

Transitivity and variational principles

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Abstract

We apply an order reasoning to mappings satisfying the triangle inequality. This general approach yields the Ekeland's variational principle as one of the consequences. In addition we obtain an extension of the Brøndsted variational principle and of the Takahashi fixed point theorem.

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The variational principles require a kind of order. One can first assume that a set is equipped with an “order” relation and then we demand the mapping(s) under consideration to fulfil some additional conditions (see e.g., [1], [6], [12], [13]). Another way is to define an “order” with the help of special mappings on a space equipped with some topological features. Such orders usually look like

$$x \leq y \text{ iff } \psi(y) + d(y, x) - \psi(x) \leq 0;$$

see, for instance, [3], [8], [9], [10], [13]. Our idea is to use a mapping $\varphi: X \times X \rightarrow R$ satisfying the triangle inequality which as well “orders” X and defines a kind of convergence in X .

Let X be a nonempty set and $\varphi: X \times X \rightarrow R$ a mapping satisfying:

$$\varphi(z, x) \leq \varphi(z, y) + \varphi(y, x), \quad x, y, z \in X. \quad (1)$$

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Then from $\varphi(x, x) \leq 2\varphi(x, x)$ and (1) we obtain

$$0 \leq \varphi(x, x) \leq \varphi(x, y) + \varphi(y, x), \quad x, y \in X. \quad (2)$$

The subsequent three definitions describe the “world” of the present paper.

Definition 1. Let X be a nonempty set and $\varphi: X \times X \rightarrow R$ a mapping satisfying (1). Then

- (i) $\mathcal{S} = (X, \varphi, x_0)$ is a local structure if $\varphi(\cdot, x_0)$ has a finite lower bound,
- (ii) $\mathcal{S} = (X, \varphi)$ is a global structure if $\varphi(\cdot, x_0)$ has a finite lower bound for each $x_0 \in X$.

By a **structure** we will mean - according to the context - either of these.

Definition 2. Let \mathcal{S} be a structure. Then $(x_n)_{n \in N}$ is a **decreasing Cauchy sequence** in X if for each $\epsilon > 0$ there exists an $n_0 \in N$ such that each $m, n \in N$, $n_0 < m < n$ yield $-\epsilon < \varphi(x_n, x_m) < 0$; if the last “ $<$ ” is replaced by “ \leq ”, then $(x_n)_{n \in N}$ is a **Cauchy sequence**.

Definition 3. A structure is **almost complete** if for each decreasing Cauchy sequence $(x_n)_{n \in N}$ in X there exists an $x \in X$ such that $\liminf_{n \rightarrow \infty} \varphi(x, x_n) \leq 0$, and it is **complete** if the same holds for each Cauchy sequence $(x_n)_{n \in N}$ in X .

Let $\psi: X \rightarrow R$ be a mapping having a finite lower bound. Then for $\varphi(y, x) = \psi(y) - \psi(x)$ (1) holds and $\varphi(\cdot, x_0)$ has a finite lower bound for each $x_0 \in X$. Therefore (X, φ) is a structure. Condition $-\epsilon < \varphi(x_n, x_m) \leq 0$, $m < n$ means $0 \leq \psi(x_m) - \psi(x_n) < \epsilon$, i.e. $(\psi(x_n))_{n \in N}$ is a bounded nonincreasing sequence in R . Then for any $s \in X$ such that $\psi(s) \leq \psi(x_n)$, $n \in N$ we obtain $\lim_{n \rightarrow \infty} \varphi(s, x_n) \leq 0$. From this reasoning follows.

Lemma 4. Let $\psi: X \rightarrow R$ be a mapping such that $\psi(s) \leq \psi(x)$, $x \in X$. Then for $\varphi(y, x) = \psi(y) - \psi(x)$, $x, y \in X$ (X, φ) is a complete structure.

In particular, Lemma 4 yields.

Lemma 5. Let (X, d) be a metric space and $\psi(x) = d(s, x)$, $x \in X$. Then for

$$\varphi(y, x) = \psi(y) - \psi(x) = d(s, y) - d(s, x), \quad x, y \in X$$

(X, φ) is a complete structure.

If $(x_n)_{n \in \mathbb{N}}$ is a Cauchy sequence in (X, φ) (see Lemma 5) then $(d(s, x_n))_{n \in \mathbb{N}}$ is nonincreasing; in particular, if $\lim_{n \rightarrow \infty} d(s, x_n) = \inf\{\limsup_{n \rightarrow \infty} d(x, x_n) : x \in X\}$ then s is an asymptotic centre of $(x_n)_{n \in \mathbb{N}}$ (see [7]).

