Contractions, inwardness, tool theorems *

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Abstract

The paper is devoted to the fixed point theory in four aspects: of contractions, nonexpansive mappings, generalized inward mappings, and of the tool theorems. The manuscript was written about ten years ago.

At first Nadler's concept of contraction for multivalued mappings is replaced here by a more general, and yet elegant condition: *for some* $\alpha + \epsilon < 1$*, and each* $x \in X$ *there exists a* $y \in F(x)$ *such that* $d(F(y), y) \leq \alpha d(y, x) \leq (\alpha + \epsilon) d(F(x), x)$.

For "nonexpansive" mappings we apply bead spaces that are more general than uniformly convex spaces, and our requirements on mappings are weaker than nonexpansivity in the sense of the Hausdorff distance.

In the last, third section the Caristi theorem is replaced by more specialized "tools", and we apply them to obtain stronger fixed point theorems on generalized inward mappings. In particular, if for each $x \in X$ a nearest point of $F(x)$ belongs to the generalized inward set, then the values of F need not to be closed.

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The following three theorems are well known:

Theorem 0.1 (Banach) *Let (X, d) be a complete metric space, $\alpha < 1$, and let a mapping $f\colon X \to X$ satisfy*

$$d(f(y), f(x)) \le \alpha d(y, x), \quad x, y \in X. \tag{1}$$

Then f has a fixed point.

Let 2^Y be the family of all subsets of Y. We say that $F\colon X \to 2^Y$ is a (multivalued) mapping if $F(x) \neq \emptyset$, $x \in X \neq \emptyset$. The graph of F is the set $graph(F) = \{(x, y)\colon x \in X,\, y \in F(x)\}$.

If (Y, d) is a metric space, $A, B \subset Y$ are nonempty, and

$$\max\{\sup_{x\in A} d(x, B), \sup_{y\in B} d(A, y)\}$$

is finite, then the latter is denoted by $D(A, B)$ (the Hausdorff distance).

Theorem 0.2 (Nadler [10, Theorem 5]) *Let (X, d) be a complete metric space, $\alpha < 1$, and let a mapping $F\colon X \to 2^X$ with bounded, closed values satisfy*

$$D(F(y), F(x)) \le \alpha d(y, x), \quad x, y \in X. \tag{2}$$

Then F has a fixed point.

A mapping satisfying (1) or (2) for $\alpha < 1$ is called α-**contraction**, and for $\alpha = 1$ such a mapping is **nonexpansive**.

Theorem 0.3 (Lim [6, Theorem 1]) *Let X be a bounded, closed, convex set in a uniformly convex Banach space, and $F\colon X \to 2^X$ a nonexpansive mapping with compact values. Then F has a fixed point.*

1 Contractions

Let us investigate conditions (1), (2). From (1) it follows that

$$d(f(y), y) \le \alpha d(y, x), \quad x \in X,\, y = f(x).$$

Similarly, from (2) we obtain

$$d(F(y), y) \le D(F(y), F(x)) \le \alpha d(y, x), \quad x \in X,\, y \in F(x).$$

Consequently, for $y \in F(x)$ such that $d(y,x) \leq (1+\epsilon/\alpha)d(F(x),x)$ ($\alpha > 0$) we get

$$d(F(y),y) \leq \alpha d(y,x) \leq (\alpha+\epsilon)d(F(x),x), \quad x \in X.$$

Now, we can see that for any $\epsilon > 0$ it follows from condition (2) that

$$\begin{array}{l}\text{for each } x \in X \text{ there exists a } y \in F(x) \text{ such} \\ \text{that } d(F(y),y) \leq \alpha d(y,x) \leq (\alpha+\epsilon)d(F(x),x),\end{array} \tag{3}$$

and the dependence is valid also for $\alpha = 0$.

