \bigcup\{A_i : t_i > 0\}$ and (5) is satisfied. □

It is seen that Theorem 15 is weaker than Theorem 11. We may assume in (8) that $x \in \bigcup\{A_i : (v_i, x) < 0\}$ (cf. (6)). If L is a subspace of a Hilbert space H, then $P_L : H \to L$ will mean the projection on the nearest point.

Now we apply Theorem 11 to prove the following:

THEOREM 16. *Let X be a bounded closed convex set in a Hilbert space H and let $f : X \to H$ be a compact map. If for each finite dimensional subspace L of H there exists a point $q \in X$, $q \in \operatorname{Int} X_L$ ($X_L = (q + L) \cap X$) in the induced topology and a sector representation $(v_0, \ldots, v_n)$ of L such that*

(9) *for each $x \in \operatorname{Fr} X_L$ such that $v = P_{q+L}(f(x)) - x \neq 0$, if $t_0, \ldots, t_n$ are sector coefficients of $x - q$, and $s_0, \ldots, s_n$ are sector coefficients of v then we have $s_i t_i > 0$ for an index i holds, then f has a fixed point.*

P r o o f . Suppose f has no fixed point. Then there exists an $r > 0$ such that $K(x, 2r) \cap K(f(x), 2r) = \emptyset$, $x \in X$ (Lemma 6). There exists a finite set $Z \subset f(X)$ such that $f(X) \subset K(Z, r)$ ($\overline{f(X)}$ is compact). Let L be the span of Z and $x \in X_L$ as to satisfy (9). For $w = x - q$ and $g(w) = P_{q+L}(f(w+q)) - q$ we have $g(w) - w = P_{q+L}(f(w + q)) - q - w = P_{q+L}(f(x)) - x = \sum s_j v_j$ and $w = x - q = \sum t_j v_j$. Thus (9) holds for g in place of f and 0 in place of q. For simplicity, let us assume $q = 0$ and $f = g$. Let $A_i = \{x \in X_L : P_L(f(x)) - x = \sum s_j v_j$ and $s_j > 0\}$ ($s_0, \ldots, s_n$ are sector coefficients of $P_L(f(x) - x)$). From the fact that $K(x, 2r) \cap K(f(x), 2r) = \emptyset$, $x \in X$ we obtain $K(x, r) \cap K(P_L \circ f)(x), r) = \emptyset$ ($K(f(x), r) \cap L \neq \emptyset$ and consequently $d((P_L \circ f)(x), f(x)) < r$, i.e., $(P_L \circ f)(x) \cap K(f(x), 2r) \neq \emptyset$). Therefore, the family $\mathcal{A} = \{A_0, \ldots, A_n\}$ covers X_L. The mapping $h : X_L \to X_L$, $h(x) = (P_L \circ f)(x) - x$ is continuous and hence $A_i = h^-(\{y \in L : y = \sum s_j v_j$ and $s_i > 0\})$ is an open set. From (9) follows (5) and in view of Theorem 11 we obtain $\bigcap \mathcal{A} \neq \emptyset$. For any $x \in \bigcap \mathcal{A}$ we have $P_L(f(x)) - x = \sum s_j v_j$ with $s_j > 0$, $j = 0, \ldots, n$ which contradicts $s_i = 0$ for an index i ($s_0, \ldots, s_n$ are sector coefficients). Thus f has a fixed point. □

In view of Remark 12 we may write $v = x - P_{q+L}(f(x))$ in (9).

By applying Theorem 15 in place of Theorem 11 in the previous proof one obtains the following

THEOREM 17. *Let X be a bounded closed convex set in a real Hilbert space H and let $f : X \to H$ be a compact map. Assume that for each finite dimensional subspace L of* Span $f(X)$, $0 \in \operatorname{Int} X_L$ ($X_L = L \cap X$) *in the induced topology and there exist $v_0, \ldots, v_n \in L$ ($n = \dim L$) such that $(v_i, v_j) \leqslant 0$, $i \neq j$ and the following is satisfied*

(10) *for each $x \in \operatorname{Fr} X_L$ there exists an index i such that $(v_i, x) > 0$ and $(v_i, P_L(f(x)) - x) > 0$.*

Then f has a fixed point.

Proof. For L defined as in the previous proof, let us adopt $A_i = \{x \in X_L : (v_i, (P_L \circ f)(x) - x) > 0\}$, $i = 1, \ldots, n$. In view of Proposition 8 $\mathcal{A} = \{A_0, \ldots, A_n\}$ covers X_L. On the other hand (10) implies (8). Therefore (Theorem 15) $\bigcap \mathcal{A}$ is nonempty. □

Clearly one may write $(v_i, x) < 0$ in condition (10).

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