Lemma 6. *Let (X, d) be a complete metric space and $\psi: X \rightarrow \mathbb{R}$ be a lower semicontinuous mapping with a finite lower bound. If each decreasing Cauchy sequence in (X, φ) ($\varphi(y, x) = \psi(y) - \psi(x)$) is a Cauchy sequence in (X, d) , then (X, φ) is an almost complete structure.*

The next example is more advanced. Let (X, d) be a metric space, $\delta = \lambda d$ for a $\lambda > 0$ and let $\psi: X \rightarrow \mathbb{R}$ be a mapping. Let us consider

$$\varphi(y, x) = \psi(y) + \delta(y, x) - \psi(x), \quad x, y \in X. \quad (3)$$

The inequality

$$-\epsilon < \varphi(x_n, x_m) = \psi(x_n) + \delta(x_n, x_m) - \psi(x_m) \leq 0$$

means

$$\psi(x_m) - \psi(x_n) - \epsilon < \delta(x_n, x_m) \leq \psi(x_m) - \psi(x_n)$$

and therefore $(\psi(x_n))_{n \in \mathbb{N}}$ is nonincreasing ($m < n$). If ψ has a finite lower bound, then $\varphi(\cdot, x_0)$ has a finite lower bound for each $x_0 \in X$ and $(\psi(x_n))_{n \in \mathbb{N}}$ is convergent. Consequently $(x_n)_{n \in \mathbb{N}}$ is a Cauchy sequence in (X, d) . If the metric space (X, d) is complete then $(x_n)_{n \in \mathbb{N}}$ converges, say to an $x \in X$. If in addition, ψ is lower semicontinuous, then $\lim_{n \rightarrow \infty} [\psi(x) + \delta(x, x_n) - \psi(x_n)] \leq 0$. This reasoning results in:

Lemma 7. *Let (X, d) be a complete metric space and let $\psi: X \rightarrow \mathbb{R}$ be a lower semicontinuous mapping with a finite lower bound. Then for φ defined by (3) (X, φ) is a complete structure.*

Let us prove some theorems for the almost complete structures.

Theorem 8. *Let the almost complete structure (X, φ, x_0) and the logical property $P(\cdot)$ about elements of X be such that: for each $x \in X$ with $\neg P(x)$ there exists a $y \in X$ satisfying $\varphi(y, x) < 0$. Then, either $P(x_0)$, or there exists a $z \in X$ such that $P(z)$, $\varphi(z, x_0) < 0$ and $0 \leq \varphi(x, z)$, for all $x \in X$. If $0 \leq \varphi(x, z)$ is valid for all $x \in X$ and φ satisfies*

$$\varphi(x, z) = 0 \text{ implies } x = z, \text{ for all } x \in X, \quad (4)$$

then $0 < \varphi(x, z)$, for all $x \in X \setminus \{z\}$.

PROOF. Assume $\neg P(x_0)$. The relation $y \prec x$ defined by $\varphi(y, x) < 0$ is transitive. In view of the Kuratowski lemma [11, p. 33] there exists a maximal chain $A \subset \{x \in X : \varphi(x, x_0) < 0\} \cup \{x_0\}$ containing x_0 . There is a $\gamma < 0$, $\gamma = \inf\{\varphi(y, x_0) : y \in A\}$, as $\varphi(\cdot, x_0)$ has a finite lower bound. There are two possibilities: either $\varphi(y, x_0) = \gamma$ for a $y \in A$, or $\gamma < \varphi(x, x_0)$, $x \in A$.

Assume $\varphi(y, x_0) = \gamma$ for a $y \in A$ ($y \neq x_0$, see (2)). Then for any $x \in A$ we have $\varphi(x, x_0) \leq \varphi(x, y) + \varphi(y, x_0)$ (see (1)), which means

$$0 \leq \varphi(x, x_0) - \varphi(y, x_0) \leq \varphi(x, y).$$

Consequently, $\varphi(y, x) < 0$ must be satisfied for $x \in A \setminus \{y\}$. Suppose $\neg P(y)$, i.e. there is a $z \in X$ such that $\varphi(z, y) < 0$. This last inequality and the following one

$$\varphi(z, x) \leq \varphi(z, y) + \varphi(y, x) < 0, \quad x \in A \setminus \{y\}$$

yield $z \in A$ (it implies $\gamma \leq \varphi(z, x_0)$) and

$$\varphi(z, x_0) \leq \varphi(z, y) + \varphi(y, x_0) \leq \varphi(z, y) + \gamma < \gamma$$

which means $\gamma \leq \varphi(z, x_0) < \gamma$ - a contradiction. Thus for $z = y$ we have $P(z)$ and $0 \leq \varphi(x, z)$, $x \in X$ ($0 \leq \varphi(z, z)$ follows from (2)).