Let us recall that a mapping $F\colon X \to 2^X$ is an α-**step** [14, Definition 17] if

$$\begin{array}{l}\text{for each } x \in X, y \in F(x) \text{ there exists} \\ \text{a } z \in F(y) \text{ such that } d(z,y) \leq \alpha d(y,x)\end{array} \tag{4}$$

holds.

Clearly, from $d(F(y),y) \leq d(z,y)$, $z \in F(y)$ it follows that (3) is more general than (4).

These considerations suggest that condition (3) is a natural component for an extension of the Nadler theorem.

Lemma 1.1 *Let (X,d) be a metric space, and $F\colon X \to 2^X$ a mapping satisfying* (3) *with $\alpha + \epsilon < 1$. Then there exists a Cauchy sequence $(x_n)_{n\in\mathbb{N}}$ in X such that $x_{n+1} \in F(x_n)$, $n \in \mathbb{N}$.*

Proof. It is sufficient to consider $\alpha > 0$. Let $x_1 \in X$ be arbitrary. Then for $x := x_n$, and $y := x_{n+1}$ as in (3), $n \in \mathbb{N}$, we obtain

$$\begin{array}{l}\alpha d(x_{n+1},x_n) \leq (\alpha+\epsilon)d(F(x_n),x_n) \leq (\alpha+\epsilon)\alpha d(x_n,x_{n-1}) \leq \\ (\alpha+\epsilon)^2 d(F(x_{n-1})),x_{n-1}) \leq \ldots \leq (\alpha+\epsilon)^n d(F(x_1),x_1).\end{array}$$

Consequently, $d(x_{n+1},x_n) \leq (\alpha+\epsilon)^n C$, $n \in \mathbb{N}$, holds, and $\sum_{n=1}^{\infty} d(x_{n+1},x_n)$ converges, which means (as in Banach's proof) that $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence. □

For $\epsilon = 0$ Lemma 1.1 can be reformulated as follows

Lemma 1.2 *Let (X,d) be a metric space, and $F\colon X \to 2^X$ a mapping satisfying*

$$\begin{array}{r}\textit{for each } x \in X \textit{ there exists a } y \in F(x) \textit{ such that} \\ d(y,x) = d(F(x),x), \textit{ and } d(F(y),y) \leq \alpha d(F(x),x)\end{array} \tag{5}$$

for an $\alpha < 1$. Then there exists a Cauchy sequence $(x_n)_{n\in\mathbb{N}}$ in X such that $x_{n+1} \in F(x_n)$, $n \in \mathbb{N}$.

Clearly, if $F(x)$ is compact (or closed convex in a Hilbert space), then it contains a point nearest to x.

In connection with Lemmas 1.1, 1.2 we are interested in the following dependence:

$$\begin{aligned}&\text{for each } (x_n)_{n\in\mathbb{N}} \text{ in } X \text{ that converges, say to an } x \in X,\\&\text{if } \lim_{n\to\infty} d(F(x_n), x_n) = 0, \text{ then } d(F(x), x) = 0.\end{aligned} \tag{6}$$

It holds, *e.g.*, if $\varphi\colon X \to [0, \infty)$ defined by $\varphi(x) = d(F(x), x)$, $x \in X$, is continuous in its zeros. This happens in particular for lower semicontinuous φ. This case requires closer attention.

Let us recall that for topological spaces X, Y a mapping $F\colon X \to 2^Y$ is **usc** if for every $x \in X$, and any neighborhood V of $F(x)$ there exists a neighborhood U of x such that $F(U) \subset V$.

