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Towards Lim

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ABSTRACT

The paper contains an elegant extension of the Nadler fixed point theorem for multivalued contractions (see Theorem 21). It is based on a new idea of the α-step mappings (see Definition 17) being more efficient than α-contractions. In the present paper this theorem is a tool in proving some fixed point theorems for "nonexpansive" mappings in the bead spaces (metric spaces that, roughly speaking, are modelled after convex sets in uniformly convex spaces). More precisely the mappings are nonexpansive on a set with respect to only one point – the centre of this set (see condition (4)). The results are pretty general. At first we assume that the value of the mapping under consideration at this central point looks "sharp" (see Definition 6). This idea leads to a group of theorems (based on Theorem 7). Their proofs are compact and the theorems, in particular, are natural extensions of the classical results for (usual) nonexpansive mappings. In the second part we apply the idea of Lim to investigate the regular sequences and here the proofs are based on our extension of Nadler's Theorem. In consequence we obtain some fixed point theorems that generalise the classical Lim Theorem for multivalued nonexpansive mappings (see e.g. Theorem 26).

Definition 1. ([8, Definition 11]) Let (X, d) be a metric space and $\mathcal{A}$ a family of bounded nonempty subsets of X. An $x \in X$ is a central point for $\mathcal{A}$ if

$$\begin{aligned} r(\mathcal{A}) &:= \inf\bigl\{t \in (0,\infty)\colon \text{ there exist } A \in \mathcal{A},\ z \in X \text{ such that } A \subset B(z,t)\bigr\} \\ &= \inf\bigl\{t \in (0,\infty)\colon \text{ there exists } A \in \mathcal{A} \text{ with } A \subset B(x,t)\bigr\}. \end{aligned} \tag{1}$$

The centre $c(\mathcal{A})$ for $\mathcal{A}$ is the set of all central points for $\mathcal{A}$, and $r(\mathcal{A})$ is the radius of $\mathcal{A}$.

It should be noted that $r(\mathcal{A})$ is defined by condition (1) also for $c(\mathcal{A}) = \emptyset$.

Lim [4, p. 314] has defined the asymptotic centre and its radius for a decreasing net of sets in locally convex linear topological space. In his idea a family of seminorms was involved. Our definition for metric space is much simpler.

If $(x_n)_{n\in N}$ is a bounded sequence of points of X then for $A_n = \{x_k\colon\ k \geqslant n\}$ and $\mathcal{A} = \{A_n\colon\ n \in N\}$, $c((x_n)_{n\in N}) := c(\mathcal{A})$, $r((x_n)_{n\in N}) := r(\mathcal{A})$ are respectively the (asymptotic) centre and the (asymptotic) radius of $(x_n)_{n\in N}$ (see [2]).

Let 2^X be the family of all subsets of X. We say that $F : X \to 2^X$ is a (multivalued) mapping if $F(x) \neq \emptyset$, $x \in X$.

If Y is a nonempty bounded set in X and $F : Y \to 2^Y$ is a mapping then for $A_n = F^n(Y)$ and $\mathcal{A} = \{A_n\colon\ n \in N\}$, $c(F) := c(\mathcal{A})$, $r(F) := r(\mathcal{A})$ are respectively the centre and the radius for F (see [8, Definition 7]).

The previous notions are useful in proving fixed point theorems.

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Definition 2. ([8, Definition 6]) A metric space (X,d) is a bead space if the following is satisfied

for every $r>0$, $\beta>0$ there exists a $\delta>0$ such that

for each $x, y\in X$ with $d(x,y)\geqslant\beta$ there exists a $z\in X$ such that

$$B(x,r+\delta)\cap B(y,r+\delta)\subset B(z,r-\delta). \tag{2}$$

In particular each uniformly convex space $(X,\|\cdot\|)$ is a bead space (see [8, Corollary 9]) and the same holds for convex subsets of X (see [7, Example 3]) with $z=(x+y)/2$ in condition (2).

The subsequent example shows that there exist bead spaces not being convex sets in uniformly convex spaces. This two dimensional case can be easily extended to the space being an intersection of a sphere and a respectively small ball in any finite dimensional Hilbert space. Would it work for the infinite dimensional Hilbert spaces?

Example 3. Let us consider an arc X being one fourth of the 1-sphere $S((0,0),\rho)$ in R^2 (Euclidean plane) for a $\rho>0$. For $x,y\in X$ let δ be such that $B(x,r+\delta)\cap B(y,r+\delta)\subset B((x+y)/2,r-\delta)$ (see condition (2)). It can be shown that $A=X\cap B(x,r+\delta)\cap B(y,r+\delta)\subset B(z,r-\delta)$ for z being a projection of $(x+y)/2$ on A (the proof is elementary and it is sufficient to consider $x=(-a,b)$, $y=(a,b)$, the closed balls and $r+\delta=2a$). Therefore X is a (complete) bead space while it is not a convex subset of the uniformly convex space R^2.

