Lech Pasicki *

Simple proofs of three fixed point theorems **

Keywords: fixed point, multivalued mapping.

Abstract. The fixed point theorems for multivalued mappings that are presented in this paper are fairly general (e.g. we do not apply the Hausdorff separation axiom). Our interest is in proving them in possibly natural and short way.

Let us recall some notions concerning multivalued mappings. For arbitrary sets X, Y the notation $F\colon X \to 2^Y$ means that $F(x) \subset Y$, $x \in X$ and $F(A) = \bigcup\{F(x)\colon x \in A\}$, $A \subset X$; F is a mapping if $F(x) \neq \emptyset$, $x \in X$. For $F\colon X \to 2^Y$ and any $B \subset Y$ we write $F^-(B) = \{x \in X\colon F(x) \cap B \neq \emptyset\}$; it can be easily checked that $F^-\colon Y \to 2^X$.

If X, Y are topological spaces then $F\colon X \to 2^Y$ is **usc** (upper semicontinuous) if for each closed set $B \subset Y$, $F^-(B)$ is closed; $F\colon X \to 2^Y$ is compact if it is usc and $\overline{F(X)}$ is compact (it does not imply $\overline{F(X)}$ is Hausdorff [3]). It can be easily seen that $F\colon X \to 2^Y$ is usc iff for each $x \in X$ and any neighbourhood W of $F(x)$ there exists a neighbourhood U of x such that $F(U) \subset W$.

The lemma to follow is a simplified version of ([10], Lemma 6) and it is related to ([6], Lemma p. 576).

Lemma 1. *Let $(X, \mathcal{V})$ be a uniform space and let $F\colon X \to 2^X$ be compact with closed values. If F has no fixed point then there exists a closed and symmetric $V \in \mathcal{V}$ such that $V \circ F$ has no fixed point.*

Proof. If $x \notin F(x)$ holds then we have $W_x \cap F(x) = \emptyset$ for a closed entourage $W_x \in \mathcal{V}$ and consequently $x \notin (F^- \circ W_x)(x)$. The point x can be separated from the closed set $(F^- \circ W_x)(x)$ means that there exists an open $V_x \in \mathcal{V}$ satisfying $V_x(x) \cap (F^- \circ V_x)(x) = \emptyset$. The family $\{V_x(x)\colon x \in \overline{F(X)}\}$ is an open cover of the

*University of Mining and Metallurgy, Department of Applied Mathematics, Kraków; e-mail: pasicki@wms.mat.agh.edu.pl

**1991 Mathematics Subject Classification: 54H25.

compact set $\overline{F(X)}$ and therefore ([3], Th. 33, p. 199) there exists a closed symmetric $V \in \mathcal{V}$ such that $\{V(x)\colon x \in \overline{F(X)}\}$ refines $\{V_x(x)\colon x \in \overline{F(X)}\}$. Thus we have $V(x) \cap (F^- \circ V)(x) = \emptyset$, $x \in \overline{F(X)}$ which means $V \circ F$ has no fixed point in $V(F(X))$. In consequence $V \circ F$ has no fixed point. □

For the notation proper to linear topological spaces the above lemma can be expressed as follows.

Lemma 2. *Let X be a set in a linear topological space and $F\colon X \to 2^X$ compact with closed values. If F has no fixed point then there exists a closed and symmetric neighbourhood V of zero such that $V + F$ has no fixed point.*

Now we concentrate on a nonempty intersection property.

Let us adopt $I = <0,1> \subset R$ and $P^n = \{(t_0, \ldots, t_n) \in I^{n+1}\colon \sum t_i = 1\}$ (standard n-simplex in R^{n+1}).

The „closed" version of the theorem to follow was given in [5] and its proof based on Sperner's lemma is a short one (see [1], p. 102). The „open" version is an almost immediate consequence of the „closed" one (see [6] the proof of Th. 1).

Theorem 1. *Let $\mathcal{A} = \{A_0, \ldots, A_n\}$ be a closed or open cover of P^n. If for each $t \in P^n$ we have $t \in \bigcup\{A_i\colon t_i \neq 0\}$ then $\bigcap \mathcal{A}$ is nonempty.*

The theorem to follow is related to ([1], Th. (1.2), p. 73). The proof is almost the same as ([10], proof of Th. 3).

