Topology and its Applications 156 (2009) 1811–1816

Contents lists available at ScienceDirect

Topology and its Applications

www.elsevier.com/locate/topol

Bead spaces and fixed point theorems

Lech Pasicki

Faculty of Applied Mathematics, AGH University of Science and Technology, Al. Mickiewicza 30, 30-059 Kraków, Poland

ARTICLE INFO

Article history:
Received 23 September 2008
Received in revised form 26 March 2009
Accepted 26 March 2009

MSC:
primary 54E35, 54H25, 47H09, 52A05
secondary 54C65

Keywords:
Bead space (new)
Uniformly convex space
Fixed point
Nonexpansive mapping

ABSTRACT

The notion of a bead metric space defined here (see Definition 6) is a nice generalization of that of the uniformly convex normed space. In turn, the idea of a central point for a mapping when combined with the "single central point" property of the bead spaces enables us to obtain strong and elegant extensions of the Browder–Göhde–Kirk fixed point theorem for nonexpansive mappings (see Theorems 14–17). Their proofs are based on a very simple reasoning. We also prove two theorems on continuous selections for metric and Hilbert spaces. They are followed by fixed point theorems of Schauder type. In the final part we obtain a result on nonempty intersection.

The definition to follow is a version of [9, Definition 5] restricted to bounded sets and it coincides with the old notion of the Chebyshev centre of A in X (see e.g. [8]).

Definition 1. Let (X,d) be a metric space and $\emptyset \neq A \subset X$ a bounded set. An $x \in X$ is a central point for A if

$$r(A) := \inf\{t \in (0,\infty)\colon \text{ there exists a } z \in X \text{ with } A \subset B(z,t)\} = \inf\{t \in (0,\infty)\colon\ A \subset B(x,t)\}. \tag{1}$$

The centre $c(A)$ for A is the set of all central points for A, and $r(A)$ is the radius of A.

Proposition 2. *Let (X,d) be a metric space satisfying*

$$\text{for each } r > 0 \text{ and } x, y \in X,\ x \neq y \text{ there exist } \delta > 0,\ z \in X \text{ such that } B(x,r) \cap B(y,r) \subset B(z, r-\delta). \tag{2}$$

If $\emptyset \neq A \subset X$ is bounded then $c(A)$ contains at most one point.

Proof. Suppose $x, y \in c(A)$ are different. Then for $r = r(A)$ we have

$$A \subset \overline{B}(x,r) \cap \overline{B}(y,r) \subset \overline{B}(z, r-\delta)$$

which means that $r(A) \leqslant r - \delta$, a contradiction. □

Our main interest are fixed point theorems. To present them we need the notion of a centre for a mapping.

E-mail address: pasicki@wms.mat.agh.edu.pl.

0166-8641/$ – see front matter
doi:10.1016/j.topol.2009.03.042

Let 2^X be the family of all subsets of X and let $F : X \to 2^X$ being a multivalued mapping mean that $F(x) \neq \emptyset$, $x \in X$.
The following is the restriction of [9, Definition 7] to the case of a bounded set Y.

Definition 3. Let (X, d) be a metric space, $\emptyset \neq Y \subset X$ a bounded set and $F : Y \to 2^Y$ a mapping. An $x \in X$ is a central point for F if

$$\begin{aligned} r(F) &:= \inf\bigl\{t \in (0,\infty):\ F^n(Y) \subset B(z,t) \text{ for a } z \in X \text{ and an } n \in N\bigr\} \\ &= \inf\bigl\{t \in (0,\infty):\ F^n(Y) \subset B(x,t) \text{ for an } n \in N\bigr\}. \end{aligned} \tag{3}$$

The centre $c(F)$ for F is the set of all central points for F, and $r(F)$ is the radius of F.