Let us consider the second case, i.e. $\gamma < \varphi(x, x_0)$, $x \in A$. Then there exists a sequence $(y_n)_{n \in \mathbb{N}}$ in A such that $(\varphi(y_n, x_0))_{n \in \mathbb{N}}$ decreases to γ . Condition (1) yields

$$\varphi(y_m, x_0) \leq \varphi(y_m, y_n) + \varphi(y_n, x_0)$$

which for $m < n$ means $0 < \varphi(y_m, y_n)$. Therefore $(y_m, y_n \in A)$ $\varphi(y_n, y_m) < 0$ is valid for $m < n$. On the other hand,

$$\varphi(y_n, x_0) \leq \varphi(y_n, y_m) + \varphi(y_m, x_0)$$

implies

$$0 < -\varphi(y_n, y_m) \leq \varphi(y_m, x_0) - \varphi(y_n, x_0) < \epsilon$$

for large m, n , $m < n$, as $\lim_{n \rightarrow \infty} \varphi(y_n, x_0) = \gamma$. Thus $(y_n)_{n \in \mathbb{N}}$ is a decreasing Cauchy sequence (Definition 2) and there exists a $y \in X$ such that $\liminf_{n \rightarrow \infty} \varphi(y, y_n) \leq 0$. The last inequality and

$$\varphi(y, x_0) \leq \varphi(y, y_n) + \varphi(y_n, x_0)$$

yield $\varphi(y, x_0) \leq \gamma$. For any $x \in A$ and large n we have $\varphi(y_n, x_0) < \varphi(x, x_0)$ and hence

$$0 < \varphi(x, x_0) - \varphi(y_n, x_0) \leq \varphi(x, y_n)$$

which implies $\varphi(y_n, x) < 0$ for large n , as $x, y_n \in A$. The preceding inequality and

$$\varphi(y, x) \leq \varphi(y, y_n) + \varphi(y_n, x)$$

yield $\varphi(y, x) \leq \liminf_{n \rightarrow \infty} \varphi(y, y_n) \leq 0$.

Suppose there is a $z \in X$ such that $\varphi(z, y) < 0$ (i.e. $\neg P(y)$). We have

$$\varphi(z, x) \leq \varphi(z, y) + \varphi(y, x) < 0, \quad x \in A$$

and

$$\varphi(z, x_0) \leq \varphi(z, y) + \varphi(y, x_0) < \varphi(y, x_0) \leq \gamma < 0,$$

i.e. $z \in A$ and consequently, $\gamma \leq \varphi(z, x_0) < \gamma$ - a contradiction. Once again for $z = y$ we obtain $P(z)$ and $0 \leq \varphi(x, z)$, $x \in X$. If (4) is satisfied and $\varphi(x, z) = 0$ for an $x \in X$, then $x = z$. Therefore (4) implies $0 < \varphi(x, z)$, for all $x \in X \setminus \{z\}$. \square

The subsequent two theorems follow from Theorem 8.

Theorem 9. *Let the almost complete structure (X, φ, x_0) and the number $\gamma \in R$ be such that: for each $x \in X$ with $\gamma \leq \varphi(x, x_0)$ there exists a $y \in X$ satisfying $\varphi(y, x) < 0$. Then, either $0 \leq \varphi(x_0, x_0) < \gamma$, or there exists a $z \in X$ such that $\varphi(z, x_0) < \min\{0, \gamma\}$ and $0 \leq \varphi(x, z)$ for all $x \in X$ ($0 < \varphi(x, z)$, for all $x \in X \setminus \{z\}$ if (4) holds).*

PROOF. Our property $P(x)$ is $\varphi(x, x_0) < \gamma$. Condition (2) yields $0 \leq \varphi(x_0, x_0)$ and we apply Theorem 8. \square

Theorem 10. *Let the almost complete structure (X, φ, x_0) and a number $\gamma \in R$ be such that: For each $x \in X$ with $\gamma < \varphi(x, x_0)$ there exists a $y \in X$ satisfying $\varphi(y, x) < 0$. Then, either $0 \leq \varphi(x_0, x_0) \leq \gamma$, or there exists a $z \in X$ such that $\varphi(z, x_0) \leq \gamma$, $\varphi(z, x_0) < 0$ and $0 \leq \varphi(x, z)$, for all $x \in X$ ($0 < \varphi(x, z)$, for all $x \in X \setminus \{z\}$ if (4) holds).*

PROOF. We adopt $P(x): \varphi(x, x_0) \leq \gamma$ in Theorem 8. \square

Let us present a consequence of Theorem 8 for a “hidden” P . This result can be treated as a weak variational principle (consider φ defined by (3)).