Theorem 2. *Let Z be a finite set in a convex subset X of a linear topological space and $H\colon Z \to 2^X$ such that for each $z \in Z$ the set $H(z) \cap \operatorname{conv} Z$ is closed (open) in $\operatorname{conv} Z$. If $\operatorname{conv} K \subset H(K)$, $K \subset Z$ holds then $\bigcap\{H(z)\colon z \in Z\} \cap \operatorname{conv} Z$ is nonempty.*

Proof. Assume $Z = \{z_0, \ldots, z_n\}$ and let $h\colon P^n \to \operatorname{conv} Z$ be defined by $h(t) = \sum t_i z_i$. Let us consider $A_i = (h^{-1} \circ H)(z_i)$, $i = 0, \ldots, n$ and $\mathcal{A} = \{A_0, \ldots, A_n\}$. Clearly all members of $\mathcal{A}$ are closed (open) in P^n, h being continuous and the other assumptions of Theorem 1 are satisfied. Therefore $\bigcap \mathcal{A}$ is nonempty. Thus we have $\emptyset \neq h(\bigcap \mathcal{A}) \subset \bigcap h(\mathcal{A}) = \bigcap\{H(z)\colon z \in Z\}$ and in view of the way h was defined, this set intersects with $\operatorname{conv} Z$. □

Now we are ready to prove the following extension of a theorem of Himmelberg.

Theorem 3. ([8], Th. 2.2.13 (a), p. 35). *Let X be a convex set in a locally convex space (not necessarily Hausdorff) and $F\colon X \to 2^X$ a compact mapping with closed convex values. Then F has a fixed point.*

Proof. We can treat X as being a uniform space $(X, \mathcal{V})$. Suppose F has no fixed point. In view of Lemma 1 $V \circ F$ has no fixed point for a closed, symmetric $V \in \mathcal{V}$ and we may require $V(x)$ to be convex, $x \in X$. There exists a finite set $Z \subset F(X)$ such that $F(X) \subset V(Z)$ ($F(X)$ is precompact), i.e. $(F^- \circ V)(Z) = X$. Let us consider $H\colon Z \to 2^X$ defined by $H(z) = X \setminus (F^- \circ V)(z)$, $z \in Z$. We show that $\bigcap\{H(z)\colon z \in Z\} \neq \emptyset$. Indeed all values of H are open and let us check the second assumption of Theorem 2. For a fixed $K \subset Z$ let x be any point of $B = \bigcap\{(F^- \circ V)(z)\colon z \in K\}$. Then we have $F(x) \cap V(z) = \emptyset$, $z \in K$ and consequently $K \subset (V \circ F)(x)$, $\operatorname{conv} K \subset \operatorname{conv}(V \circ F)(x) = (V \circ F)(x)$. From $x \notin (V \circ F)(x)$ it follows that $x \notin \operatorname{conv} K$ and therefore $B \cap \operatorname{conv} K = \emptyset$, x being arbitrary. Thus we have $\operatorname{conv} K \subset X \setminus B = \bigcup\{X \setminus (F^- \circ V)(z)\colon z \in K\} = H(K)$. Now by Theorem 2 $\emptyset \neq \bigcap\{H(z)\colon z \in Z\} = X \setminus (F^- \circ V)(Z)$ holds. This fact contradicts to $X = (F^- \circ V)(Z)$ and thus F has a fixed point. □

The ideas presented above can be extended to the case of a convex structure called *weed.*

For any point $x = (x_1, \ldots, x_n) \in X^n$ let us write $\partial_i x = (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n) \in X^{n-1}$.

Definition 1. ([8], Def. 1.2.7, p. 18). *A pair (E, Q) is a* **weed** *in a topological space X if $Q = (Q_n)_{n \in N}$ is a sequence of mappings $Q_n\colon E^n \times P^{n-1} \to X$ satisfying*

$$\text{if } t_i = 0 \text{ then } Q_n(e,t) = Q_{n-1}(\partial_i e, \partial_i t),\ e \in E^n,\ i = 1, \ldots, n,\ n \in N \tag{1}$$

and $Q_n(e, \cdot)\colon P^{n-1} \to X$ is continuous, $e \in E^n$. (X, Q) is a weed if it is a weed in X.

For Q_n as in Definition 1 we adopt $Q_n(e) = Q_n(e, P^{n-1})$.

A convex set E in a linear topological space, and Q_n being convex combination of n points of E present a natural „linear" example of a weed.

Definition 2. ([8], Def. 1.2.2, p. 16). *If (E, Q) is a weed in X then $cQA = \bigcup\{Q_n((E \cap A)^n)\colon n \in N\}$; A is an underhull if $A \subset cQA$ and A is an overhull if $cQA \subset A$; A is convex whenever $A = cQA$.*

For a „linear" example of a weed cQA is just the convex hull of A.