In connection with condition (3) let us observe that from $F : Y \to 2^Y$ and $F^n(Y) \subset B(z,t)$ follows $F^p(Y) \subset F^n(Y) \subset B(z,t)$ for all $p \geqslant n$.
The proof of the theorem to follow is based on

Remark 4. If x is the only point satisfying condition P and $f(x)$ satisfies P then we have $f(x) = x$.

Theorem 5. *([10, Theorem 13]) Let (X, d) be a metric space and let $f : X \to X$ be a mapping. Assume that $\emptyset \neq Y \subset X$ is a bounded set such that $f(Y) \subset Y$ and $c(f|_Y) = \{x\}$ (a singleton). If the following*

$$d\bigl(f(x), f(y)\bigr) \leqslant d(x, y) \quad \text{for all } y \in Y \tag{4}$$

holds then x is a fixed point for f.

Proof. For $f^{n-1}(Y) \subset B(x,t)$ we have $f^n(Y) \subset f(Y \cap B(x,t))$. In view of (4) $f(Y \cap B(x,t)) \subset B(f(x),t)$ holds and consequently $f^n(Y) \subset B(f(x),t)$ which implies $f(x) \in c(f|_Y)$. Now it is clear that $f(x) = x$ as $c(f|_Y)$ is a singleton. □

The set $c(f|_Y)$ can be a singleton because of a special feature of f. We are interested in the case when this property depends on the space.

Definition 6. A metric space (X, d) is a bead space if the following is satisfied

$$\begin{aligned} &\text{for every } r > 0,\ \beta > 0 \text{ there exists a } \delta > 0 \text{ such that} \\ &\quad \text{for each } x, y \in X \text{ with } d(x,y) \geqslant \beta \text{ there exists a } z \in X \\ &\quad \text{such that } B(x, r+\delta) \cap B(y, r+\delta) \subset B(z, r-\delta). \end{aligned} \tag{5}$$

The previous definition looks much better than the following one:

Definition 7. ([9, Definition 1]) A metric space (X, d) is a discus space if there exists a mapping $\rho : [0,\infty) \times (0,\infty) \to [0,\infty)$ such that

$$\rho(\beta, r) < \rho(0, r) = r, \quad \beta, r > 0, \tag{6}$$

$$\rho(\cdot, r) \text{ is nonincreasing}, \quad r > 0, \tag{7}$$

$$\rho(\delta, \cdot) \text{ is upper semicontinuous}, \quad \delta \geqslant 0, \tag{8}$$

$$\text{for each } x, y \in X,\ r, \epsilon > 0 \text{ there exists a } z \in X \text{ such that } B(x,r) \cap B(y,r) \subset B\bigl(z, \rho\bigl(d(x,y), r\bigr) + \epsilon\bigr). \tag{9}$$

Lemma 8. *Each discus space is a bead space.*

Proof. Let $r > 0$, $x, y \in X$ with $d(x,y) = \beta > 0$ be arbitrary. In view of (9) for each $\epsilon, \kappa > 0$ there exists a $z \in X$ such that we have

$$B(x, r+\kappa) \cap B(y, r+\kappa) \subset B\bigl(z, \rho(\beta, r+\kappa) + \epsilon\bigr).$$

Let us adopt $2\gamma = r - \rho(\beta, r) = \rho(0, r) - \rho(\beta, r) > 0$ (see (6)). For sufficiently small κ, ϵ we have $\rho(\beta, r+\kappa) + \epsilon \leqslant \rho(\beta, r) + \gamma$ (see (8)). Now we obtain

$$\rho(\beta, r+\kappa) + \epsilon \leqslant \rho(\beta, r) + \gamma = r - 2\gamma + \gamma = r - \gamma$$

and

$$B(x, r+\kappa) \cap B(y, r+\kappa) \subset B(z, r-\gamma).$$

From this last inclusion for $\delta = \min\{\kappa, \gamma\}$ we easily get

$$B(x, r+\delta) \cap B(y, r+\delta) \subset B(z, r-\delta).$$

If $d(x, y) \geqslant \beta$ then by (7) $\rho(d(x, y), r+\kappa) \leqslant \rho(\beta, r+\kappa)$ and our inclusion holds for the same value of δ. $\square$

It was shown in [9, Example 3] that each uniformly convex space is a discus space; in particular each inner product space is a discus space.

