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Meir and Keeler were right

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ABSTRACT

It is shown here that the celebrated fixed point theorem of Meir and Keeler is equivalent to formally more general result of Matkowski and Ćirić. In addition, a short proof of an extension of these theorems is given, and a tool theorem with applications is presented. Also some fixed point theorems for commuting (and cyclic) mappings in dislocated metric spaces are proved.

1. Introduction

Meir and Keeler in [6] applied the following condition to a selfmapping f on a metric space:

$$\begin{aligned} &\textit{for each } \alpha > 0, \textit{ there exists an } \epsilon > 0 \textit{ such that} \\ &\alpha \le p(y,x) < \alpha + \epsilon \textit{ implies } p(fy,fx) < \alpha. \end{aligned} \tag{1}$$

This condition was later extended by Matkowski in [4], Theorem 1.5.1, and by Ćirić [1]. They used two conditions. One of them has the following form:

$$p(fy, fx) < p(y, x), \quad x \neq y.$$

If p is a metric, then the above condition is equivalent to the following one:

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$$p(fy, fx) > 0 \textit{ yields } p(fy, fx) < p(y, x), \tag{2}$$

and clearly, (1) implies (2) (for $\alpha = p(y, x) > 0$).

The second condition of Matkowski and Ćirić is

$$\begin{aligned} &\textit{for each } \alpha > 0 \textit{ there exists an } \epsilon > 0 \textit{ for which} \\ &\alpha < p(y, x) < \alpha + \epsilon \textit{ yields } p(fy, fx) \le \alpha. \end{aligned}$$

If we assume that (2) holds, then the above condition is equivalent to the following one:

$$\begin{aligned} &\textit{for each } \alpha > 0 \textit{ there exists an } \epsilon > 0 \textit{ for which} \\ &p(y, x) < \alpha + \epsilon \textit{ yields } p(fy, fx) \le \alpha, \end{aligned} \tag{3}$$

as for $p(y, x) \le \alpha$ we have $p(fy, fx) < p(y, x) \le \alpha$. Obviously, (1) implies (2) and (3).

In Section 2 the equivalence of the respective theorems is proved (see Corollary 2.7), and a tool for proving fixed point theorems (*i.e.* Theorem 2.8) is presented. Our tool theorem is applied to obtain two elegant fixed point results, *i.e.* Theorem 2.9 and Theorem 2.11.

Section 3 is devoted to common fixed point theorems for more mappings. Theorem 3.2 extends the Matkowski and Ćirić theorem to the case of two commuting mappings. Consequently, a strong fixed point theorem for single mapping is obtained (see Theorem 3.3). Theorems 3.4 and 3.5 concern fixed points of families of mappings. The final three theorems are devoted to cyclic mappings.

2. On the Meir–Keeler theorem

First, we present a short proof of an extension of the Meir–Keeler theorem.

Let us recall that a mapping $p\colon X \times X \to [0, \infty)$ is a *d-metric* (or *dislocated metric*) [2], if the following conditions are satisfied:

$$p(x, y) = 0 \textit{ yields } x = y, \quad x, y \in X, \tag{4a}$$

$$p(x, y) = p(y, x), \quad x, y \in X, \tag{4b}$$

$$p(x, z) \le p(x, y) + p(y, z), \quad x, y, z \in X. \tag{4c}$$

Then, (X, p) is a *d-metric space* (topology is generated by balls).

If $f\colon X \to X$ is a mapping, then for $x_n = f^n x_0$, $n \in \mathbb{N}$, the set $Z[x_0] = \{x_0, x_1, \ldots, x_n, \ldots\}$ is an *orbit* of f.

Lemma 2.1. *Let f be a selfmapping on a d-metric space (X, p), and let $Z = Z[x_0]$ be an orbit of f. If the following conditions are satisfied:*

$$p(f^2x, fx) > 0 \textit{ yields } p(f^2x, fx) < p(fx, x), \quad x \in Z, \tag{5}$$

$$\begin{aligned} &\textit{for each } \alpha > 0 \textit{ there exists an } \epsilon > 0 \textit{ for which} \\ &p(fx, x) < \alpha + \epsilon \textit{ yields } p(f^2x, fx) \le \alpha, \quad x \in Z, \end{aligned} \tag{6}$$

then $\lim_{n\to\infty} p(x_{n+1}, x_n) = 0$. If $\lim_{n\to\infty} p(x_{n+1}, x_n) = 0$ and (3) holds for $y = x_n$, $x = x_m$ and large $m, n \in \mathbb{N}$, then $\lim_{m,n\to\infty} p(x_n, x_m) = 0$.