The subsequent lemma extends [3, Lemma 2]:

Lemma 1.3 *Let X be a set in a metric space (Y, d), and $F\colon X \to 2^Y$ a mapping that is usc or continuous with respect to the Hausdorff metric in Y. Then $d(F(\cdot), \cdot)$ is lower semicontinuous.*

Proof. In view of the continuity assumption on F, if $(x_n)_{n\in\mathbb{N}}$ converges in X to a point x, then for $y_n \in F(x_n)$ there exist $z_n \in F(x)$ satisfying $\lim_{n\to\infty} d(z_n, y_n) = 0$. Hence, for $(y_n)_{n\in\mathbb{N}}$ such that $\liminf_{n\to\infty} d(y_n, x_n) = \liminf_{n\to\infty} d(F(x_n), x_n)$ we obtain

$$\begin{aligned}&d(F(x), x) \le \liminf_{n\to\infty} d(z_n, x) \le \liminf_{n\to\infty}[d(z_n, y_n)+\\&d(y_n, x_n) + d(x_n, x)] = \liminf_{n\to\infty} d(F(x_n), x_n). \qquad \square\end{aligned}$$

Another way of obtaining (6) is to assume that the graph of F is closed. Clearly, all values of such a mapping must be closed. If Y is a regular space, $F\colon X \to 2^Y$ is usc, and all values of F are closed, then the graph of F is closed (see the proof of [1, Lemma, p. 285]). The same holds for (Y, d) being a metric space, and $F\colon X \to 2^Y$ a closed-valued mapping that is continuous with respect to the Hausdorff metric in 2^Y. Therefore, it is better to assume that the graph of F is closed.

In view of the considerations preceding Lemma 1.1, the next theorem extends the theorem of Nadler, and [14, Theorem 21]. It seems to be well placed between the Banach theorem and the Nadler one.

Theorem 1.4 *Let (X, d) be a complete metric space, and $F\colon X \to 2^X$ a closed-valued mapping satisfying* (6) *(e.g. if the graph of F is closed). If* (3) *for some $\alpha + \epsilon < 1$ is satisfied, then F has a fixed point.*

Proof. Let $(x_n)_{n\in\mathbb{N}}$ be a Cauchy sequence as in Lemma 1.1. Consequently, $(x_n)_{n\in\mathbb{N}}$ converges, say to x, (X, d) being complete. Then $d(F(x), x) = 0$ (see (6)), and $x \in F(x)$ this last being closed. $\square$

Theorem 1.4 is more general than [3, Theorem 2], as we do not assume $d(F(\cdot), \cdot)$ to be lower semicontinuous.

The subsequent theorem is more elegant but weaker:

Theorem 1.5 *Let (X, d) be a complete metric space, and $F\colon X \to 2^X$ a mapping satisfying* (6)*. If* (5) *for an $\alpha < 1$ holds, then F has a fixed point.*

Proof. For $(x_n)_{n\in\mathbb{N}}$ as in Lemma 1.2 there exists an $x \in X$ such that $d(F(x), x) = 0$, and in view of (5) $x \in F(x)$. $\square$

For a relaxed version of condition (5) we obtain

Lemma 1.6 *Let (X, d) be a metric space, and $F\colon X \to 2^X$ a mapping satisfying*

$$\begin{aligned} &\textit{for each } x \in X \textit{ there exists a } y \in F(x) \\ &\textit{such that } d(F(y), y) \le \alpha d(F(x), x) \end{aligned} \tag{7}$$

for an $\alpha < 1$. Then there exist $x_n \in X$ such that $\lim_{n\to\infty} d(F(x_n), x_n) = 0$.

Proof. For arbitrary $x_1 \in X$, and $x := x_n$, $y := x_{n+1}$ as in (7), $n \in \mathbb{N}$, we obtain

$$d(F(x_{n+1}), x_{n+1}) \le \alpha d(F(x_n), x_n) \le \cdots \le \alpha^n d(F(x_1), x_1). \qquad \square$$

Clearly, condition (7) does not guarantee $(x_n)_{n\in\mathbb{N}}$ to be a Cauchy sequence. We have the following:

Theorem 1.7 *Let (X, d) be a complete metric space, and $F\colon X \to 2^X$ a closed-valued mapping satisfying* (6) *(e.g. if the graph of F is closed). If* (7) *holds for some $\alpha < 1$, and $\overline{F(X)}$ is compact, then F has a fixed point.*