Corollary 9. *Each inner product space or uniformly convex space is a discus space which in turn is a bead space.*

If condition (5) is satisfied then (2) holds and in view of Proposition 2 for each bounded and nonempty set A in a bead space $c(A)$ contains at most one point.

Lemma 10. *Let (X, d) be a bead space and let $\emptyset \neq A \subset X$ be bounded. Then $c(A)$ consists of at most one point. If in addition (X, d) is complete then $c(A)$ is a singleton.*

Proof. Let $(r_n)_{n\in N}$ decrease to $r = r(A)$ while $A \subset B(x_n, r_n)$. Suppose $(x_n)_{n\in N}$ is not a Cauchy sequence, i.e. $d(x_n, x_k) \geqslant \beta > 0$ for infinitely many $k < n$. In view of (5) for $r_n \leqslant r_k < r + \delta$ we have

$$A \subset B(x_n, r_k) \cap B(x_k, r_k) \subset B(x_n, r+\delta) \cap B(x_k, r+\delta) \subset B(z, r-\delta).$$

It means that $r(A) \leqslant r(A) - \delta$, a contradiction. Therefore $(x_n)_{n\in N}$ is a Cauchy sequence. If (X, d) is complete then $(x_n)_{n\in N}$ converges, say, to x. Then for any $\delta > 0$ and all sufficiently large n we have $A \subset B(x_n, r_n) \subset B(x, r+\delta)$ which means that $x \in c(A)$. In view of Proposition 2 $c(A)$ is a singleton. $\square$

Before proving other fixed point theorems let us present more properties of the bead spaces.

Definition 11. Let (X, d) be a metric space and $\mathcal{A}$ a family of bounded nonempty subsets of X. An $x \in X$ is a central point for $\mathcal{A}$ if

$$\begin{aligned} r(\mathcal{A}) &:= \inf\bigl\{t \in (0, \infty)\colon \text{ there exist } A \in \mathcal{A},\ z \in X \text{ such that } A \subset B(z, t)\bigr\} \\ &= \inf\bigl\{t \in (0, \infty)\colon \text{ there exists } A \in \mathcal{A} \text{ with } A \subset B(x, t)\bigr\}. \end{aligned} \tag{10}$$

The centre $c(\mathcal{A})$ for $\mathcal{A}$ is the set of all central points for $\mathcal{A}$, and $r(\mathcal{A})$ is the radius of $\mathcal{A}$.

In fact $r(F)$, $c(F)$ (see Definition 3) can be defined as $r(\mathcal{A})$, $c(\mathcal{A})$ for $\mathcal{A} = \{F^n(Y)\colon n \in N\}$.

It should be noted that $r(\mathcal{A})$, $c(\mathcal{A})$ as in Definition 11 are simplified versions of the notions presented in [7, p. 314] (asymptotic radius/center of decreasing net of sets).

Lemma 12. *Let (X, d) be a bead space and let $\mathcal{A}$ be a family of nonempty and bounded subsets of X directed by $\supset$. Then $c(\mathcal{A})$ consists of at most one point. If $\{x_A\} = c(A)$, $A \in \mathcal{A}$ then $(x_A)_{A\in\mathcal{A}}$ is a Cauchy net and $\lim_{A\in\mathcal{A}} x_A = x$ means $\{x\} = c(\mathcal{A})$. In particular if (X, d) is a complete bead space then $c(\mathcal{A})$ is a singleton. If $\mathcal{A}$ consists of compact sets then $c(\mathcal{A}) = c(\bigcap \mathcal{A})$, $r(\mathcal{A}) = r(\bigcap \mathcal{A})$ hold.*