Proof. If $p(x_{n+1}, x_n) = 0$ for an $n \in \mathbb{N}$, then $x_{n+1} = x_n$ and

$$x_{n+2} = f(x_{n+1}) = f(x_n) = x_{n+1} = x_n.$$

By induction we obtain $x_{n+k} = x_n$, $k \in \mathbb{N}$, and $\lim_{m,n\to\infty} p(x_n, x_m) = 0$. Assume, $a_n = p(x_{n+1}, x_n) > 0$, $n \in \mathbb{N}$. In view of (5) we have $a_{n+1} < a_n$, $n \in \mathbb{N}$, and therefore $\lim_{n\to\infty} a_n = \alpha \geq 0$. Suppose, $\alpha > 0$. Then for large n we obtain $\alpha < a_{n+1} < a_n < \alpha + \epsilon$, and (6) yields $a_{n+1} \leq \alpha$, a contradiction. Now, it is clear that $\lim_{n\to\infty} a_n = 0$ (this fact is also a consequence of [11], Lemma 2.1).

Assume $\lim_{n\to\infty} p(x_{n+1}, x_n) = 0$, and suppose $\lim_{m,n\to\infty} p(x_n, x_m) = 0$ is false. Then there exist $\alpha > 0$ and an infinite set $\mathbb{K} \in \mathbb{N}$, such that for all $k \in \mathbb{K}$ there exist $n \in \mathbb{N}$ for which $\alpha < p(x_{k+n+1}, x_{k+1})$ holds. Let $n = n(k)$ be the smallest such number. We have

$$p(x_{k+n}, x_k) \leq p(x_{k+n}, x_{k+1}) + p(x_{k+1}, x_k) \leq$$
$$\alpha + p(x_{k+1}, x_k) < \alpha + \epsilon$$

for large k. Now, (3) yields $p(x_{k+n+1}, x_{k+1}) \leq \alpha$, a contradiction. □

A d-metric space (X, p) is 0-*complete* ([8], Definition 2.3) if for each sequence $(x_n)_{n\in\mathbb{N}}$ in X, such that $\lim_{m,n\to\infty} p(x_n, x_m) = 0$, there exists an $x \in X$ for which $\lim_{n\to\infty} p(x_n, x) = 0$.

From Lemma 2.1 and the triangle inequality we obtain

Corollary 2.2. *Let f be a selfmapping on a d-metric space (X, p), and let $Z = Z[x_0]$ be an orbit of f. Then (5), (6) yield $\lim_{n\to\infty} p(x_{n+1}, x_n) = 0$. If $\lim_{n\to\infty} p(x_{n+1}, x_n) = 0$, (3) holds for $y = x_n$, $x = x_m$ and large $m, n \in \mathbb{N}$, and (X, p) is 0-complete, then there exists an $x \in X$ such that $\lim_{n\to\infty} p(x_n, x) = p(x, x) = 0$.*

Clearly, our corollary works also for f orbitally 0-complete d-metric spaces (the respective sequences consist of points of any orbit of f). Further results of the present section can be extended in the following way: replace "0-complete" by "f orbitally 0-complete".

A selfmapping f on a d-metric space (X, p) is 0-*continuous at an* $x \in X$ ([11], Definition 2.10), if $\lim_{n\to\infty} p(x_n, x) = 0$ implies $\lim_{n\to\infty} p(fx_n, fx) = 0$ for each sequence $(x_n)_{n\in\mathbb{N}}$ in X; f is 0-*continuous* if it is 0-continuous at each point $x \in X$.