If (E, Q) is a weed in X then a Q' can be defined ([8], Ex. 1.2.12, p. 19) in such a way that

$$\text{if } e_i = e_j \text{ then } Q'_n(e,t) = Q'_{n-1}(\partial_i e, t'),\ (\text{a } t' \in P^{n-2}),\ e \in E^n,\ n \in N \tag{2}$$

holds, $Q'_n(e) \subset Q_n(e)$ and (E, Q') is a weed in X.

Clearly if (X, Q) is a weed then (X, Q') preserves this property. What is more $cQ' Z = Q'_n(Z^n)$ for any set Z consisting of n points. These properties are natural for the "linear" example.

The above nice properties simplify the notation and therefore we require Q to satisfy(2) in what follows.

The theorem below extends ([8], Th. 1.4.3, p. 23).

Theorem 4. *Let (Z, Q) be a weed in topological space X and Z finite. Assume that $H\colon Z \to 2^X$ is such that for each $z \in Z$, $H(z) \cap cQZ$ is closed (open) in cQZ. If for each set $K \subset Z$ we have $cQK \subset H(K)$ then $\bigcap\{H(z)\colon z \in Z\} \cap cQZ$ is nonempty.*

Proof. Let us assume $Z = \{z_0, \dots, z_n\}$ and let us consider $h: P^n \to \mathrm{cQ}\, Z$ defined by $h(t) = Q_{n+1}((z_0, \dots, z_n), t)$. Now we follow the proof of Theorem 2 for cQ in place of conv. □

Let us prove the following being a particular case of ([8], Th. 2.2.11, p. 33).

Theorem 5. *Let $(X, \mathcal{V})$ be a uniform space, $F: X \to 2^X$ a compact mapping with closed values and $(F(X), Q)$ a weed in X. If in addition*

$$\textit{for each } W \in \mathcal{W} \textit{ there is a } V \in \mathcal{V} \textit{ with } cQ(V \circ F)(x) \subset (W \circ F)(x),\ x \in X \qquad (3)$$

holds then F has a fixed point.

Proof. Suppose F has no fixed point. Then for a $W \in \mathcal{W}$ such that $W \circ F$ has no fixed point (Lemma 1) let V be a closed symmetric entourage as in (3). There exists a finite set $Z \subset F(X)$ such that $F(X) \subset V(Z)$, i.e. $(F^- \circ V)(Z) = X$. We adopt $H(z) = X \setminus (F^- \circ V)(z)$, $z \in Z$. Now for a fixed $K \subset Z$ and any $x \in B = \bigcap\{(F^- \circ V)(z): z \in K\}$ we have $K \subset (V \circ F)(x)$ and by (3) $\mathrm{cQ}\, K \subset (W \circ F)(x)$. From $x \notin (W \circ F)(x)$ it follows that $x \notin \mathrm{cQ}\, K$ and consequently $B \cap \mathrm{cQ}\, K = \emptyset$ for arbitrary x. Thus we have $\mathrm{cQ}\, K \subset X \setminus B = \bigcup X \setminus \{(F^- \circ V)(z): z \in K\} = H(K)$. In view of Theorem 4 we have $\emptyset \neq \{H(z): z \in Z\} = X \setminus (F^- \circ V)(Z)$ which contradicts $(F^- \circ V)(Z) = X$. □

Now we extend Lemma 1 for the needs of the weak topology in locally convex spaces.

Lemma 3. *Let $(X,\mathcal{U})$ be a uniform space and $F: X \to 2^X$ with compact $A = \overline{F(X)}$. For $S = X$ let $(S, \mathcal{V})$ be a uniform space with the topology weaker than the topology of $(X,\mathcal{U})$ and assume that $F: X \to 2^S$ is usc and closed valued on a neighbourhood of A. If F has no fixed point then there exists a closed symmetric $V \in \mathcal{V}$ such that $V \circ F$ has no fixed point.*

Proof. Let us adopt $Y = F(X)$ and suppose $Fix(V \circ F) \neq \emptyset$, $V \in \mathcal{V}$. The condition $x \in (V \circ F)(x)$ implies $Fix(V \circ F) \subset V(Y)$ which is equivalent to $V(Fix(V \circ F)) \cap Y \neq \emptyset$ (we consider symmetric entourages). The family $\{Fix(V \circ F): V \in \mathcal{V}\}$ has the finite intersection property as $\mathcal{V}$ is a filter. In consequence the family $\{\overline{V(Fix(V \circ F)) \cap Y}: V \in \mathcal{V}\} = \mathcal{B}$ (closures in $(X,\mathcal{U})$) consisting of compact sets has nonempty intersection. We will show that $Fix F = \bigcap \mathcal{B}$. For any $x \in \bigcap \mathcal{B}$ we have $\emptyset \neq (V \circ F)(U(x)) \cap V(U(x))$, $U \in \mathcal{U}$, $V \in \mathcal{V}$. For „small" U we have $(V \circ F)(U(x)) \subset (2V \circ F)(x)$ as F is usc on a neighbourhood of $\overline{Y}$, and $V(U(x)) \subset 2V(x)$ as the topology of $(S, \mathcal{V})$ is weaker than the topology of $(X,\mathcal{U})$.

