

Multivalued Selections

by

Lech PASICKI

Presented by C. BESSAGA on October 21, 1996

Summary. Michael's basic theorems on selections are extended to the case of pseudo-metrizable spaces. Natural consequences such as theorems on extensions, retractions and fixed points are included. In addition a version of Dugundji's theorem is given.

We adopt Kelley's terminology concerning topology. It should be stressed that in this paper topological spaces are not assumed to be Hausdorff.

Let us recall some notions for multivalued mappings. For arbitrary sets X, Y the notation $F : X \rightarrow 2^Y$ means that $F(x) \subset Y$, $x \in X$ and $F(A) = \bigcup \{F(x) : x \in A\}$, $A \subset X$; F is a mapping if $F(x) \neq \emptyset$, $x \in X \neq \emptyset$. For $F : X \rightarrow 2^Y$ and any $C \subset Y$ we write $F^-(C) = \{x \in X : F(x) \cap C \neq \emptyset\}$ and $F^+(C) = \{x \in X : F(x) \subset C\}$ (cf. [1], 26). It can be easily checked that $F^- : Y \rightarrow 2^X$. If for each $x \in X$, $F(x)$ is a singleton or empty, then we write $F : X \rightarrow Y$ instead of $F : X \rightarrow 2^Y$, and $F(x) = \{y\}$ is replaced by $F(x) = y$ ($\{y\}$ must not be an element of Y).

Let X, Y be topological spaces. We say that $F : X \rightarrow 2^Y$ is lsc if for each open $C \subset Y$, $F^-(C)$ is open; F is usc if for each closed $C \subset Y$, $F^-(C)$ is closed. Clearly $F^-(C)$ is open (closed) iff $F^+(Y \setminus C)$ is closed (open).

The lemma to follow extends ([6] Ex. 1.3*, 362).

LEMMA 1. *Let X, Y be topological spaces, $F : X \rightarrow 2^Y$ lsc, $A \subset X$ a closed set and $S : A \rightarrow 2^Y$ lsc with $S(x) \subset F(x)$, $x \in A$. Then $G : X \rightarrow 2^Y$ defined by*

$$G(x) = \begin{cases} S(x) & x \in A \\ F(x) & x \in X \setminus A \end{cases}$$

1991 MS Classification: 54C65, 54C15, 54C20, 54H25.

Key words: selection, extension, retraction, fixed point.

is lsc.

P r o o f. Let $W \subset Y$, $U_1, U_2 \subset X$ be open sets such that $X \setminus A \subset U_1$, $U_1 \cap A = S^-(W)$ and $U_2 = F^-(W)$. Then for $U = U_1 \cap U_2$ we have $U \cap A = S^-(W) = U_1 \cap A$ as $S^-(W) \subset F^-(W)$. Hence $G^-(W) = S^-(W) \cup [F^-(W) \cap (X \setminus A)] = (U \cap A) \cup (U \cap X \setminus A) = U$ is an open set.

LEMMA 2. *Let X, Y be topological spaces, and mapping $S : X \rightarrow 2^Y$ with values contained in minimal closed, nonempty sets. Then for each closed or open set $C \subset Y$ we have $S^+(C) = S^-(C)$. Consequently S is lsc iff it is usc.*

P r o o f. Let $C \subset Y$ be closed and assume $S(x) \cap C \neq \emptyset$. Let B be a minimal closed set containing $S(x)$. Then $B \cap C$ is closed, minimal and consequently $B \cap C = B \subset C$ which means $S(x) \subset C$ and $S^-(C) = S^+(C)$. If C is open then $Y \setminus C$ is closed and we have $B \subset Y \setminus C$ or $B \subset C$. Therefore for any closed or open set $C \subset Y$, $S^+(C) = S^-(C)$ holds.

Remark 3. From Lemma 2 it follows that if all values of mappings $G, H : X \rightarrow 2^Y$ are contained in the same minimal closed sets then G is continuous iff H is lsc or usc.

The preceding remark suggests to adopt the following definition which will be applied in this paper.