Lemma 2.3. *Let f be a selfmapping on a d-metric space (X, p), and let $Z = Z[x_0]$ be an orbit of f. If (5), (6) are satisfied, then $\lim_{n\to\infty} p(x_{n+1}, x_n) = 0$. If $\lim_{n\to\infty} p(x_{n+1}, x_n) = 0$, (3) holds for $y = x_n$, $x = x_m$ and large $m, n \in \mathbb{N}$, and (X, p) is 0-complete, then there exists an $x \in X$ such that $\lim_{n\to\infty} p(x_n, x) = p(x, x) = 0$; if in addition, f is 0-continuous at x, then $fx = x$.*

Proof. For x as in Corollary 2.2 we have

$$p(fx, x) \leq p(fx, x_{n+1}) + p(x_{n+1}, x) = p(fx, fx_n) + p(x_{n+1}, x),$$

and the 0-continuity of f at x implies $p(fx, x) = 0$. □

Now, we are ready to present an extension of the Matkowski and the Ćirić theorem.

Theorem 2.4. *Let f be a selfmapping on a 0-complete d-metric space (X, p). If (2), (3) are satisfied for all $x, y \in X$, then f has a unique fixed point x, and in addition, $\lim_{n\to\infty} p(f^n x_0, x) = p(fx, x) = 0$, $x_0 \in X$.*

Proof. If (2) is satisfied, then f is 0-continuous and (5) holds. Clearly, (3) yields (6). Now, in view of Lemma 2.3, for each $x_0 \in X$ there exists a point x such that $\lim_{n\to\infty} p(f^n x_0, x) = \lim_{n\to\infty} p(fx, x) = 0$. Suppose x, y are two fixed points of f. From (2) we get $0 < p(y, x) = p(fy, fx) < p(y, x)$, a contradiction. Consequently, $x_0 \in X$ can be arbitrary. □

Clearly, the above theorem implies the next one, as condition (1) yields (2) and (3). If (X, p) is a metric space, then Theorem 2.5 becomes the Meir–Keeler theorem.

Theorem 2.5. *Let f be a selfmapping on a 0-complete d-metric space (X, p). If (1) is satisfied for all $x, y \in X$, then f has a unique fixed point x, and $\lim_{n\to\infty} p(f^n x_0, x) = p(fx, x) = 0$, $x_0 \in X$.*

Let us show that Theorem 2.5 implies Theorem 2.4.

Proposition 2.6. *If a mapping $f : X \to X$ satisfies (2) and (3) for all $x, y \in X$, then (1) holds for f replaced by f^2, and all $x, y \in X$.*

Proof. If $p(f^2 y, f^2 x) > 0$, then from (2) we get

$$p(f^2 y, f^2 x) < p(fy, fx) < p(y, x),$$

which for $p(y, x) = \alpha$ means that $p(f^2 y, f^2 x) < \alpha$. In turn, for $\alpha < p(y, x) < \alpha + \epsilon$ conditions (2), (3) yield

$$p(f^2 y, f^2 x) < p(fy, fx) \le \alpha,$$

i.e. $p(f^2 y, f^2 x) < \alpha$. □

Now, it is clear that if the assumptions of Theorem 2.4 hold, then we have the assumptions of Theorem 2.5 for f replaced by f^2. Let us recall ([7], Lemma 29) that if f^t has a unique fixed point, then f has a unique fixed point. Similar dependency concerns the convergences of $((f^t)^n x_0)_{n\in\mathbb{N}}$ and $(f^n x_0)_{n\in\mathbb{N}}$. Consequently, Theorem 2.5 implies Theorem 2.4.

Corollary 2.7. *Theorems 2.5 and 2.4 are equivalent and the same concerns the classical results of Meir–Keeler, and Matkowski, Ćirić.*

The next theorem is a tool in proving fixed point theorems, and it is a consequence of Lemma 2.3.