Proof. Set $r = r(\mathcal{A})$. Suppose $x, y \in c(\mathcal{A})$ and $d(x, y) \geqslant \beta > 0$. In view of (10) there exist $C, D \in \mathcal{A}$ such that $C \subset B(x, r+\delta)$ and $D \subset B(y, r+\delta)$ for δ as in (5). There is a set $E \in \mathcal{A}$ such that

$$E \subset C \cap D \subset B(x, r+\delta) \cap B(y, r+\delta) \subset B(z, r-\delta)$$

which means $r(\mathcal{A}) < r$, a contradiction. Now assume $\{x_A\} = c(A)$, $A \in \mathcal{A}$ and suppose $(x_A)_{A\in\mathcal{A}}$ is not a Cauchy net. Then for x_C, x_D such that $d(x_C, x_D) \geqslant \beta$ we set $x = x_C$, $y = x_D$ and we repeat the previous reasoning. Now it is clear that for our bead space being complete $c(\mathcal{A})$ is a singleton. If $\mathcal{A}$ consists of compact sets and $\bigcap \mathcal{A} \subset U = B(x, t)$ then by [4, F(a), p. 163] an intersection of finitely many members of $\mathcal{A}$ is contained in U. Consequently there exists a member of $\mathcal{A}$ contained in U and we get the final equalities. $\square$

From Lemma 12 we obtain the following extension of [9, Lemma 8].

Lemma 13. *Assume that (X, d) is a bead space, $\emptyset \neq Y \subset X$ a bounded set and $F : Y \to 2^Y$ a mapping. Then $c(F)$ consists of at most one point. If $\{x_n\} = c(F^n(Y))$, $n \in N$ then $(x_n)_{n\in N}$ is a Cauchy sequence and $\lim_{n\to\infty} x_n = x$ means $c(F) = \{x\}$. In particular if (X, d) is a complete bead space then $c(F)$ is a singleton.*

Proof. We apply Lemma 12 to $A_n = F^n(Y)$. □

From Lemma 13 and Theorem 5 we obtain the following two theorems:

Theorem 14. *Let (X,d) be a bead space and let $f : X \to X$ be a mapping. Assume that $\emptyset \neq Y \subset X$ is a bounded set such that $f(Y) \subset Y$ and $x \in c(f|_Y)$. If condition (4) is satisfied then x is a fixed point for f.*

Proof. In view of Lemma 13 $c(f|_Y)$ is a singleton and we apply Theorem 5. □

Theorem 15. *Let (X,d) be a complete bead space and let $f : X \to X$ be a mapping. Assume that $\emptyset \neq Y \subset X$ is a bounded set such that $f(Y) \subset Y$. Then there exists a unique point $x \in c(f|_Y)$; if condition (4) is satisfied then x is a fixed point for f.*

Each nonexpansive mapping satisfies (4) and in consequence from the previous two theorems we obtain respectively.

Theorem 16. *Let (X,d) be a bead space and let $f : X \to X$ be a nonexpansive mapping. Assume that $\emptyset \neq Y \subset X$ is a bounded set such that $f(Y) \subset Y$ and $x \in c(f|_Y)$. Then x is a fixed point for f.*

Theorem 17. *Let (X,d) be a complete bead space and let $f : X \to X$ be a nonexpansive mapping. Assume that $\emptyset \neq Y \subset X$ is a bounded set such that $f(Y) \subset Y$. Then $c(f|_Y)$ is a singleton consisting of a fixed point for f.*

Each of the previous four theorems extends the well-known theorem of Browder–Göhde–Kirk for Hilbert spaces [3, Theorem (1.3), p. 52] and for uniformly convex spaces [3, (C.1)(b), p. 76].

Now let us consider multivalued mappings.

In what follows if $\{x\} = c(F(z))$ then we adopt $(c \circ F)(z) = x$.