Proof. In view of Lemma 1.6 there exist a sequence $(x_n)_{n\in\mathbb{N}}$, and $y_n \in F(x_n)$ such that $\lim_{n\to\infty} d(x_n, y_n) = 0$. The relative compactness of $\{y_n\colon n \in \mathbb{N}\}$ yields the relative compactness of $\{x_n\colon n \in \mathbb{N}\}$. Therefore, $(x_n)_{n\in\mathbb{N}}$ has a convergent subsequence, say $(x_{k_n})_{n\in\mathbb{N}}$. Then for $x = \lim_{n\to\infty} x_{k_n} = \lim_{n\to\infty} y_{k_n}$ we have $x \in F(x)$ (see (6)). $\square$

2 Nonexpansive mappings

Lim's theorem (Theorem 0.3) concerns uniformly convex spaces. Let us present a similar idea for metric spaces.

Definition 2.1 ([13, Definition 6]) *A metric space* (X, d) *is a* **bead space** *if the following is valid:*

$$\begin{aligned}&\textit{for every } r > 0, \beta > 0 \textit{ there exists a } \delta > 0 \textit{ such that}\\ &\textit{for each } x, y \in X \textit{ with } d(x, y) \geq \beta \textit{ there exists a } z \in X\\ &\textit{such that } B(x, r+\delta) \cap B(y, r+\delta) \subset B(z, r-\delta).\end{aligned} \tag{8}$$

In particular, each convex set in a uniformly convex space is a bead space [12, Example 3] with $z = (x + y)/2$ in condition (8) (*cf.* [15, Theorem 4]).

A normed space is a bead space if and only if it is uniformly convex [15, Theorem 14]. On the other hand, there exist bead spaces which are not convex sets in uniformly convex spaces [14, Example 3].

Definition 2.2 ([13, Definition 11]) *Let* (X, d) *be a metric space, and* $\mathcal{A}$ *a family of nonempty bounded subsets of* X*. An* $x \in X$ *is a* **central point** *for* $\mathcal{A}$ *if*

$$\begin{aligned}&r(\mathcal{A}) := inf\{t \in (0, \infty) : \textit{ there exist } A \in \mathcal{A}, z \in X \textit{ such that } A \subset\\ &B(z, t)\} = inf\{t \in (0, \infty) : \textit{ there exists } A \in \mathcal{A} \textit{ with } A \subset B(x, t)\}.\end{aligned} \tag{9}$$

The **centre** $c(\mathcal{A})$ *for* $\mathcal{A}$ *is the set of all central points for* $\mathcal{A}$*, and* $r(\mathcal{A})$ *is the* **radius** *of* $\mathcal{A}$*.*

It should be noted that $r(\mathcal{A})$ is defined by condition (9) even if $c(\mathcal{A}) = \emptyset$.

If $\mathcal{A}$ is a family of nonempty, and bounded sets directed by $\supset$ in a bead space, then $c(\mathcal{A})$ consists of at most one point, and it is a singleton if the bead space under consideration is complete (*cf.* [13, Lemma 12]).

If $(x_n)_{n\in\mathbb{N}}$ is a bounded sequence of points of X, then for $A_n = \{x_k : k \geq n\}$, and $\mathcal{A} = \{A_n : n \in \mathbb{N}\}$, $c((x_n)_{n\in\mathbb{N}}) := c(\mathcal{A})$, $r((x_n)_{n\in\mathbb{N}}) := r(\mathcal{A})$ are, respectively, the **(asymptotic) centre**, and the **(asymptotic) radius** of $(x_n)_{n\in\mathbb{N}}$ (see [4]).

From [13, Lemma 12] it follows that the centre of any bounded sequence in a bead space is at most a singleton, and that it is nonempty if such a space is complete (*cf.* [14]).