The following is a part of [8, Lemma 12].

Lemma 4. *Let (X,d) be a bead space and let $\mathcal{A}$ be a family of nonempty and bounded subsets of X directed by $\supset$. Then $c(\mathcal{A})$ consists of at most one point. If in addition (X,d) is complete then $c(\mathcal{A})$ is a singleton.*

Remark 5. From the previous lemma we obtain the respective results for sets, sequences of points, and mappings (see [8, Lemmas 10, 13]).

Definition 6. Let (X,d) be a metric space and for an $x\in X$ let $F(x)$ be a nonempty subset of X. We say that $F(x)$ is r-pointed for an $r\geqslant 0$ if

$x\in\overline{F}(x)$ or there exist $\delta>0$, $z\in X$ such that

$$B(x,r+\delta)\cap B\big(F(x),r+\delta\big)\subset B(z,r-\delta). \tag{3}$$

The set $F(x)$ is pointed if it is r-pointed for each $r\geqslant 0$. A mapping $F:X\to 2^X$ is r-pointed (pointed) if $F(x)$ is r-pointed (pointed) for each $x\in X$.

Intuitively condition (3) means that $F(x)$ does not look “flat” when observed from x.

Clearly if (X,d) is a bead space and $F(x)$ is a singleton then it is pointed and therefore any mapping $F:X\to X$ is pointed.

If $(X,\|\cdot\|)$ is uniformly convex and $F(x)$ is a closed subset of $\{x+\alpha(y-x):\ \alpha\geqslant 0\}$ for a $y\in X$, $y\neq x$ (ray) then $F(x)$ is pointed.

Let $F:X\to 2^X$ be a mapping and let $\emptyset\neq Y\subset X$. We say that $G:Y\to 2^X$ is a (multivalued) selection for $F_{|Y}$ if for each $x\in Y$ we have $\emptyset\neq G(x)\subset F(x)$.

Theorem 7. *Let (X,d) be a metric space and $F:X\to 2^X$ a mapping. Assume that $\emptyset\neq Y\subset X$ is a bounded set, $G:Y\to 2^Y$ is a selection for $F_{|Y}$ and $x\in c(G)$. If $F(x)$ is closed, $r(G)$-pointed and satisfies*

$$G(y)\subset\overline{B}\big(F(x),d(x,y)\big)\quad\textit{for all } y\in Y \tag{4}$$

then $x\in F(x)$.

Proof. Let us adopt $r=r(G)$. Then for any $\delta>0$, large n and $y\in G^n(Y)\subset Y$, $z\in G(y)$ we have $z\in G^{n+1}(Y)\subset G^n(Y)\subset B(x,r+\delta)$. On the other hand (4) implies $d(F(x),z)\leqslant d(x,y)<r+\delta$ (i.e. $G^{n+1}(Y)\subset B(F(x),r+\delta)$). Now for $r=0$ we obtain $d(x,F(x))\leqslant d(x,z)+d(z,F(x))<2\delta$ which implies $x\in F(x)$. Suppose $x\notin F(x)$ and consequently $r>0$. From condition (3) we obtain $G^{n+1}(Y)=G^n(Y)\cap G^{n+1}(Y)\subset B(x,r+\delta)\cap B(F(x),r+\delta)\subset B(z,r-\delta)$ for some $z\in X$. It means that $r(G)=r\leqslant r-\delta$, a contradiction. □

One of the ways a selection G can be defined is to choose any point $y_1\in X$ and $y_{n+1}\in F(y_n)$, $n\in N$. Then we adopt $G(y_n)=y_{n+1}$ and $Y=\{y_n:\ n\in N\}$ – an orbit of G.

For $G=F$ condition (4) means that F is “nonexpansive” with respect to one point x.

From Theorem 7 follows

Theorem 8. *Let (X, d) be a complete bead space and $F : X \to 2^X$ a mapping. Assume that $\emptyset \neq Y \subset X$ is a bounded set and $G : Y \to 2^Y$ is a selection for $F_{|Y}$. If from $x \in c(G)$ it follows that $F(x)$ is closed, $r(G)$-pointed and (4) holds then F has a fixed point in $c(G)$.*

Proof. In view of [8, Lemma 13] $c(G)$ is a singleton and we apply Theorem 7. □

Remark 9. As regards Theorems 7 and 8, if (4) is satisfied for G in place of F ($G(x)$ must be defined) then we get $x \in F(x)$; if in addition $G(x)$ is closed and $r(G)$-pointed then $x \in G(x) \subset F(x)$.