Theorem 2.8. *Assume that f is a selfmapping on a d-metric space (X, p). If (5), (6) are satisfied for an orbit Z of f, then $\lim_{n\to\infty} p(f^{n+1} z, f^n z) = 0$, $z \in Z$. If (X, p) is 0-complete, $\lim_{n\to\infty} p(f^{n+1} x_0, f^n x_0) = 0$, and (3) holds for $y = f^n x_0$, $x = f^m x_0$ and large $m, n \in \mathbb{N}$, then there exists an x such that $\lim_{n\to\infty} p(f^n x_0, x) = p(x, x) = 0$; if in addition, f is 0-continuous at x, then $fx = x$. If (X, p) is 0-complete, f and each $x_0 \in X$ fulfil the above requirements and x is the unique fixed point of f, then $\lim_{n\to\infty} p(f^n x_0, x) = p(fx, x) = 0$, $x_0 \in X$.*

Proof. In view of Lemma 2.3, for any $x_0 \in X$ there exists an x such that $\lim_{n\to\infty} p(f^n x_0, x) = p(fx, x) = 0$. Such point is unique and therefore x_0 can be arbitrary. □

Now, we can present another extension of Theorem 2.4

Theorem 2.9. *Let f be a selfmapping on a 0-complete d-metric space (X, p), and let (2) hold for all $x, y \in X$. If (3) is satisfied for each orbit Z of f, then f has a unique fixed point x, and $\lim_{n\to\infty} p(f^n x_0, x) = p(fx, x) = 0$, $x_0 \in X$.*

Proof. Conditions (2), (3) imply (5), (6), and f is 0-continuous (see (2)). In view of Theorem 2.8 there exists an x such that $\lim_{n\to\infty} p(f^n x_0, x) = p(fx, x) = 0$. Suppose x, y are two fixed points. Then (2) implies $0 < p(y, x) = p(fy, fx) < p(y, x)$, a contradiction. Therefore, x is unique, and x_0 can be arbitrary. □

Remark 2.10. Let us note that $p(y, x)$ in (2) or (3) for Theorem 2.9 can be replaced by

$$c_f(y, x) = \max\{p(y, x), p(fy, y), p(fx, x)\}$$

if $p(y, y), p(x, x) \le p(y, x)$ for c_f used in (2), and if f is 0-continuous (see the reasoning below).

Let us present a more advanced application of Theorem 2.8.

A mapping $p\colon X \times X \to [0, \infty)$ is a *partial metric* ([5], Definition 3.1) if the following conditions are satisfied:

$$x = y \text{ iff } p(x, x) = p(x, y) = p(y, y), \quad x, y \in X, \tag{7a}$$

$$p(x, x) \le p(x, y), \quad x, y \in X, \tag{7b}$$

$$p(x, y) = p(y, x), \quad x, y \in X, \tag{7c}$$

$$p(x, z) \le p(x, y) + p(y, z) - p(y, y), \quad x, y, z \in X. \tag{7d}$$

From (7b) it follows that $p(x, y) = 0$ implies $p(x, x) = p(y, y) = 0$ and $x = y$ (see (7a)). Consequently, each partial metric is a d-metric (see (4a)).

Let us consider the following conditions

$$p(fy, fx) > 0 \text{ yields } p(fy, fx) < m_f(y, x), \quad x, y \in X, \tag{8}$$

$$\begin{aligned} &\text{for each } \alpha > 0 \text{ there exists an } \epsilon > 0 \text{ for which} \\ &m_f(y, x) < \alpha + \epsilon \text{ yields } p(fy, fx) \le \alpha, \quad x, y \in Z, \end{aligned} \tag{9}$$

where

$$\begin{aligned} m_f(y, x) = \max\{p(y, x), p(fy, y), p(fx, x), [p(fy, x) + p(fx, y)]/2\} = \\ \max\{c_f(y, x), [p(fy, x) + p(fx, y)]/2\}. \end{aligned}$$

If p is a partial metric, $f\colon X \to X$ is a mapping, and $x_n = f^n x_0$, $n \in \mathbb{N}$, then we have (see (7d))

$$\begin{aligned} &[p(x_{n+2}, x_n) + p(x_{n+1}, x_{n+1})]/2 \le \\ &[p(x_{n+2}, x_{n+1}) + p(x_{n+1}, x_n) - p(x_{n+1}, x_{n+1}) + p(x_{n+1}, x_{n+1})]/2 = \\ &[p(x_{n+2}, x_{n+1}) + p(x_{n+1}, x_n)]/2 \le \max\{p(x_{n+2}, x_{n+1}), p(x_{n+1}, x_n)\}. \end{aligned}$$