In consequence $(V \circ F)(x) \cap V(x) \neq \emptyset$, i.e. $F(x) \cap 2V(x) \neq \emptyset$, $V \in \mathcal{V}$ and $x \in F(x)$ as $(S, \mathcal{V})$ is regular and $F(x)$ closed. This contradiction proves that $V \circ F$ has no fixed point for a $V \in \mathcal{V}$. □

Definition 3. ([9], Def. 4, p. 71). *Let X be a topological space and Y a set in a locally convex space M. We say that $F: X \to 2^Y$ is* **wusc** *(weakly usc) if it is usc for Y equipped with the topology induced by the weak topology of M.*

Theorem 6. ([9], Th. 2, p. 72). *Let X be a convex set in a locally convex space M (not necessarily Hausdorff) and $F\colon X \to 2^X$ a mapping with closed convex values and $\overline{F(X)}$ compact. If F is wusc then it has a fixed point.*

Proof. Let us consider a uniformity $\mathcal{V}$ in X which determines the weak topology. In view of ([4] Corol. 14.4, p. 119) the values of F are closed in this topology. Suppose F has no fixed point. Now by Lemma 3 the mapping $V \circ F$ has no fixed point for a closed, symmetric $V \in \mathcal{V}$ and we may require $V(x)$ to be convex, $x \in X$. There exists a finite set $Z \subset F(X)$ such that $F(X) \subset V(Z)$ ($F(X)$ is precompact), i.e. $(F^- \circ V)(Z) = X$. Now we follow the remaining part of the proof of Theorem 3. □

The wusc mappings with closed convex values seem to be less general than the so called uhc (upper hemicontinuous) mappings ([7], p. 1435), nevertheless we do not loose the generality in the previous theorem (see [9], Lemma 3, p. 71), while the formulation itself is more natural.

For the completeness we quote the Definition (extended to the case of complex spaces) and the Lemma.

Definition 4. ([9], Def. 2, p. 70). *Let X be a topological space and M a linear topological space. We say that a mapping $F\colon X \to 2^M$ is* **uhc** *if for each continuous linear functional m on M (i.e. $m \in M^*$)* $\sup \operatorname{Re} m \circ F : X \to \overline{R}$ *is upper semicontinuous (*$\operatorname{Re} m$ *is the real part of m).*

Lemma 4. ([9], Lemma 3, p. 71). *Let M be a linear topological space and let $F\colon X \to 2^M$ be an uhc mapping. If $Z \subset M$ is such that all values of $Z \cap F$ are convex and weakly compact then $Z \cap F\colon X \to 2^M$ is wusc.*

To see how the above lemma applies to Theorem 6 one may set $X = Z = \operatorname{conv} Z$.

References

[1] Dugundji J., Granas A.: *Fixed point theory, Vol. I.* Warszawa, PWN 1982

[2] Himmelberg C.J.: *Fixed points of compact multifunctions.* J. Math. Anal. Appl., 38, 1972, 205–207

[3] Kelley J.L.: *General topology.* New York, Heidelberg, Berlin, Springer 1975

[4] Kelley J.L., Namioka I.: *Linear topological spaces.* Berlin, Springer 1976

[5] Knaster B., Kuratowski K., Mazurkiewicz S.: *Ein Beweis des Fixpunktsatzes für n-dimensionale Simplexe.* Fund. Math., 14, 1929, 132–137

[6] Lassonde M.: *Sur le principe KKM.* C. R. Acad. Sci. Paris, 310, 1990, 573–576

[7] Lasry J.M., Robert R.: *Degré pour les fonctions multivoques et applications.* C. R. Acad. Sci. Paris, 280, 1975, 1435–1438

[8] Pasicki L.: *A fixed point theory and some other applications of weeds.* Opuscula Math., 7, 1990, 1–98

[9] Pasicki L.: *Fixed point theorems for uhc mappings.* Opuscula Math., 12, 1993, 69–75

[10] Pasicki L.: *On the KKM theorem.* Bull. Pol. Ac.: Math., 43, 1995, 1–8

Received
December 11, 2001