DEFINITION 4. Let X, Y be topological spaces and $F : X \rightarrow 2^Y$. A continuous mapping $S : X \rightarrow 2^Y$ is a multivalued selection for F if $S(x) \subset F(x)$, $x \in X$ and all values of S are contained in minimal closed sets.

Let (Y, \mathcal{V}) be a uniform space and $C \subset Y$. If $V \in \mathcal{V}$ and there exists a $y \in Y$ such that $C \subset V(y)$ then we write $\text{dia } C \leq V$. As regards uniform spaces, in the proofs of this paper we will take only symmetric entourages into account; nV replaces $V \circ \dots \circ V$ (n members).

THEOREM 5. *Let X be a topological space and (Y, \mathcal{V}) a uniform pseudo-metrizable space. Assume that \mathcal{H} is a family of lsc mappings $H : X \rightarrow 2^Y$ with closed, complete values satisfying*

- (1) *for each $H \in \mathcal{H}$, $V \in \mathcal{V}$ there exists a $G \in \mathcal{H}$ such that $G(x) \subset H(x)$ and $\text{dia } G(x) \leq V$, $x \in X$.*

Then any member $H \in \mathcal{H}$ admits a multivalued selection with closed, complete values.

P r o o f. Let \mathcal{S} be any chain of members of \mathcal{H} such that for each $V \in \mathcal{V}$ there exists a $G \in \mathcal{S}$ with $\text{dia } G(x) \leq V$, $x \in X$. Such a chain exists as Y is pseudo-metrizable ([4] Thm 13, 186). Now in view of ([4] Thm 23, 193)

for each $x \in X$ there exists a nonempty set $S(x) = \bigcap \mathcal{S}(x)$. Suppose C is a closed nonempty subset of $S(x)$ and $z \in S(x) \setminus C$. There exists a $V \in \mathcal{V}$ such that $2V(z) \cap C = \emptyset$ - Y being regular. In view of (1) for a $y \in Y$ we have $z \in V(y)$ and $C \subset V(y)$ which imply $y \in V(z) \cap V(C)$ which is a contradiction. Thus all values of S consist of minimal nonempty, closed sets which clearly are complete. Now let us show that S is usc. Assume $C \subset Y$ is closed and $S(x_0) \cap C = \emptyset$. Then there exists a $V \in \mathcal{V}$ such that $(3V \circ S)(x_0) \cap C = \emptyset$ ($S(x_0)$ is minimal). Let $G \in \mathcal{S}$ be as in (1). By lsc of G the set $U_{x_0} = G^{-}((V \circ S)(x_0))$ is a neighbourhood of x_0 and we have $G(x) \cap (V \circ S)(x_0) \neq \emptyset, x \in U_{x_0}$. Let $V \in \mathcal{V}, G \in \mathcal{S}$ be as in (1). Then $G(x) \subset V(y)$ holds for a $y \in Y$, i.e. $V(y) \cap (V \circ S)(x_0) \neq \emptyset$ and $y \in (2V \circ S)(x_0)$, which in turn implies $G(x) \subset V(y) \subset (3V \circ S)(x_0)$ and $S(x) \cap C \subset G(x) \cap C = \emptyset, x \in U_{x_0}$. This result means that $X \setminus S^{-}(C)$ is open and S is usc. In view of Lemma 2 S is continuous. \square

The author apologizes for not using pseudo-metrizable notations (balls) in the initial part of the preceding proof. The other proofs concern uniform spaces and the part in question limits the results of the paper (except for Theorems 19-21).

In the theorems to follow we may require all values of multivalued $S : X \rightarrow 2^Y$ to be closed, complete and convex (see Remark 3).

LEMMA 6. *Let X be a topological space and (Y, \mathcal{V}) a uniform topological subspace of a locally convex space. Assume $F : X \rightarrow 2^Y$ is a lsc mapping with closed, complete and convex values satisfying*

- (2) *for each $V \in \mathcal{V}$ there exists a locally finite partition of unity \mathcal{L} on X being subordinated to $\{F^{-}(V(y)) : y \in F(X)\}$.*

Then there exists a family \mathcal{H} of lsc mappings with closed, complete and convex values, $H(x) \subset F(x), H \in \mathcal{H}, x \in X$ satisfying (1).