Let us recall that a mapping $F : X \to 2^X$ is nonexpansive if (4) holds for $G = F$, all $x \in X$ and $Y = X$.
If (X, d) is a metric space, $A, B \subset X$ are nonempty and

$$\max\left\{\sup_{x\in A} d(x, B), \sup_{y\in B} d(A, y)\right\}$$

is finite then we denote it by $D(A, B)$ (the Hausdorff distance).
It can be stated that $F : X \to 2^X$ is nonexpansive if and only if the following is satisfied

$$D\big(F(x), F(y)\big) \leqslant d(x, y) \quad \text{for all } x, y \in X.$$

Now we present the respective versions of Theorem 7, Theorem 8 for F being nonexpansive.

Theorem 10. *(Cf. Theorem 7.) Let (X, d) be a metric space and $F : X \to 2^X$ a nonexpansive mapping. Assume that $\emptyset \neq Y \subset X$ is a bounded set, $G : Y \to 2^Y$ is a selection for $F_{|Y}$ and $x \in c(G)$. If $F(x)$ is closed and $r(G)$-pointed then $x \in F(x)$.*

Theorem 11. *(Cf. Theorem 8.) Let (X, d) be a complete bead space and $F : X \to 2^X$ a nonexpansive mapping. Assume that $\emptyset \neq Y \subset X$ is a bounded set and $G : Y \to 2^Y$ is a selection for $F_{|Y}$. If from $x \in c(G)$ it follows that $F(x)$ is closed and $r(G)$-pointed then F has a fixed point in $c(G)$.*

The proof of Theorem 7 is compact and the subsequent results seem to be fairly general. They raise an interesting question:

Problem 12. What are the shapes of pointed sets (e.g. in uniformly convex spaces)?

Now we are going to prove some results related to Lim's Theorem [5, Theorem 1].
Let us recall that a bounded sequence $(x_n)_{n\in N}$ in a metric space (X, d) is regular if for each of its subsequences $(x_{k_n})_{n\in N}$ we have $r((x_{k_n})_{n\in N}) = r((x_n)_{n\in N})$; $(x_n)_{n\in N}$ is almost convergent if in addition $c((x_{k_n})_{n\in N}) = c((x_n)_{n\in N})$ (see [3]). It is known that any bounded sequence in a metric space contains a regular subsequence [3, Lemma 2].

Lemma 13. *Let (X, d) be a bead space. If $(x_n)_{n\in N}$ is a regular sequence in X then it is almost convergent.*

Proof. Suppose $c((x_{k_n})_{n\in N}) = \{x\} \neq \{y\} = c((x_n)_{n\in N})$ and let $r = r((x_{k_n})_{n\in N}) = r((x_n)_{n\in N})$. From $x \neq y$ it follows that $r > 0$ (otherwise $(x_n)_{n\in N}$ would converge to y). We have $x_{k_n} \in B(x, r+\delta) \cap B(y, r+\delta) \subset B(z, r-\delta)$ for a $\delta > 0$ and large n (see (2)). Hence follows $r((x_{k_n})_{n\in N}) = r \leqslant r - \delta$, a contradiction. □

Theorem 14. *Let (X, d) be a bead space and $F : X \to 2^X$ a mapping. Assume that $(x_n)_{n\in N}$ is a regular sequence in X, $Y = \{x_n\colon n \in N\}$ and $G : Y \to X$ is a selection for $F_{|Y}$ such that*

$$\lim_{n\to\infty} d\big(x_n, G(x_n)\big) = 0 \tag{5}$$

holds. If $x \in c((x_n)_{n\in N})$, $F(x)$ is compact and (4) is satisfied then $x \in F(x)$.

Proof. In view of (4) we have

$$d\big(x_n, F(x)\big) \leqslant d\big(x_n, G(x_n)\big) + d\big(G(x_n), F(x)\big) \leqslant d\big(x_n, G(x_n)\big) + d(x_n, x).$$

Therefore (see (5)) there exist $z_n \in F(x)$, $n \in N$ such that $\limsup_{n\to\infty} d(x_n, z_n) \leqslant r = r((x_n)_{n\in N})$. The sequence $(z_n)_{n\in N}$ has a subsequence $(z_{k_n})_{n\in N}$ that converges, say to a point z in the compact set $F(x)$. Consequently (Lemma 13) $z \in c((x_{k_n})_{n\in N}) = c((x_n)_{n\in N}) = \{x\}$ as $(x_n)_{n\in N}$ is regular. Thus we have $x = z \in F(x)$. □