Now, we obtain

$$\begin{aligned} &m_f(x_{n+1}, x_n) = \max\{p(x_{n+1}, x_n), p(x_{n+2}, x_{n+1}), p(x_{n+1}, x_n), \\ &[p(x_{n+2}, x_n) + p(x_{n+1}, x_{n+1})]/2\} = \max\{p(x_{n+1}, x_n), p(x_{n+2}, x_{n+1})\} = \\ &c_f(x_{n+1}, x_n), \end{aligned}$$

and (8) yields

$$p(x_{n+2}, x_{n+1}) > 0 \textit{ implies } p(x_{n+2}, x_{n+1}) < m_f(x_{n+1}, x_n) =$$
$$c_f(x_{n+1}, x_n) = \max\{p(x_{n+1}, x_n), p(x_{n+2}, x_{n+1})\} = p(x_{n+1}, x_n)$$

($p(x_{n+1}, x_n) < p(x_{n+2}, x_{n+1})$ would mean a contradiction), *i.e.* (5) holds, and moreover, (9) yields (6). Now, we get $\lim_{n\to\infty} p(x_{n+1}, x_n) = 0$ (see Lemma 2.1). We also have (large $m, n \in \mathbb{N}$)

$$c_f(x_n, x_m) = \max\{p(x_n, x_m), p(x_{n+1}, x_n), p(x_{m+1}, x_m)\} < p(x_n, x_m) + \epsilon,$$

and (see (7d), (7b))

$$m_f(x_n, x_m) = \max\{c_f(x_n, x_m), [p(x_{n+1}, x_m) + p(x_{m+1}, x_n)]/2\} \leq$$
$$\max\{c_f(x_n, x_m), [p(x_{n+1}, x_n) + p(x_n, x_m) - p(x_n, x_n)+$$
$$p(x_{m+1}, x_m) + p(x_m, x_n) - p(x_m, x_m)]/2\} =$$
$$\max\{c_f(x_n, x_m), p(x_n, x_m) + [p(x_{n+1}, x_n) - p(x_n, x_n)+$$
$$p(x_{m+1}, x_m) - p(x_m, x_m)]/2\} < p(x_n, x_m) + \epsilon$$

Consequently, (9) implies (3) for $y = x_n$, $x = x_m$ and large $m, n \in \mathbb{N}$. Now, if f is 0-continuous, then Lemma 2.3 applies, and f has a fixed point.

Suppose x, y are different fixed points. Then (8), (7b) imply

$$0 < p(y, x) = p(fy, fx) < m_f(y, x) =$$
$$\max\{p(y, x), p(fy, y), p(fx, x), [p(fy, x) + p(fx, y))]/2\} =$$
$$\max\{p(y, x), p(y, y), p(x, x), [p(y, x) + p(x, y))]/2\} = c_f(y, x) = p(y, x),$$

a contradiction. Now, Theorem 2.8 yields the following strong result.

Theorem 2.11. *Let f be a selfmapping on a 0-complete partial metric space (X, p), and let (8) hold. If (9) is satisfied for each orbit Z of f, and f is 0-continuous, then f has a unique fixed point x, and $\lim_{n\to\infty} p(f^n x_0, x) = p(fx, x) = 0$, $x_0 \in X$.*

Remark 2.12. Let us note, that m_f in (8) or (9) in Theorem 2.11 can be replaced by p or c_f (see also [11], Remark 3.5).

3. Common fixed points

Lemma 3.1. *Let g, h be commuting selfmappings on a d-metric space (X, p), and let the following conditions hold:*

$$p(hy, gx) > 0 \textit{ yields } p(hy, gx) < p(y, x), \quad x, y \in X, \tag{10}$$

$$\begin{aligned} &\textit{for each } \alpha > 0 \textit{ there exists an } \epsilon > 0 \textit{ for which} \\ &p(y, x) < \alpha + \epsilon \textit{ yields } p(hy, gx) \leq \alpha, \quad x, y \in X. \end{aligned} \tag{11}$$

Then $h \circ g$ satisfies (1) for all $x, y \in X$.