Let us recall (see [5]) that a bounded sequence $(x_n)_{n\in\mathbb{N}}$ in a metric space (X,d) is **regular** if $r((x_{k_n})_{n\in\mathbb{N}}) = r((x_n)_{n\in\mathbb{N}})$ holds for each of its subsequences $(x_{k_n})_{n\in\mathbb{N}}$; a regular sequence $(x_n)_{n\in\mathbb{N}}$ is **almost convergent** if $c((x_{k_n})_{n\in\mathbb{N}}) = c((x_n)_{n\in\mathbb{N}})$. It is known that any bounded sequence in a metric space contains a regular subsequence [5, Lemma 2]. On the other hand, any regular sequence in a bead space is almost convergent [14, Lemma 13].

Now, we are ready to prove the following:

Theorem 2.3 *Let (X,d) be a bead space, and $F\colon X\to 2^X$ a mapping. Assume that $(x_n)_{n\in\mathbb{N}}$ is a regular sequence, $x\in c((x_n)_{n\in\mathbb{N}})$, $y_n\in F(x_n)$ are such that $\lim_{n\to\infty} d(y_n,x_n)=0$, and*

$$\limsup_{n\to\infty} d(F(x),y_n) \le \limsup_{n\to\infty} d(x,x_n) \tag{10}$$

holds. If $F(x)$ is compact, then $x\in F(x)$.

Proof. Let us adopt $r = r(((x_n)_{n\in\mathbb{N}}))$. There exist $z_n\in F(x)$ such that

$$\begin{aligned}&\limsup_{n\to\infty} d(z_n,x_n) \le \limsup_{n\to\infty} [d(z_n,y_n)+d(y_n,x_n)] =\\ &\limsup_{n\to\infty} d(F(x),y_n) \le \limsup_{n\to\infty} d(x,x_n) = r\end{aligned}$$

(see (10)). The sequence $(z_n)_{n\in\mathbb{N}}$ has a subsequence $(z_{k_n})_{n\in\mathbb{N}}$ that converges, say to a point z, in the compact set $F(x)$. Hence, $\limsup_{n\to\infty} d(z,x_{k_n})\le r$, and in view of [14, Lemma 13] $z\in c((x_{k_n})_{n\in\mathbb{N}}) = c((x_n)_{n\in\mathbb{N}}) = \{x\}$ as $(x_n)_{n\in\mathbb{N}}$ is regular. Thus, we have $x=z\in F(x)$. $\square$

Condition (10) is satisfied, *e.g.*, if

$$y_n\in F(x_n) \text{ are such that } d(F(x),y_n)\le d(x,x_n),\quad n\in\mathbb{N} \tag{11}$$

holds. In turn, (11) follows from

$$F(x_n)\subset \overline{B}(F(x),d(x,x_n)),\quad n\in\mathbb{N}, \tag{12}$$

and (12) is clearly satisfied for nonexpansive F.

Let us recall once again that any bounded sequence in a metric space has a regular subsequence, and the centre of any bounded sequence in a complete bead space is a singleton.

By putting these facts together we obtain the following simplified version of Theorem 2.3:

Theorem 2.4 *Let* (X,d) *be a complete bead space, and* $F\colon X \to 2^X$ *a bounded mapping satisfying* (7) *for an* $\alpha < 1$. *Assume that a regular sequence* $(x_n)_{n\in\mathbb{N}}$, *and* $y_n \in F(x_n)$ *are such that* $\lim_{n\to\infty} d(y_n, x_n) = 0$ *(in view of Lemma 1.6 such* y_n *exist), and for* $x \in c((x_n)_{n\in\mathbb{N}})$ (11) *holds. If* $F(x)$ *is compact, then* $x \in F(x)$.