Proof. Let us consider a $V \in \mathcal{V}$ such that $V(y)$ is open, $y \in F(X)$ and induced by a convex set in the original locally convex topology. For \mathcal{L} as in (2) we write $h_V = \Sigma\{\alpha(x)y_\alpha : \alpha \in \mathcal{L}\}, x \in X$ where $\alpha^{-1}((0, 1 >) \subset F^{-}(V(y_\alpha)), \alpha \in \mathcal{L}$. Clearly h_V is continuous and $V(y_\alpha) \cap F(x) \neq \emptyset, x \in F^{-}(V(y_\alpha))$. Let us consider $z_\alpha \in V(y_\alpha) \cap F(x)$ for a fixed x . From the convexity of $F(x)$ it follows that $\Sigma\{\alpha(x)z_\alpha : \alpha \in \mathcal{L}\} \in F(x)$ and consequently for $H_V(x) = F(x) \cap \Sigma\{\alpha(x)V(y_\alpha) : \alpha \in \mathcal{L}\} = F(x) \cap V(\Sigma\{\alpha(x)y_\alpha : \alpha \in \mathcal{L}\})$ we have $H_V(x) \neq \emptyset, x \in X$. In view of ([6] Prop. 2.5, 366) H_V is lsc, F, h_V being lsc and so is $\overline{H_V}$ ([6] Prop. 2.3, 366). On the other hand, $\text{dia} H_V(x) \leq V, x \in X$. Let \mathcal{H} consist of mappings H_V for all open $V \in \mathcal{V}$. For any $V, W \in \mathcal{V}$ there exists a $U \in \mathcal{V}$ such that $U \subset V \cap W$. Therefore, the above reasoning when applied to $\overline{H_V}$ in place of F (all values of $\overline{H_V}$

are closed, complete and convex) gives a $\overline{H_U}$ being lsc and $\overline{H_U}(x) \subset \overline{H_V}(x)$, $\text{dia } \overline{H_U}(x) \leq U \subset W$, $x \in X$. Thus (1) is satisfied. \square

Now let us present the following two natural extensions of the well-known theorems of Michael ([6] Thm 3.2'', 3.1', 367).

THEOREM 7. *Let X be a paracompact space and Y pseudo-metrizable topological subspace of a locally convex space. Assume $F : X \rightarrow 2^Y$ is an lsc mapping with closed, complete and convex values. Then F admits a multi-values selection (with closed, complete, convex values if required).*

P r o o f . We apply Lemma 6 and Theorem 5 (see Remark 3).

THEOREM 8. *Let X be a normal space and Y pseudo-metrizable topological subspace of a locally convex space. Assume $F : X \rightarrow 2^Y$ is an lsc mapping with closed, complete and convex values and $F(X)$ is precompact (i.e. totally bounded). Then F admits a multivalued selection.*

P r o o f . The family $\{V(y) : y \in F(X)\}$ contains a finite cover, say $\{V(y_i) : i = 1, \dots, n\}$ of $F(X)$. Now $\{F^{-1}(V(y_i)) : i = 1, \dots, n\}$ is a locally finite open cover of X and in view of ([4] W, 171) there exists a locally finite partition of unity subordinated to it. Now we apply Lemma 6 and Theorem 5. \square

A natural consequence of Theorems 7, 8 (see [6] Prop. 1.4, 363) is the following.