Proof. Let us adopt $f = h \circ g$. Then for $p(fy, fx) > 0$ we have (see (10))

$$p(fy, fx) = p((h \circ g)y, (h \circ g)x) = p(hgx, ghy) < p(gx, hy) < p(y, x).$$

Now, $p(y, x) = \alpha$ yields $p(fy, fx) < \alpha$. For $\alpha < p(y, x) < \alpha + \epsilon$ the above inequality and (11) imply $p(fy, fx) < p(hy, gx) \leq \alpha$. □

Let us note, that Proposition 2.6 is a consequence of Lemma 3.1 for $g = h = f$.

Theorem 3.2. *Let g, h be commuting selfmappings on a 0-complete d-metric space (X, p). If (10), (11) hold, then g, h have a unique and common fixed point x, and $\lim_{n\to\infty} p((h \circ g)^n x_0, x) = p(gx, x) = p(hx, x) = 0$, $x_0 \in X$.*

Proof. In view of Lemma 3.1 $f = h \circ g$ satisfies (1). From Theorem 2.5 it follows that f has a unique fixed point x, and $\lim_{n\to\infty} p(f^n x_0, x) = p(fx, x) = 0$, $x_0 \in X$. We have

$$fgx = (h \circ g)gx = (g \circ (h \circ g))x = gfx = gx,$$

i.e. gx is also a fixed point of f. Now, by the uniqueness we obtain $gx = x$. Moreover, $hx = hgx = fx = x$ means that x is a fixed point of h. Suppose, *e.g.* that y is another fixed point of h. Then (10) implies $0 < p(y, x) = p(hy, gx) < p(y, x)$, a contradiction. □

The next theorem is a far extension of [10], Theorem 2.6.

Theorem 3.3. *Let f be a selfmapping on a 0-complete d-metric space (X, p). Assume that for some $r, s \in \mathbb{N}$ and $g = f^r$, $h = f^s$ conditions (10), (11) are satisfied. Then f has a unique fixed point x, and $\lim_{n\to\infty} p(f^n x_0, x) = p(fx, x) = 0$, $x_0 \in X$.*

Proof. Clearly, g, h commute. In view of Theorem 3.2 f^{r+s} has a unique fixed point x, and $\lim_{n\to\infty} p((f^{r+s})^n x_0, x) = p(f^{r+s}x, x) = 0$, $x_0 \in X$. Now, we apply [7] Lemma 29. □

Theorem 3.4. *Let $\mathcal{F}$ be a family of selfmappings on a 0-complete d-metric space (X, p). Assume that (10), (11) are satisfied for all $g, h \in \mathcal{F}$. Then all members of $\mathcal{F}$ have the same unique fixed point x, and $\lim_{n\to\infty} p(f^n x_0, x) = p(fx, x) = 0$, $f \in \mathcal{F}$, $x_0 \in X$.*

Proof. In view of Theorem 2.4 each member f of $\mathcal{F}$ has a unique fixed point x, and $\lim_{n\to\infty} p(f^n x_0, x) = p(fx, x) = 0$, $x_0 \in X$. Suppose $hy = y$, $gx = x$, and $x \neq y$. Then (10) implies

$$0 < p(y, x) = p(hy, gx) < p(y, x),$$

a contradiction. □

The next theorem extends [10], Theorem 2.12.

Theorem 3.5. *Let $\mathcal{F}$ be a family of selfmappings on a 0-complete d-metric space (X, p). Assume that (10), (11) are satisfied for all $g, h \in \mathcal{F}$ replaced by g^r, h^s ($r, s \in \mathbb{N}$ are fixed). Then all members of $\mathcal{F}$ have the same unique fixed point x, and $\lim_{n\to\infty} p(f^n x_0, x) = p(fx, x) = 0$, $f \in \mathcal{F}$, $x_0 \in X$.*

Proof. In view of Theorem 3.4 g^r, h^s have the same unique fixed point. Now, we apply [7], Lemma 29. □

The notion of cyclic mappings was formalized by Rus in [12] and the idea itself is due to Kirk, Srinivasan and Veeramani [3]. We adopt the notations from [9], Definition 2.5. For a $t \in \mathbb{N}$ we put $t++ = 1$, and $j++ = j+1$, for $j \in \{1, \dots, t-1\}$. Then $f\colon X \to X$ is *cyclic* if $X = X_1 \cup \dots \cup X_t$, and $f(X_j) \subset X_{j++}$, $j = 1, \dots, t$.