Theorem 18. *Let (X,d) be a metric space and let $F : X \to 2^X$ be a mapping with $c(F(x)) \subset F(x)$, $x \in X$. Assume that $f = c \circ F : Z \to X$ is a mapping. If the other assumptions of one of Theorems 14–17 are satisfied then F has a fixed point.*

Proof. If x is a fixed point for f then we have $x = f(x) = (c \circ F)(x) \in F(x)$. □

Another idea leads to some fixed point theorems for continuous multivalued mappings.

Lemma 19. *Let (X,d) be a bead space and let $\mathcal{A} = \{A_n\colon\ n \in N\}$ be a family of nonempty and bounded subsets of X such that $\lim_{n\to\infty} D(A_n, A) = 0$. Then we have $\lim_{n\to\infty} r(A_n) = r(A)$. If $\{x_n\} = c(A_n)$, $n \in N$ then $(x_n)_{n\in N}$ is a Cauchy sequence and $\lim_{n\to\infty} x_n = x$ means that $c(A) = \{x\}$.*

Proof. Let us consider $C_n = \bigcup_{k=n}^{\infty} A_k$. Clearly $\mathcal{C} = \{C_n\colon\ n \in N\}$ is a decreasing family of nonempty and bounded subsets of X. In view of Lemma 12 $c(\mathcal{C})$ consists of at most one point. For $r_n = r(A_n)$ there exists $\epsilon_n > 0$ such that $A_n \subset C_n \subset B(x_n, r_n + \epsilon_n)$, $n \in N$ and $\lim_{n\to\infty} \epsilon_n = 0$. Therefore $r = r(A) = r(\mathcal{C}) = \lim_{n\to\infty} r_n$ and $(x_n)_{n\in N}$ is a Cauchy sequence (the proof as in Lemma 12). For $x = \lim_{n\to\infty} x_n$ and any $\delta > 0$ we have $A_n \subset C_n \subset B(x, r+\delta)$, $n \geqslant n_0$ which means that $\{x\} = c(A) = c(\mathcal{C})$. □

Lemma 20. *Let (Z,ρ) be a metric space and (X,d) a bead space. If $F : (Z,\rho) \ni z \mapsto F(z) \in (2^X, D)$ is a continuous mapping, $F(z)$, $z \in Z$ are bounded and $c(F(z)) \neq \emptyset$, $z \in Z$ (e.g. if (X,d) is complete) then $c \circ F : Z \to X$ is continuous.*

Proof. By continuity of F from $\lim_{n\to\infty} z_n = z$ follows $\lim_{n\to\infty} F(z_n) = F(z)$ in $(2^X, D)$ and then by Lemma 19 we have $\lim_{n\to\infty} (c \circ F)(z_n) = (c \circ F)(z)$ which means the continuity of $c \circ F$. □

As a corollary from the previous lemma we obtain

Theorem 21. *Let (Z,ρ) be a metric space and (X,d) a bead space. If $F : (Z,\rho) \ni z \mapsto F(z) \in (2^X, D)$ is a continuous mapping, $F(z)$, $z \in Z$ are bounded and $\emptyset \neq c(F(z)) \subset F(z)$, $z \in Z$ then $c \circ F$ is a continuous selection for F.*

Another consequence of Lemma 20 is the following

Theorem 22. *Let X be a nonempty convex set in a normed bead space $(Y, \|\cdot\|)$. If $F : X \ni x \mapsto F(x) \in (2^Y, D)$ is a continuous mapping, $\emptyset \neq c(F(x)) \subset F(x)$, $x \in X$ and $\overline{\{(c \circ F)(x)\colon\ x \in X\}} \subset X$ is compact then F has a fixed point.*

Proof. In view of Theorem 21 $c \circ F$ is a continuous selection for F. Consequently $c \circ F : X \to X$ is a compact map and by Schauder Theorem it has a fixed point. □