The formulation is clearer in the case of condition (12):

Theorem 2.5 *Let* (X,d) *be a complete bead space, and* $F\colon X \to 2^X$ *a bounded mapping satisfying* (7) *for an* $\alpha < 1$. *If for the centre* $\{x\}$ *of any regular sequence* $(x_n)_{n\in\mathbb{N}}$ *such that* $\lim_{n\to\infty} d(F(x_n), x_n) = 0$ *the set* $F(x)$ *is compact, and* (12) *holds, then* F *has a fixed point.*

The theorem also follows from [14, Theorem 14].

If we assume that all values of F are compact, then Theorem 2.5 yields

Theorem 2.6 *Let* (X,d) *be a complete bead space, and* $F\colon X \to 2^X$ *a bounded, compact-valued mapping satisfying* (7) *for an* $\alpha < 1$. *If for the centre* $\{x\}$ *of any regular sequence* $(x_n)_{n\in\mathbb{N}}$ *such that* $\lim_{n\to\infty} d(F(x_n), x_n) = 0$ (12) *holds, then* F *has a fixed point.*

The next theorem is a consequence of Theorem 2.5, and it has a "classical" outlook.

Theorem 2.7 *Let* (X,d) *be a complete bead space, and* $F\colon X \to 2^X$ *a bounded, compact-valued nonexpansive mapping. If there exists a sequence* $(x_n)_{n\in\mathbb{N}}$ *such that* $\lim_{n\to\infty} d(F(x_n), x_n) = 0$ *(e.g.* (7) *holds for an* $\alpha < 1$*), then* F *has a fixed point.*

Theorem 2.7 is close to Lim's Theorem 0.3, and for condition (7) it is situated between the Nadler Theorem 0.2, and the Lim theorem.

Remark 2.8 *If for each* $x \in X$ *there exists a* $y \in F(x)$ *such that* $d(y,x) = d(F(x),x)$, *then we define a selection* $g\colon X \to X$ *for* F *by taking* $g(x) = y$. *Our paper [14] contains some theorems concerning the case of (multivalued) selections.*

Problem 2.9 *In his original proof Lim has shown that for any bounded nonexpansive mapping in a uniformly convex space* X *there exist* $x_n \in X$ *with* $\lim_{n\to\infty} d(F(x_n), x_n) = 0$. *If this was true for the bead spaces, condition* (7) *could be disregarded. A proof or a counterexample is needed.*

3 Tool theorems, and inward mappings

Let us recall the Caristi theorem.

Theorem 3.1 (Caristi [2, Theorem (2.1)']**)** *Let* (X, d) *be a complete metric space, and* $g\colon X \to X$ *a mapping. If* $\varphi\colon X \to [0, \infty)$ *is a lower semicontinuous mapping such that*

$$d(x, g(x)) \leq \varphi(x) - \varphi(g(x)), \quad x \in X,$$

then g *has a fixed point.*

Our first "tool" theorem is the following:

Theorem 3.2 *Let* (X, δ) *be a complete metric space, and* $X \subset Y$*. Assume that* $\varphi\colon X \to \mathbb{R}$ *is a lower semicontinuous mapping having a finite lower bound. If* $F\colon X \to 2^Y$ *is a mapping satisfying*

$$\begin{aligned} &\textit{for each } x \in X \setminus F(x) \textit{ there exists a } z \in X \setminus \{x\} \\ &\textit{such that } \delta(x, z) \leq \varphi(x) - \varphi(z), \end{aligned} \tag{13}$$

then F *has a fixed point.*

Proof. We apply the idea from [11]. In view of the Teichmüller-Tukey lemma there exists an x_0 belonging to a maximal set $A \subset X$ such that all $x, z \in A$ satisfy

$$\delta(x, z) \leq \mid \varphi(x) - \varphi(z) \mid .$$

Let us adopt $\gamma = \inf\{\varphi(z)\colon z \in A\}$, and suppose that $\gamma < \varphi(z)$, $z \in A$. Then there exists a sequence $(x_n)_{n\in\mathbb{N}}$ such that $(\varphi(x_n))_{n\in\mathbb{N}}$ decreases to γ. For each $m < n$ we have