Now, let us present a version of Lemma 3.1 for cyclic mappings.

Lemma 3.6. *Let g, h be commuting cyclic selfmappings on a d-metric space (X, p), and let the following conditions hold:*

$$\begin{aligned} &p(hy, gx) > 0 \text{ yields } p(hy, gx) < p(y, x),\\ &x \in X_j,\ y \in X_{j++},\ j = 1, \dots, t, \end{aligned} \tag{12}$$

$$\begin{aligned} &\text{for each } \alpha > 0 \text{ there exists an } \epsilon > 0 \text{ for which}\\ &p(y, x) < \alpha + \epsilon \text{ yields } p(hy, gx) \le \alpha,\\ &x \in X_j,\ y \in X_{j++},\ j = 1, \dots, t. \end{aligned} \tag{13}$$

Then for $f = h \circ g$ the following condition is satisfied:

$$\begin{aligned} &\text{for each } \alpha > 0, \text{ there exists an } \epsilon > 0 \text{ such that}\\ &\alpha \le p(y, x) < \alpha + \epsilon \text{ implies } p(fy, fx) < \alpha,\\ &x \in X_j,\ y \in X_{j++},\ j = 1, \dots, t. \end{aligned} \tag{14}$$

Proof. We follow the proof of Lemma 3.1. It works because for $x \in X_j$, $y \in X_{j++}$ we have $gx \in X_{j++}$, $hy \in X_{(j++)++}$. □

Now we can prove an analog of Theorem 3.2 for cyclic mappings.

Theorem 3.7. *Let g, h be commuting cyclic selfmappings on a 0-complete d-metric space (X, p). If (12), (13) hold, then g, h have a unique and common fixed point x, and $\lim_{n\to\infty} p((h \circ g)^n x_0, x) = p(gx, x) = p(hx, x) = 0$, $x_0 \in X$.*

Proof. In view of Lemma 3.6 our mapping $f = h \circ g$ satisfies the assumptions of [11], Theorem 3.3 (see [11], Remark 3.5), as condition (14) implies [11], (20) and (21). □

In turn, Lemma 3.6, [11], Theorem 3.3, and [7], Lemma 29 yield the following analog of Theorem 3.3 (the proof is similar to the proof of Theorem 3.3).

Theorem 3.8. *Let f be a selfmapping on a 0-complete d-metric space (X, p). Assume that for some $r, s \in \mathbb{N}$ and $g = f^r$, $h = f^s$, mappings g, h are cyclic and conditions (12), (13) are satisfied. Then f has a unique fixed point x, and $\lim_{n\to\infty} p(f^n x_0, x) = p(fx, x) = 0$, $x_0 \in X$.*

The next theorem is an analog of Theorem 3.4.

Theorem 3.9. *Let $\mathcal{F}$ be a family of cyclic selfmappings on a 0-complete d-metric space (X, p). Assume that (12), (13) are satisfied for all $g, h \in \mathcal{F}$. Then all members of $\mathcal{F}$ have the same unique fixed point x, and in addition, $\lim_{n\to\infty} p(f^n x_0, x) = p(fx, x) = 0$, $f \in \mathcal{F}$, $x_0 \in X$.*

Proof. In view of [11], Theorem 3.3 each member f of $\mathcal{F}$ has a unique fixed point x, and $\lim_{n\to\infty} p(f^n x_0, x) = p(fx, x) = 0$, $x_0 \in X$. All fixed points belong to $X_1 \cap \cdots \cap X_t$, and therefore the remaining part of the proof of Theorem 3.4 works. □

One can transform Theorem 3.9 to become an analog of Theorem 3.5 (g^r, h^s in place of g, h in (12), (13)). Such result is a consequence of Theorem 3.9 and [7], Lemma 29.

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