THEOREM 9. *Let A be a closed set in a paracompact space X and Y a pseudo-metrizable topological subspace of a locally convex space. Assume $S : A \rightarrow 2^Y$ is a multivalued selection for $F|_A$, where $F : X \rightarrow 2^Y$ is a lsc mapping with closed, complete and convex values. Then S has a continuous extension to X , $S(x) \subset F(x)$, $x \in X$. The same holds for X being normal in place of paracompact if $F(X)$ is precompact.*

P r o o f . Let us consider G as in Lemma 1. It is lsc and we may assume all values of G to be closed, complete and convex (Remark 3). Now we apply Theorem 7 or Theorem 8, respectively. \square

Corollary 15 ([6], 363) suggests the following.

THEOREM 10. *Let A be a closed set in a paracompact space X and Y a set in a locally convex space, $\overline{\text{conv } Y}$ complete and pseudo-metrizable. Then any continuous mapping $S : A \rightarrow 2^Y$ with values contained in minimal closed sets can be extended to a continuous mapping $T : X \rightarrow 2^{\overline{\text{conv } Y}}$ with values possessing the same property. If in addition $\text{conv } Y$ is precompact then X can be assumed to be normal instead of being paracompact.*

P r o o f. Minimal closed sets in locally convex space are convex and therefore (see Remark 3) we may require all values of S to be closed, complete and convex. Let us consider mapping $F : X \rightarrow 2^{\overline{\text{conv} Y}}$, $F(x) = S(x)$, $x \in A$, and $F(x) = \overline{\text{conv} Y}$, $x \in X \setminus A$. In view of Theorem 9 (see Lemma 1) F has a multivalued selection. As regards $\text{conv} Y$ being precompact, we apply Lemma 6. \square

The theorem to follow is a consequence of Theorem 10.

THEOREM 11. *Let $Y \neq \emptyset$ be a closed, complete and convex pseudo-metrizable topological subspace of a locally convex paracompact space X . Then there exists a continuous retraction $r : X \rightarrow Y$. If Y is precompact then X can be assumed normal instead of being paracompact.*

P r o o f. Let us apply Theorem 10 to $S : Y \rightarrow Y$ defined by $S(x) = x$, $x \in Y$. Then for $r(x) \in T(x)$, $x \in X$, r is a retraction (see Remark 3). \square

Remark 3 suggests to adopt the following

DEFINITION 12. Let Y be a subset of a topological space X . A continuous mapping $S : X \rightarrow 2^Y$ is a multivalued retraction if all values of S are contained in minimal closed sets and $x \in \overline{S}(x)$, $x \in Y$. If there exists a multivalued retraction $S : X \rightarrow 2^Y$ then Y is a retract of X .

From Theorems 7, 8 we obtain the following extension of the Bartle-Graves theorem (see [6] Corol. p. 364).

THEOREM 13. *Let X be a paracompact space and Y a pseudo-metrizable topological subspace of a locally convex space. Assume that $H : Y \rightarrow 2^X$ is an open mapping such that $H(Y) = X$ and all values of H^- are closed, complete and convex. Then there exists a continuous mapping $G : X \rightarrow 2^Y$ such that $x \in (H \circ G)(x)$, $x \in X$. If $H : Y \rightarrow X$ then G can be considered $X \rightarrow Y$, i.e. G is then the right inverse of H . If in addition Y is precompact then X can be assumed to be normal instead of being paracompact.*

P r o o f. For simplicity of notations let us assume that $H(Y) = X$. Let us consider $F = H^- : X \rightarrow 2^Y$. If $W \subset Y$ is open then $F^-(W) = H(W)$ is open, i.e. F is lsc. Now in view of Theorem 7 F admits a multivalued selection, say G and we have $G(x) \subset H^-(x)$ which means $x \in (H \circ G)(x)$. The remaining part follows from Remark 3. \square

Theorem 8 is applied in proving the following fixed point theorem.