Theorem 11. *Let the almost complete structure (X, φ, x_0) and a $u \in X$ be such that $\varphi(u, x_0) < 0$. Then there exists a $z \in X$ such that $\varphi(z, x_0) < 0 \leq \varphi(x_0, x_0)$ and $0 \leq \varphi(x, z)$, for all $x \in X$ ($0 < \varphi(x, z)$, for all $x \in X \setminus \{z\}$ if (4) holds).*

PROOF. Let $A \neq \emptyset$, $\gamma < 0$ be defined as in the proof of Theorem 8. Let us adopt $P(x) : x \notin A$ or $\varphi(x, x_0) \leq \gamma$. Then $\neg P(x)$ means $x \in A$ and $\gamma < \varphi(x, x_0)$. For such an x there exists a $y \in A$ such that $\varphi(y, x_0) < \varphi(x, x_0)$ and hence

$$0 < \varphi(x, x_0) - \varphi(y, x_0) \leq \varphi(x, y) \text{ (see (1))}.$$

Consequently, we have $\varphi(y, x) < 0$, as $x, y \in A$. Theorem 8 yields $P(x_0)$, i.e. $0 \leq \varphi(x_0, x_0) \leq \gamma < 0$ (see (2)), which is impossible or there exists a $z \in X$ such that $\varphi(z, x_0) \leq \gamma < 0$ (see the proof of Theorem 8) and $0 \leq \varphi(x, z)$, $x \in X$. \square

The next lemma is a useful criterion of (almost) completeness.

Lemma 12. *Let (X, φ, x_0) be a structure. If for each decreasing sequence $(\varphi(x_n, x_0))_{n \in \mathbb{N}}$ there exists an $x \in X$ such that $\liminf_{n \rightarrow \infty} \varphi(x, x_n) \leq 0$, then (X, φ, x_0) is almost complete; if the same holds for each nonincreasing sequence $(\varphi(x_n, x_0))_{n \in \mathbb{N}}$, then (X, φ, x_0) is complete.*

PROOF. Assume $\varphi(x_n, x_m) < 0$, $m < n$, $m, n \in \mathbb{N}$. Then in view of (1)

$$\varphi(x_n, x_0) - \varphi(x_m, x_0) \leq \varphi(x_n, x_m) < 0,$$

i.e. $(\varphi(x_n, x_0))_{n \in \mathbb{N}}$ is decreasing and clearly, bounded below. Therefore

$$-\epsilon < \varphi(x_n, x_0) - \varphi(x_m, x_0) \leq \varphi(x_n, x_m) < 0$$

holds for large $m < n$, $m, n \in \mathbb{N}$. Consequently, for each decreasing Cauchy sequence $(x_n)_{n \in \mathbb{N}}$ in X (see Definition 2) there exists an $x \in X$ with $\liminf_{n \rightarrow \infty} \varphi(x, x_n) \leq 0$, i.e. (X, φ, x_0) is almost complete. A similar reasoning for $\varphi(x_n, x_m) \leq 0$ proves the completeness of (X, φ, x_0) . \square

Lemma 12 and Theorem 11 yield the Brézis-Browder theorem [2, Theorem 1]. Let us present a shorter reasoning.

Theorem 13. *Let $\varphi : X \times X \rightarrow \mathbb{R}$ be a mapping satisfying (1). If for each nonincreasing and bounded sequence $(\varphi(x_n, x_0))_{n \in \mathbb{N}}$ there exists an $x \in X$ such that $\liminf_{n \rightarrow \infty} \varphi(x, x_n) < 0$, then $\varphi(\cdot, x_0)$ is unbounded below.*

PROOF. Suppose that there exists a bounded nonincreasing sequence $(\varphi(x_n, x_0))_{n \in \mathbb{N}}$ convergent to $\gamma = \inf\{\varphi(x, x_0) : x \in X\}$. For $x \in X$ such that $\liminf_{n \rightarrow \infty} \varphi(x, x_n) < 0$ we have (see (1))

$$\gamma \leq \varphi(x, x_0) \leq \liminf_{n \rightarrow \infty} \varphi(x, x_n) + \lim_{n \rightarrow \infty} \varphi(x_n, x_0) < \gamma$$

- a contradiction. \square

In particular, Theorem 13 yields the following compact version of the Brézis-Browder theorem (we disregard the ordering of X):

Theorem 14. *Let $\psi : X \rightarrow \mathbb{R}$ be such a mapping that for each nonincreasing and bounded sequence $(\psi(x_n))_{n \in \mathbb{N}}$ there exists an $x \in X$ with $\psi(x) < \lim_{n \rightarrow \infty} \psi(x_n)$. Then ψ is unbounded below.*

PROOF. We apply Theorem 13 to $\varphi(y, x) = \psi(y) - \psi(x)$, $x, y \in X$. \square

If (X, φ, x_0) is a structure, then $y \preceq x$ defined by $\varphi(y, x) \leq 0$ is also a transitive relation. In what follows X is ordered by \preceq .