From Theorem 14 we obtain

Theorem 15. *Let (X,d) be a bead space and $F: X \to 2^X$ a nonexpansive mapping. Assume that $(x_n)_{n\in N}$ is a regular sequence in X, $Y = \{x_n\colon n \in N\}$ and $G: Y \to X$ is a selection for $F_{|Y}$ as to satisfy* (5)*. If $x \in c((x_n)_{n\in N})$ and $F(x)$ is compact then $x \in F(x)$.*

Proof. If F is nonexpansive then (4) holds and we apply Theorem 14. □

Remark 16. If (X,d) is a complete bead space then for bounded $(x_n)_{n\in N}$ the set $c((x_n)_{n\in N})$ is a singleton and the conclusion of Theorems 14, 15 can be modified: If from $x \in c((x_n)_{n\in N})$ it follows that $F(x)$ is compact then F has a fixed point in $c((x_n)_{n\in N})$.

Let (X,d) be a metric space. Let us recall that a mapping $F: X \to 2^X$ is an α-contraction if $0 \leqslant \alpha < 1$ and $D(F(x), F(y)) \leqslant \alpha d(x,y)$ for all $x, y \in X$.

Definition 17. Let (X,d) be a metric space and $F: X \to 2^X$ a mapping. We say that F is an α-step if $\alpha \geqslant 0$ and the following is satisfied

$$\text{for each } x \in X,\ y \in F(x) \text{ there exists a } z \in F(y) \text{ such that } d(z,y) \leqslant \alpha d(y,x). \tag{6}$$

Lemma 18. *Let (X,d) be a metric space and $F: X \to 2^X$ an α-contraction. Then for any $\epsilon > 0$ the mapping F is $(\alpha+\epsilon)$-step. If all values of F are compact then F is an α-step. If F is nonexpansive and compact valued then it is a 1-step.*

Proof. Let $y \in F(x)$ be arbitrary. It is sufficient to consider $y \neq x$. Then we have $d(F(y), y) \leqslant D(F(y), F(x)) \leqslant \alpha d(y,x)$. For any $\epsilon > 0$ there exists a $z \in F(y)$ such that $d(z,y) \leqslant d(F(y), y) + \epsilon d(y,x)$ (or $\epsilon = 0$ if $F(y)$ is compact) and we obtain (6) with $(\alpha+\epsilon)$ in place of α. □

The subsequent example shows that the α-step mappings can be more efficient than the α-contractions.

Example 19. For $X = [0,\infty)$ and a $\beta \in (0,1)$ let us adopt $F(x) = [\beta x, x]$, $x \in X$. We have $D(F(x), F(y)) = \max\{|\beta x - \beta y|, |x-y|\} = |x-y|$, i.e. F is nonexpansive. On the other hand for any $y \in [\beta x, x] = F(x)$, $z = y \in [\beta y, y] = F(y)$ and any $\alpha \geqslant 0$ we have $d(z,y) = 0 \leqslant \alpha d(x,y)$, i.e. F is an α-step mapping.

There is a natural question:

Problem 20. Under which assumptions on the bead space (or on F) can it be guaranteed the existence of a sequence and a selection G satisfying (5) for F being a 1-step (or even nonexpansive)?

Let us recall that for X, Y being topological spaces a mapping $F: X \to 2^Y$ is usc if for each $x \in X$ and any neighbourhood V of $F(x)$ there exists a neighbourhood U of x such that $F(U) \subset V$.

If Y is a regular space, $F: X \to 2^Y$ is usc and all values of F are closed then the graph of F is closed (see the proof of [1, Lemma, p. 285]). This holds in particular for multivalued mappings F into a metric space (Y,d) where F is closed-valued and continuous with respect to the Hausdorff metric of 2^Y.

In view of Example 19, Lemma 18 and the previous comment the following extends the well-known Nadler Theorem for contractions.