The property to follow is certainly known:

Proposition 23. *Let $(X, \|\cdot\|)$ be a normed space and $\emptyset \neq A \subset X$ a bounded set. Then $c(A)$ is closed and convex.*

Proof. If $x \in \overline{c(A)}$ then $A \subset B(x, r(A) + \delta)$ for each $\delta > 0$ and consequently we obtain $x \in c(A)$. If $x, y \in c(A)$ then for any $z \in A$, $\alpha \in [0, 1]$ we have

$$\big\|z - \big(\alpha x + (1-\alpha)y\big)\big\| \leqslant \alpha\|z - x\| + (1-\alpha)\|z - y\| \leqslant r(A).$$

Thus $c(A)$ is convex. □

An interesting problem is the relation between A and $c(A)$.

For $(X, \|\cdot\|)$ being a normed space, $x \in X$, $g \in X^*$, $r > 0$ let us write $B_g(x, r) = \{y \in B(x, r):\ g(x)g(y) \leqslant 0\}$.

Lemma 24. *Let $(X, \|\cdot\|)$ be a normed space. Assume that the following*

$$\text{for each } r > 0,\ x \in X,\ g \in X^* \text{ such that } g(x) \neq 0 \text{ there exist } \delta > 0,\ z \in X,\ \text{with } B_g(x, r) \subset B(z, r - \delta) \tag{11}$$

holds. Then for each $\emptyset \neq A \subset X$ being bounded closed and convex we have $c(A) \subset A$.

Proof. Suppose $c(A)$ contains a point $x \notin A$. Then x can be strongly separated from A by a $g \in X^*$ [5, 14.4, p. 119]. We may assume that $g(x) > 0$ and $g(y) \leqslant 0$ is satisfied for each $y \in A$. Then for $r = r(A)$ we have $\overline{A} \subset \overline{B_g(x, r)}$ and in view of (11) we obtain $A \subset B(z, r - \delta)$ for a $\delta > 0$. It means that $r(A) \leqslant r(A) - \delta$, a contradiction. □

With the help of the Pythagoras Theorem we state that for X being an inner product space condition (11) holds. On the other hand V. Klee in [6] has proved that if $(X, \|\cdot\|)$ is a Banach space with dimension at least 3 then $c(A) \subset A$ for all bounded closed convex subsets of X implies X is a Hilbert space. Hence we obtain

Corollary 25. *Let $(X, \|\cdot\|)$ be a Banach space with dimension at least* 3. *Then* (11) *holds iff X is a Hilbert space.*

In the context of Lemma 24 it is worth of being noted that the last two pages of [2] contain some examples of bounded closed convex sets (in normed spaces) which do not contain their centres.

From Theorem 21 follows

Theorem 26. *Let (Z, ρ) be a metric space and X a Hilbert space. If $F : (Z, \rho) \ni z \mapsto F(z) \in (2^X, D)$ is a continuous mapping* (*for D being the Hausdoff metric*) *and all values of F are bounded closed convex then $c \circ F$ is a continuous selection for F.*

From Theorem 22 we obtain

Theorem 27. *Let X be a nonempty convex set in a Hilbert space Y. If $F : X \ni x \mapsto F(x) \in (2^Y, D)$ is a continuous mapping, all values of F are bounded closed convex and $\overline{\{(c \circ F)(x):\ x \in X\}} \subset X$ is compact then F has a fixed point.*

The theorem to follow seems to be of a particular interest though the first part of it is known:

Theorem 28. *Let X be a Hilbert space and let $\mathcal{C}$ be a family of bounded closed convex sets having the finite intersection property. Then $\bigcap \mathcal{C}$ is nonempty. If $\mathcal{C}$ is directed by $\supset$ then $\emptyset \neq c(\mathcal{C}) \subset \bigcap \mathcal{C}$ holds*; *if in addition $\mathcal{C}$ consists of compact sets then $c(\mathcal{C}) = c(\bigcap \mathcal{C})$ and $r(\mathcal{C}) = r(\bigcap \mathcal{C})$.*