For any structure (X, φ, x_0) and $A \subset X$ let us adopt

$$m(A, \varphi) = \{x \in A : \varphi(x, x_0) \leq \varphi(z, x_0), \text{ for each } z \in A\}.$$

Theorem 15. *Let (X, φ, x_0) be a complete structure satisfying*

$$\begin{aligned} &\text{for each maximal chain } A \text{ containing } x_0, \text{ if } m(A, \varphi) \\ &\text{is nonempty, then it has a unique smallest element.} \end{aligned} \quad (5)$$

Assume that the logical property $P(\cdot)$ about elements of X is such that: for each $x \in X$ with $\neg P(x)$ there exists a $y \in X \setminus \{x\}$ satisfying $\varphi(y, x) \leq 0$. Then, either $P(x_0)$, or there exists a minimal (for \preceq) $z \in X$ such that $P(z)$, $\varphi(z, x_0) \leq 0$ and $0 < \varphi(x, z)$, for all $x \in X \setminus \{z\}$.

PROOF. The reasoning is similar to the one presented in the proof of Theorem 8. Assume $\neg P(x_0)$. The relation $y \preceq x$ is transitive and in view of the Kuratowski lemma [11, p. 33] there exists a maximal chain $A \subset \{x \in X : \varphi(x, x_0) \leq 0\} \cup \{x_0\}$ containing x_0 . Let us adopt $\gamma = \inf\{\varphi(y, x_0) : y \in A\}$ ($\gamma \leq 0$).

Suppose $\gamma < \varphi(x, x_0)$, $x \in A$. Then there exists a sequence $(y_n)_{n \in \mathbb{N}}$ in A such that $(\varphi(y_n, x_0))_{n \in \mathbb{N}}$ decreases to γ . Condition (1) yields

$$\varphi(y_m, x_0) \leq \varphi(y_m, y_n) + \varphi(y_n, x_0)$$

which for $m < n$ means $0 < \varphi(y_m, y_n)$. Therefore $(y_m, y_n \in A) \varphi(y_n, y_m) \leq 0$ is valid for $m < n$. On the other hand,

$$\varphi(y_n, x_0) \leq \varphi(y_n, y_m) + \varphi(y_m, x_0)$$

implies

$$0 \leq -\varphi(y_n, y_m) \leq \varphi(y_m, x_0) - \varphi(y_n, x_0) < \epsilon$$

for large m, n , $m < n$, as $\lim_{n \rightarrow \infty} \varphi(y_n, x_0) = \gamma$. Thus $(y_n)_{n \in \mathbb{N}}$ is a Cauchy sequence (Definition 2) and there exists a $y \in X$ such that $\liminf_{n \rightarrow \infty} \varphi(y, y_n) \leq 0$. The last inequality and

$$\varphi(y, x_0) \leq \varphi(y, y_n) + \varphi(y_n, x_0)$$

yield $\varphi(y, x_0) \leq \gamma$. For any $x \in A$ and large n we have $\varphi(y_n, x_0) < \varphi(x, x_0)$, hence

$$0 < \varphi(x, x_0) - \varphi(y_n, x_0) \leq \varphi(x, y_n)$$

which implies $\varphi(y_n, x) \leq 0$ for large n , as $x, y_n \in A$. Thus in view of

$$\varphi(y, x) \leq \varphi(y, y_n) + \varphi(y_n, x)$$

we obtain $\varphi(y, x) \leq \liminf_{n \rightarrow \infty} \varphi(y, y_n) \leq 0$. Now it is seen that $y \in A$ and therefore $\varphi(y, x_0) = \gamma$. For any $x \in A$ with $\gamma < \varphi(x, x_0)$,

$$0 < \varphi(x, x_0) - \varphi(y, x_0) \leq \varphi(x, y)$$

yields $\varphi(y, x) \leq 0$, i.e. $y \in m(A, \varphi)$. In view of (5) we may assume that y is the only smallest element of $m(A, \varphi)$ (and of A).

Suppose there is a $z \in X \setminus \{y\}$ such that $\varphi(z, y) \leq 0$ (i.e. $\neg P(y)$). We have

$$\varphi(z, x) \leq \varphi(z, y) + \varphi(y, x) \leq 0, x \in A,$$

i.e. $z \neq y$ is another smallest element of A , which contradicts (5). By replacing y with z we obtain $P(z)$, $\varphi(z, x_0) = \gamma \leq 0$ and $0 < \varphi(x, z)$, $x \in X \setminus \{z\}$. Clearly, z is a minimal element in X , as A is a maximal chain. \square

Let us prove an analog of Theorem 10 for Theorem 15.