$$\begin{aligned} &\delta(x_m, x_n) \leq \delta(x_m, x_{m+1}) + \cdots + \delta(x_{n-1}, x_n) \leq \\ &\varphi(x_m) - \varphi(x_{m+1}) + \cdots + \varphi(x_{n-1}) - \varphi(x_n) = \varphi(x_m) - \varphi(x_n), \end{aligned}$$

and therefore $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence converging, say to an $x \in X$ (X is complete). For any $z \in A$, and large n we have $\delta(x_n, z) \leq \varphi(z) - \varphi(x_n)$,

$$\delta(x, z) \leq \delta(x, x_n) + \delta(x_n, z) \leq \delta(x, x_n) + \varphi(z) - \varphi(x_n) \leq \delta(x, x_n) + \varphi(z) - \varphi(x),$$

(φ is lower semicontinuous), and consequently, $\delta(x, z) \leq \varphi(z) - \varphi(x)$, *i.e.* $x \in A$. In addition, $\gamma \leq \varphi(x) < \varphi(x_n)$ means $\varphi(x) = \gamma$. Suppose $x \notin F(x)$, and let $z \neq x$ be as in (13), *i.e.*

$$\varphi(z) \leq \varphi(x) - \delta(x, z) < \varphi(x) = \gamma.$$

On the other hand, for any $y \in A$ we have

$$\delta(y,z) \le \delta(y,x) + \delta(x,z) \le \varphi(y) - \varphi(x) + \varphi(x) - \varphi(z) = \varphi(y) - \varphi(z),$$

i.e. $z \in A$, and therefore, $\gamma \le \varphi(z)$ which contradicts $\varphi(z) < \gamma$. $\square$

The direct proof of the preceding general theorem is short, and the theorem itself is a convenient tool. On the other hand, Theorems 3.1, 3.2 are equivalent as shown by the following reasoning:

Theorem 3.1 is a consequence of Theorem 3.2 for $F(x) := \{g(x)\}$, and $z := g(x)$. On the contrary, assume F as in Theorem 3.2 has no fixed point. Then for $g(x) = z$ (z as in (13)), $x \in X$, $g\colon X \to X$ is a fixed point free mapping satisfying the assumptions of Theorem 3.1.

The method presented in the proof of Theorem 3.2 is used, in a modified way, to prove theorems 3.8, 3.12.

It is worth noting that [17, Theorem 23] is a far extension of Theorem 3.2, and maybe it could a be starting point to modernize the present paper.

As a consequence of Theorem 3.2 we obtain the following:

Theorem 3.3 *Let (Y,d) be a metric space, $X \subset Y$, and let (X,δ) be a complete metric space. If $F\colon X \to 2^Y$ is a mapping such that $d(\cdot, F(\cdot))$ is lower semicontinuous, and*

$$\text{for each } x \in X \setminus F(x) \text{ there exists a } z \in X \setminus \{x\}$$
$$\text{such that } \delta(x,z) \le d(x,F(x)) - d(z,F(z))$$

holds, then F has a fixed point.

Proof. We apply Theorem 3.2 to $\varphi(x) = d(x, F(x))$, $x \in X$. $\square$

Theorem 3.3 is also a far consequence of [16, Theorem 24] which, alas, has a much longer proof.

Theorem 3.3 applies, *e.g.*, in proving the following:

Theorem 3.4 *Let X be a complete set in a metric space (Y,d). If $F\colon X \to 2^Y$ is an α-contraction, $\alpha + \epsilon < 1$, and*

$$\begin{aligned}&\text{for each } x \in X \setminus F(x) \text{ there exists a } z \in X \setminus \{x\}\\ &\text{such that } (1-\epsilon)d(x,z) \le d(x,F(x)) - d(z,F(x))\end{aligned} \tag{14}$$

is satisfied, then

$$\begin{aligned}&\text{for each } x \in X \setminus F(x) \text{ there exists a } z \in X \setminus \{x\}\\ &\text{such that } (1-\alpha-\epsilon)d(x,z) \le d(x,F(x)) - d(z,F(z))\end{aligned} \tag{15}$$

holds, and consequently, F has a fixed point.