THEOREM 14. *Let X be a normal convex topological subspace of a locally convex space. Assume $F : X \rightarrow 2^X$ is lsc with closed, convex values and $F(X)$ pseudo-metrizable, relatively compact. Then F has a fixed point.*

P r o o f. From the fact that $\overline{F(X)}$ is compact ([4] B (b), 161) it follows that this set is precompact ([4] Thm 32, 198) and all values of F are complete ([4] Thm 22, 192). Now Theorem 8 applies. If S is a usc selection for F and all values of S are closed and convex, then in view of ([7] Thm 2.2.13 (a), 35) S has a fixed point (a simple proof is given in [8] Thm 7). \square

A direct consequence of Theorem 14 and [4] Corol. 35, 160) is the following

THEOREM 15. *Let X be a pseudo-metrizable convex topological subspace of a locally convex space and $F : X \rightarrow 2^X$ a lsc mapping with closed, convex values and $F(X)$ relatively compact. Then F has a fixed point.*

Theorem 14 can be extended to the following theorem on composition of mappings.

THEOREM 16. *Let $X_1 \dots X_n$ be normal convex topological subspaces of locally convex spaces and $F_i : X_i \rightarrow 2^{X_{i+1}}$, $F_n : X_n \rightarrow 2^{X_1}$ lsc mappings with closed, convex values and $F_n(X_n)$, $F_i(X_i)$ pseudo-metrizable and relatively compact, $i = 1 \dots n - 1$. Then $F_n \circ \dots \circ F_1$ has a fixed point.*

P r o o f. Let us consider $X = X_1 \times \dots \times X_n$ and $F : X \rightarrow 2^X$ defined as follows ([7], 39), $F(x_1, \dots, x_n) = F_n(x_n) \times F_1(x_1) \times \dots \times F_{n-1}(x_{n-1})$. It can be easily checked that F has a fixed point iff $F_n \circ \dots \circ F_1$ has one. For multivalued selections $S_i : X_i \rightarrow 2^{X_{i+1}}$, $S_n : X_n \rightarrow 2^{X_1}$, $i = 1 \dots n - 1$ which are usc we apply ([7] Thm 3.1.8, 42) and consequently $S_n \circ \dots \circ S_1$ has a fixed point. \square

Let us present a simplified version of Theorem 16

THEOREM 17. *Let $X_1 \dots X_n$ be pseudo-metrizable convex topological subspaces of locally convex spaces and $F_i : X_i \rightarrow 2^{X_{i+1}}$, $F_n : X_n \rightarrow 2^{X_1}$ lsc mappings with closed, convex values and $F_n(X_n)$, $F_i(X_i)$ relatively compact, $i = 1 \dots n - 1$. Then $F_n \circ \dots \circ F_1$ has a fixed point.*

Remark 18. Theorems 16, 17 have their “mixed up” versions for some (or all) F_i being usc (with respectively relaxed assumptions on X_i and $F_i(X_i)$), and the others having multivalued selections. The idea is to apply ([7] Thm 3.1.8, 42) to the Cartesian product of usc mappings and selections.

The theorem to follow is an auxiliary result; the original one ([7] Thm 6.1.1, 77) concerned metric spaces.

THEOREM 19. *Let A be a closed set in a pseudo-metric space (X, d) and $p : X \setminus A \rightarrow (0, \infty)$ a lower semicontinuous mapping. Then for each*

nonempty set $C \subset A$ there exists a locally finite open cover \mathcal{U} of $X \setminus A$ and a set $Z = \{z_U : U \in \mathcal{U}\} \subset C$ such that

$$(3) \quad d(x, z_U) < p(x) + d(x, C), \quad x \in St(U, \mathcal{U}), \quad U \in \mathcal{U}$$

holds.