Theorem 16. *Let the complete structure (X, φ, x_0) satisfying (5) and the number $\gamma \in R$ be such that: For each $x \in X$ with $\gamma < \varphi(x, x_0)$ there exists a $y \in X \setminus \{x\}$ satisfying $\varphi(y, x) \leq 0$. Then, either $0 \leq \varphi(x_0, x_0) \leq \gamma$, or there exists a $z \in X$ such that $\varphi(z, x_0) \leq \min\{\gamma, 0\}$ and $0 < \varphi(x, z)$, for all $x \in X \setminus \{z\}$.*

PROOF. We adopt $P(x): \varphi(x, x_0) \leq \gamma$ in Theorem 15. □

An analog of Theorem 11 is the following variational principle:

Theorem 17. *Let the complete structure (X, φ, x_0) satisfying (5) and a $u \in X$ be such that $\varphi(u, x_0) \leq 0$. Then there exists a $z \in X$ such that $\varphi(z, x_0) \leq 0 \leq \varphi(x_0, x_0)$ and $0 < \varphi(x, z)$, $x \in X \setminus \{z\}$.*

PROOF. Let $A \neq \emptyset$, $\gamma \leq 0$ be defined as in the proof of Theorem 15. Let us adopt $P(x): x \notin A$ or x is the smallest element of A (with respect to \preceq). If $\neg P(x)$ then $x \in A$ and $\varphi(y, x) \leq 0$ for a $y \in A \setminus \{x\}$. Theorem 15 yields $P(x_0)$, i.e. the smallest element z of A is equal to x_0 and $0 \leq \varphi(z, x_0) = \gamma (= 0)$ (see (2)) or $z \neq x_0$ and $\varphi(z, x_0) = \gamma \leq 0$. The smallest element of A is minimal in X and therefore $0 < \varphi(x, z)$, $x \in X \setminus \{z\}$. □

A way to guarantee the existence of a smallest element of A in the proof of Theorem 15 (without applying condition (5)) is to change the sense of completeness (Definitions 2, 3) by considering nets in place of sequences. The next lemma presents another idea.

Lemma 18. *Let x_0 be an element of a maximal chain A in a complete structure (X, φ, x_0) . Assume that $m(A, \varphi)$ is nonempty and*

$$\begin{aligned} & \text{there exists a countable set } B \subset m(A, \varphi) \text{ such that} \\ & \text{for each } x \in m(A, \varphi) \setminus B \text{ there is a } y \in B \text{ with } \varphi(y, x) \leq 0 \end{aligned} \quad (6)$$

holds. Then $m(A, \varphi)$ contains a smallest element of A .

PROOF. Let us adopt $\gamma = \varphi(y, x_0)$ for any $y \in m(A, \varphi)$. Then for $x \in A$ such that $\gamma < \varphi(x, x_0)$ and any $y \in m(A, \varphi)$ we have

$$0 < \varphi(x, x_0) - \varphi(y, x_0) \leq \varphi(x, y)$$

and therefore $\varphi(y, x) \leq 0$ ($x, y \in A$). Consequently, it is sufficient to find a $y \in A$ preceding all elements of $m(A, \varphi) \setminus \{y\}$. If B is finite then it contains

a smallest element of $m(A, \varphi)$. Assume B is infinite. Let $(y_n)_{n \in N}$ be such a sequence that $B = \{y_n : n \in N\}$ and $y_m \neq y_n$ for all $m \neq n$, $m, n \in N$. The equality $\gamma = \varphi(y_n, x_0) = \varphi(y_m, x_0)$ yields the inequalities:

$$\begin{aligned} 0 &= \varphi(y_n, x_0) - \varphi(y_m, x_0) \leq \varphi(y_n, y_m), \\ 0 &= \varphi(y_m, x_0) - \varphi(y_n, x_0) \leq \varphi(y_m, y_n). \end{aligned}$$

Consequently, (B is a chain) $\varphi(y_n, y_m) = 0$ or $\varphi(y_m, y_n) = 0$ is valid for $m \neq n$. One can easily construct a “nonincreasing” subsequence $(y_{k_n})_{n \in N}$ of $(y_n)_{n \in N}$, i.e. such that $\varphi(y_{k_{n+1}}, y_{k_n}) = 0$, $n \in N$ and (see (1)) $\varphi(y_{k_n}, y_{k_m}) = 0$, $m < n$, $m, n \in N$. Thus $(y_{k_n})_{n \in N}$ is a Cauchy sequence and there exists a $y \in X$ such that $\liminf_{n \rightarrow \infty} \varphi(y, y_{k_n}) \leq 0$ and clearly, $\liminf_{n \rightarrow \infty} \varphi(y, y_n) \leq 0$. For any $x \in m(A, \varphi) \setminus B$ and $\varphi(y_n, y_m) = 0$, $\varphi(y_m, x) \leq 0$ from

$$\varphi(y, x) \leq \varphi(y, y_n) + \varphi(y_n, y_m) + \varphi(y_m, x)$$

it follows that $\varphi(y, x) \leq \liminf_{n \rightarrow \infty} \varphi(y, y_n) = 0$. Similarly, for $x \in B \setminus \{y\}$, i.e. $x = y_m \neq y$ from

$$\varphi(y, y_m) \leq \varphi(y, y_n) + \varphi(y_n, y_m)$$

we obtain $\varphi(y, y_m) \leq 0$. Consequently, y precedes all elements of $m(A, \varphi) \setminus \{y\}$ and of $A \setminus \{y\}$. On the other hand, $y \in A$, as A is a maximal chain. In addition, $\varphi(y, x_0) \leq \varphi(y, y_m) + \varphi(y_m, x_0) \leq \gamma$ means $y \in m(A, \varphi)$. \square

From Lemma 18 we obtain:

Lemma 19. *Let (X, φ, x_0) be a complete structure for a φ satisfying*

$$\varphi(x, y) \leq 0 \text{ and } \varphi(y, x) \leq 0 \text{ imply } x = y, \text{ for all } x, y \in X \quad (7)$$

(i.e. \preceq is antisymmetric). Assume that for each maximal chain $A \subset X$ containing x_0 , if $m(A, \varphi)$ is nonempty, then (6) holds. Then condition (5) is satisfied.