Proof. Let $\mathcal{A}$ be the family of all finite intersections of the sets from $\mathcal{C}$. Clearly $\mathcal{A}$ is a family of nonempty bounded closed convex sets. From Corollary 9 and Lemma 10 it follows that for each $A \in \mathcal{A}$ there exists $\{x_A\} = c(A) \subset A$. The family $\mathcal{A}$ is directed by $\supset$ and in view of Lemma 12 $(x_A)_{A \in \mathcal{A}}$ is a Cauchy net which converges, say, to x. We have $x \in \bigcap \mathcal{A} \subset \bigcap \mathcal{C}$ and once again by Lemma 12 we get $\{x\} = c(\mathcal{A})$ and the remaining part of our theorem. □

Remark 29. By the Mazur–Šmulian Theorem (see [3, Theorem (C.9)(b), p. 605]) every bounded closed convex set in Hilbert space is weakly compact. So the first part of Theorem 28 follows from the Mazur–Šmulian Theorem.

We should give an additional comment to [1,2]. Both papers are devoted to the problem of the existence of central points. They contain some conditions of spaces under consideration and it is intersting to compare them with our condition (5). The paper [2] concerns normed spaces. The authors use the notions of strict convexity, uniform convexity in every direction (Definition 1.3) and the crucial "property H in every direction" (Definition 1.7). All these ideas strongly depend on the linear structure of the space and therefore they do not correspond to our condition (5). Now let us consider [1]. A metric space X is said to be (netwise) c.p. complete if every bounded net has at least one central point. The authors prove that every complete metric space having the chained exchangeability property is netwise c.p. complete (Theorem 3). The notion of the "chained exchangeability property" is based on "ϵ-exchangeability". Two balls $B(x, r + \delta)$ and $B(y, r)$ for $\delta \geqslant 0$ are ϵ-exchangeable if there exists a ball $B(z, r)$ such that $B(x, r + \delta) \cap B(y, r) \subset B(z, r)$ and $d(x, z) \leqslant \epsilon$. The just mentioned inclusion does not imply the inclusion from (5) (e.g. for R^2 normed by $\max\{|x|, |y|\}$). On the other hand from (5) it does not even follow that $d(x, z) < d(x, y)$ and abstract bead space fails to have the chained exchangeability property (though finding the respective example can be exhausting).

As regards our condition (11), in view of Corollary 25 it is just another way to get a Hilbert space and the answer to the question of "how far is the chained exchangeability property from (11)" is "very".

References

[1] C.L. Anderson, W.H. Hyams, C.K. McKnight, Center points of nets, Canad. J. Math. 27 (1975) 418–422.
[2] J.R. Calder, W.P. Coleman, R.L. Harris, Centers of infinite bounded sets in normed spaces, Canad. J. Math. 25 (1973) 986–999.
[3] A. Granas, J. Dugundji, Fixed Point Theory, Springer, New York, 2003.
[4] J.L. Kelley, General Topology, Springer, New York, 1975.
[5] J.L. Kelley, I. Namioka, Linear Topological Spaces, Springer, New York, 1976.
[6] V. Klee, Circumspheres and inner products, Math. Scand. 8 (1960) 363–370.
[7] T.C. Lim, Characterizations of normal structure, Proc. Amer. Math. Soc. 43 (1974) 313–319.
[8] T.C. Lim, On asymptotic centers and fixed points of nonexpansive mappings, Canad. J. Math. 32 (1980) 421–430.
[9] L. Pasicki, A basic fixed point theorem, Bull. Pol. Acad. Sci. Math. 54 (2006) 85–88.
[10] L. Pasicki, On discus spaces, Lect. Notes Nonlinear Anal., in press.