P r o o f. Let us consider $h(x) = p(x)/10$, $x \in X \setminus A$ and $V_x = h^{-}((h(x)/2, \infty))$. Clearly $x \in V_x$ and $V_x \subset X \setminus A$ is open, consequently $\mathcal{V} = \{B(x, h(x)) \cap V_x : x \in X\}$ is an open cover of $X \setminus A$. There exists a locally finite star refinement \mathcal{U} of \mathcal{V} (see [4] Thm 28(c), 156, [3] Lemmas 5.1.13, 5.1.15, p. 377). For $U \in \mathcal{U}$ let $y_U \in X \setminus A$ be such that $St(U, \mathcal{U}) \subset B(y_U, h(y_U)) \cap V_{y_U}$ and for $x_U \in U$ $z_U \in B(x_U, d(x_U, C) + h(y_U)) \cap C$. Then for $x \in St(U, \mathcal{U})$ we have $d(x, z_U) \leq d(x, x_U) + d(x_U, z_U) \leq \text{dia } St(U, \mathcal{U}) + d(x_U, C) + h(y_U) \leq 3h(y_U) + d(x_U, C) \leq 3h(y_U) + d(x_U, x) + d(x, C) \leq 5h(y_U) + d(x, C)$. From the fact that $x \in h^{-}((h(y_U)/2, \infty))$, i.e. $h(y_U)/2 < h(x)$ we obtain $d(x, z_U) < 10h(x) + d(x, C) \leq p(x) + d(x, C)$. \square

Now let us present an extension of Dugundji's theorem ([2] Thm 4.1, 357). This result is independent of Theorem 10.

THEOREM 20. *Let A be a closed set in a pseudo-metrizable space X and Z a locally convex space. Then any continuous mapping $S : A \rightarrow 2^Z$ with values contained in minimal closed sets, can be extended to a continuous mapping T with values preserving the same property and $T(X) \subset S(A) \cup \text{conv } S(\text{Fr } A)$.*

P r o o f. In view of Remark 3 it suffices to consider the case of $S : A \rightarrow Z$. Let \mathcal{U} be as in Theorem 19 for $C = \text{Fr } A$ and $p(x) = d(x, C)$ (d - pseudo-metric for X). For \mathcal{L} being a locally finite partition of unity subordinated to \mathcal{U} let us consider

$$T(x) = \begin{cases} S(x) & x \in A \\ \Sigma\{\alpha(x)S(z_\alpha) : \alpha \in \mathcal{L}\} & x \in X \setminus A, \end{cases}$$

where $z_\alpha := z_U$ for $U \in \mathcal{U}$ such that $\text{supp } \alpha \subset U$.

Clearly T is continuous on $X \setminus A$ and $T(A) \subset S(A) \cup \text{conv } S(\text{Fr } A)$. If $x \in X \setminus A \cap B(x_0, r)$ for a $x_0 \in \text{Fr } A$, then in view of (3) we have $T(x) \in S(B(x, 2d(x, \text{Fr } A)) \cap \text{Fr } A) \subset \text{conv } S(B(x_0, 3r) \cap \text{Fr } A)$. Consequently T is continuous on $\text{Fr } A$. \square

Theorem 20 (cf. Thm 11, Def. 12) implies

THEOREM 21. *Any closed, convex and nonempty subset of a locally convex pseudo-metrizable space X is a retract of X .*

P r o o f. We apply Thm 20 to A being closed and convex, $Z = X$ and $S(x) = x$, $x \in A$.

DEPARTMENT OF MATHEMATICS, ACADEMY OF MINING AND METALLURGY, AL. MICKIEWICZA 30, 30-059 KRAKÓW
(INSTYTUT MATEMATYKI, AGH)
E-mail @uci.agh.edu.pl

REFERENCES

- [1] C. Berge, *Espaces topologiques, fonctions multivoques*, Dunod, Paris 1959.
- [2] J. Dugundji, *An extension of Tietze's theorem*, Pacific J. Math., **1** (1951) 353–367.
- [3] R. Engelking, *General topology*, PWN, Warszawa 1977.
- [4] J. L. Kelley, *General topology*, Springer-Verlag, Berlin 1975.
- [5] J. L. Kelley, I. Namioka, *Linear topological spaces*, Springer-Verlag, Berlin 1976.
- [6] E. Michael, *Continuous selections I*, Ann. Math., **63** (1956) 361–382.
- [7] L. Pasicki, *A fixed point theory and some other applications of weeds*, Opuscula Math., **7** (1990) 1–98.
- [8] L. Pasicki, *On KKM mappings*, Bull. Pol. Ac.: Math., **43** (1995) 1–8.