PROOF. In view of Lemma 18 A has a smallest element and by (7) such an element is unique. \square

In view of Lemma 19, we can reformulate Theorem 17 as follows:

Theorem 20. *Let (X, φ, x_0) be a complete structure for a φ satisfying (7). Assume that for each maximal chain $A \subset X$ containing x_0 , if $m(A, \varphi)$ is nonempty, then (6) holds. If in addition, $\varphi(u, x_0) \leq 0$ is satisfied for an $u \in X$, then there exists a $z \in X$ such that $\varphi(z, x_0) \leq 0 \leq \varphi(x_0, x_0)$ and $0 < \varphi(x, z)$, for all $x \in X \setminus \{z\}$.*

It is shown by the next lemma that condition (5) is not so abstract.

Lemma 21. *Let (X, φ) be a complete structure for the mapping φ defined by*

$$\varphi(y, x) = \psi(y) + d(y, x) - \psi(x), \quad x, y \in X,$$

while ψ has a finite lower bound, d satisfies the triangle inequality and

$$d(x, y) \leq 0 \text{ implies } x = y, \text{ for all } x, y \in X.$$

Then condition (5) is satisfied for each $x_0 \in X$.

PROOF. Clearly, φ satisfies (7), as from $\varphi(x, y) \leq 0$ and $\varphi(y, x) \leq 0$ it follows that $d(x, y) + d(y, x) \leq 0$, i.e. $d(x, y) \leq 0$ or $d(y, x) \leq 0$. Let $A \subset X$ be a maximal chain containing x_0 . Assume that $m(A, \varphi)$ is nonempty and let $\beta = \inf\{\psi(x) : x \in m(A, \varphi)\}$. If ψ is constant on $m(A, \varphi)$, then the chain $m(A, \varphi)$ consists of one element. Assume $\beta < \psi(x)$ for an $x \in m(A, \varphi)$. Then for $y_n \in m(A, \varphi)$ such that $(\psi(y_n))_{n \in \mathbb{N}}$ is nonincreasing and convergent to β , $B = \{y_n : n \in \mathbb{N}\}$, any $x \in m(A, \varphi) \setminus B$ and large n we have $\psi(y_n) < \psi(x)$. Clearly,

$$\psi(x) + d(x, y_n) - \psi(y_n) \leq 0$$

implies $d(x, y_n) < 0$ which is impossible, as $x \neq y_n$. Therefore $\varphi(y_n, x) \leq 0$ must be satisfied ($m(A, \varphi)$ is a chain). Now it is seen that (6) holds, and we apply Lemma 19. \square

A direct consequence of Lemma 21 and Theorem 17 is the following result of Bishop-Phelps [9, Theorem 2, p. 206]:

Theorem 22. *Let (X, d) be a complete metric space and let $\psi : X \rightarrow \mathbb{R}$ be a lower semicontinuous mapping with a finite lower bound. Then for any $\lambda > 0$, $\delta := \lambda d$ and any $x_0 \in X$ there exists a $z \in X$ such that*

$$\psi(z) + \delta(z, x_0) \leq \psi(x_0)$$

and

$$\psi(z) < \psi(x) + \delta(x, z), \quad x \in X \setminus \{z\}.$$

PROOF. In view of Lemma 7, (X, φ) for φ defined by (3) is a complete structure. Now we apply Lemma 21 and Theorem 17 ($\varphi(x_0, x_0) = 0$). \square

It was proved in [5], that the Bishop-Phelps theorem is equivalent to the famous Ekeland's variational principle [8, Theorem 1]. Our Theorem 17 yields the subsequent extension of a more general theorem of Brøndsted [3, Theorem 2]:

Theorem 23. *Let (X, \mathcal{U}) be a uniform space and $d: X \times X \rightarrow R$ a mapping satisfying:*

- (i) $d(z, x) \leq d(z, y) + d(y, x)$, for all $x, y, z \in X$,
- (ii) $0 \leq d(y, x)$, for all $x, y \in X$,
- (iii) $d(x, y) = 0$ implies $x = y$, for all $x, y \in X$ (Brøndsted assumes equivalence)
- (iv) for each $U \in \mathcal{U}$ there exists a $\delta > 0$ such that $d^{-1}([0, \delta]) \subset U$,
- (v) $d(\cdot, x)$ is lower semicontinuous, for each $x \in X$.