Theorem 21. *(Cf. [6, Theorem 5].) Let (X,d) be a metric space and $F: X \to 2^X$ an α-step with $\alpha < 1$. Then for each $\delta > 0$ there exists an $x \in X$ such that $d(x, F(x)) < \delta$. If in addition X is complete and the graph of F is closed then F has a fixed point.*

Proof. With the help of (6) we define a Cauchy sequence $(x_n)_{n\in N}$ such that $x_{n+1} \in F(x_n)$ and $d(x_{n+1}, x_n) \leqslant \alpha^n k$, $n \in N$. If X is complete then $(x_n)_{n\in N}$ converges, say to an $x \in X$. If the graph of F is closed then $x_{n+1} \in F(x_n)$ tend to a point of $F(x)$. □

Lemma 22. *Let X be a bounded convex set in a normed space, $x_0 \in X$ and let $F: X \to 2^X$ be a mapping satisfying*

$$\begin{aligned}&\text{for each } x \in X,\ u \in F(x),\ \kappa > 1 \text{ and } \lambda \in (0,1)\\ &\quad \text{there exists a } v \in F\big((1-\lambda)x_0 + \lambda u\big) \text{ such that } d(v,u) \leqslant \kappa d\big((1-\lambda)x_0 + \lambda u, x\big).\end{aligned} \tag{7}$$

Then there exist a regular sequence $(x_n)_{n\in N}$ in X and, for $Y = \{x_n\colon n \in N\}$, a selection G for $F_{|Y}$ as to satisfy (5)*.*

Proof. Let us adopt $F_\lambda(x) = (1-\lambda)x_0 + \lambda F(x)$, $x \in X$. For the simplicity of formulas we assume $x_0 = 0$ (do not agree if you are ambitious :)). Then for x, u, v as in (7) we put $z = \lambda v$ and $y = \lambda u$. Now for $y \in F_\lambda(u) = \lambda F(u)$, $z \in F_\lambda(y) = \lambda F(y)$ we have

$$d(z,y) = d(\lambda v, \lambda u) = \lambda d(v,u) \leqslant \lambda\kappa d(\lambda u, x) = \lambda\kappa d(y,x)$$

i.e. F_λ is a $\lambda\kappa$-step. In view of Theorem 21 for $\lambda\kappa < 1$ and any $\delta > 0$ there exists an x such that $d(x, F_\lambda(x)) < \delta$, i.e. $d(x, \lambda y) < \delta$ for a $y \in F(x)$. We have

$$\begin{aligned}\|x-y\| &= \big\|x-\lambda y-(1-\lambda)y\big\| \leqslant \|x-\lambda y\| + (1-\lambda)\|y\| \\ &< \delta + (1-\lambda)\|y\| \leqslant \delta + (1-\lambda)k,\end{aligned}$$

X being bounded. Now for $\lambda_n \to 1$ and $\delta_n \to 0$ we obtain $y_n = G(x_n) \in F(x_n)$ as to satisfy (5). □

Remark 23. If F for X as in Lemma 22 is nonexpansive then (7) holds; if in addition all values of F are compact then (7) holds for $\kappa = 1$.

Any nonempty convex set X in a uniformly convex space is a bead space. Therefore (see Remark 16) Theorems 14, 15 extend the following well-known theorem of Lim.

Theorem 24. *([5, Theorem 1]) Let X be a closed bounded and convex set in a uniformly convex Banach space and let $F: X \to 2^X$ be a nonexpansive mapping with compact values. Then F has a fixed point.*

Let us present the respective version of Theorem 14.

Theorem 25. *Let X be a bounded complete convex set in a uniformly convex space. Assume that, for an $x_0 \in X$, $F: X \to 2^X$ is a mapping satisfying (7). Then there exist a regular sequence $(x_n)_{n\in N}$ in X and, for $Y = \{x_n\colon n \in N\}$, a selection $G: Y \to X$ for $F_{|Y}$ satisfying (5). If from $x \in c((x_n)_{n\in N})$ (in X) it follows that $F(x)$ is compact and (4) holds then F has a fixed point in $c((x_n)_{n\in N})$.*

Proof. In view of Lemma 22 there exist a regular sequence $(x_n)_{n\in N}$ and a selection G as to satisfy (5). X is a complete bead space and therefore $c((x_n)_{n\in N})$ is a singleton in X. Now we apply Theorem 14 (see Remark 16). □

It should be stressed that for X being a complete convex subset of a uniformly convex space the centres of its subsets may be placed outside X, but for X treated as a bead space itself the centres are relative to X and they belong to X. This comment concerns also Theorem 26.

Let us formulate a more classical version of Theorem 25. It also extends the Lim Theorem as (4) is much more general than the nonexpansivity condition.

Theorem 26. *Let X be a bounded complete convex set in a uniformly convex space. Assume that $F: X \to 2^X$ is a mapping and $x_n \in X$, $y_n \in F(x_n)$ are such that (cf. (5))*

$$\lim_{n\to\infty} d(x_n, y_n) = 0$$

and for $x \in c((x_n)_{n\in N})$ (with respect to X)

$$d\big(y_n, F(x)\big) \leqslant d(x_n, x) \quad \text{for each } n \in N$$

(cf. (4)) hold. If $F(x)$ is compact then $x \in F(x)$.

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