Assume that $\psi: X \rightarrow R$ is a mapping with a finite lower bound and such that $\psi^{-1}((-\infty, \gamma])$ is complete, for each $\gamma \in R$. Then for

$$\varphi(y, x) = \psi(y) + d(y, x) - \psi(x), \quad x, y \in X$$

(X, φ) is a complete structure and for any $x_0 \in X$ there exists a $z \in X$ such that $\varphi(z, x_0) \leq 0 \leq \varphi(x_0, x_0)$ and $0 < \varphi(x, z)$, for all $x \in X \setminus \{z\}$.

PROOF. Clearly, (X, φ) is a structure as d satisfies the triangle inequality, ψ has a finite lower bound and $0 \leq d(x, y)$. The inequality (see Definition 2)

$$-\epsilon < \varphi(x_n, x_m) = \psi(x_n) + d(x_n, x_m) - \psi(x_m) \leq 0$$

means

$$\psi(x_m) - \psi(x_n) - \epsilon < d(x_n, x_m) \leq \psi(x_m) - \psi(x_n)$$

and therefore $(\psi(x_n))_{n \in N}$ is nonincreasing ($m < n$), as $0 \leq d(x_n, x_m)$. Consequently $(\psi(x_n))_{n \in N}$ is convergent. From $d(x_n, x_m) < \delta$, for large m, n it follows that $(x_n)_{n \in N}$ is a Cauchy sequence in (X, \mathcal{U}) . For any fixed m the inequality $d(x_n, x_m) \leq \psi(x_m) - \psi(x_n)$ yields $x_n \in \{z \in X : \psi(z) \leq \psi(x_m)\}$ and this last set is complete. Therefore $(x_n)_{n \in N}$ is convergent, say to an

x , and $\psi(x) \leq \psi(x_n)$, $n \in N$. From the lower semicontinuity of $d(\cdot, z)$ it follows that $d(x, x_m) \leq d(x_n, x_m)$, $n \in N$. Now it is clear that

$$\psi(x) + d(x, x_m) - \psi(x_m) \leq \psi(x_n) + d(x_n, x_m) - \psi(x_m) \leq 0$$

for large n , i.e. $\varphi(x, x_n) \leq 0$, $n \in N$. Therefore, (X, φ) is a complete structure and we apply Lemma 21. Now Theorem 17 works. \square

Let 2^Y be the family of all subsets of Y . We say that $F: X \rightarrow 2^Y$ is a (multivalued) mapping if $F(x) \neq \emptyset$, for all $x \in X \neq \emptyset$.

As regards the fixed point theory, a direct consequence of our Theorem 15 is the following result:

Theorem 24. *Let (X, φ, x_0) be a complete structure for a φ satisfying (5). Assume $X \subset Y$ and $F: X \rightarrow 2^Y$ is a multivalued mapping such that for each $x \in X \setminus F(x)$ there exists a $y \in X \setminus \{x\}$ satisfying $\varphi(y, x) \leq 0$. Then $x_0 \in F(x_0)$ or there exists a $z \in F(z)$ such that $\varphi(z, x_0) \leq 0$ and $0 < \varphi(x, z)$, for all $x \in X \setminus \{z\}$.*

PROOF. We adopt $P(x): x \in F(x)$ in Theorem 15. \square

The next theorem extends the theorems of Takahashi [9, Theorem 5] and of Caristi [4, Theorem (2.1)'] (see Lemmas 7, 21).

Theorem 25. *Let (X, φ, x_0) be a complete structure for a φ satisfying (5). Assume $X \subset Y$ and $F: X \rightarrow 2^Y$ is a multivalued mapping such that for each $x \in X$ there exists a $y \in F(x)$ satisfying $\varphi(y, x) \leq 0$. Then there exists a $z \in F(z)$ such that $\varphi(z, x_0) \leq 0$ and $0 < \varphi(x, z)$, for all $x \in X \setminus \{z\}$.*

PROOF. All the assumptions of Theorem 17 are satisfied and the existing minimal point of any maximal chain containing x_0 , is a fixed point of F . \square

From Theorem 8 we obtain:

Theorem 26. *Let (X, φ, x_0) be an almost complete structure. Assume $X \subset Y$ and $F: X \rightarrow 2^Y$ is a multivalued mapping such that for each $x \in X \setminus F(x)$ there exists a $y \in X$ satisfying $\varphi(y, x) < 0$. Then $x_0 \in F(x_0)$ or there exists a $z \in F(z)$ such that $\varphi(z, x_0) < 0$ and $0 \leq \varphi(x, z)$, for all $x \in X$ ($0 < \varphi(x, z)$, for all $x \in X \setminus \{z\}$ if (4) holds).*

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