Proof. We have

$$d(z, F(z)) \leq d(z, F(x)) + D(F(x), F(z)) \leq d(z, F(x)) + \alpha d(x, z) =$$
$$d(z, F(x)) - d(x, F(x)) + \alpha d(x, z) + d(x, F(x)), \quad x, z \in X.$$

Consequently, in view of (14) for $x \in X \setminus F(x)$ there exists a $z \in X \setminus \{x\}$ such that

$$d(z, F(z)) \leq \alpha d(x, z) + d(x, F(x)) - (1 - \epsilon) d(x, z),$$

i.e. (15) holds, and we can apply Theorem 3.3 for $\delta = (1 - \alpha - \epsilon) d$ (see Lemma 1.3). $\square$

Condition (14) has much in common with generalized inwardness. Let (Y, d) be a metric space. Then for $x, t \in Y$ a metric segment is defined by

$$[x, t] = \{s \in Y \colon d(x, s) + d(s, t) = d(x, t)\},$$

and $(x, t] = [x, t] \setminus \{x\}$. For a nonempty set $X \subset Y$ a generalized inward set $\tilde{I}_X(x)$ can be defined by (*cf.* [9, p. 1209])

$$\tilde{I}_X(x) = \{t \in Y \colon \text{ for each } \beta > 0 \text{ there exists an } s \in (x, t] \text{ such that } d(z, s) \leq \beta d(x, s)\}.$$

The inward set $I_X(x)$ in a normed space is defined by

$$I_X(x) = x + \{\lambda(z - x) \colon z \in X, \ \lambda \geq 1\},$$

and the weakly inward set is $\overline{I_X(x)}$.

Maciejewski has proved [9, Lemma 1.3] that if Y is a normed space, then $\overline{I_X(x)} \subset \tilde{I}_X(x)$.

Lemma 3.5 *Let (Y, d) be a metric space, X, C nonempty subsets of Y, and $x \in X \subset C$. Then for each $\epsilon > 0$, $t \in C \cap \tilde{I}_X(x)$ there exists a $z \in X \setminus \{x\}$ such that*

$$(1 - \epsilon) d(x, z) \leq d(x, t) - d(z, t).$$

If t is the nearest point of C to x, and $t \in \tilde{I}_X(x)$, then

$$(1 - \epsilon) d(x, z) \leq d(x, C) - d(z, C).$$

Proof. For a $t \in C$ let $s \in (x, t]$, $z \in X$ be such that $d(z, s) \le \beta d(x, s)$. We have

$$d(x, z) + d(z, t) - d(x, t) \le d(x, s) + d(s, z) + d(z, s) + d(s, t) - d(x, t) \le$$
$$2d(z, s) + d(x, s) + d(s, t) - d(x, t) = 2d(z, s) \le 2\beta d(x, s).$$

From

$$d(x, s) \le d(x, z) + d(z, s) \le d(x, z) + \beta d(x, s)$$

we obtain $d(x, s) \le d(x, z)/(1 - \beta)$ for $\beta < 1$. Hence it follows that

$$d(x, z) + d(z, t) - d(x, t) \le 2\beta d(x, z)/(1 - \beta) \le \epsilon d(x, z),$$

i.e.

$$(1 - \epsilon)d(x, z) \le d(x, t) - d(z, t)$$

for small $\beta > 0$. If t is the nearest point to x in C, then we have

$$(1 - \epsilon)d(x, z) \le d(x, C) - d(z, t) \le d(x, C) - d(z, C). \qquad \square$$

From Theorem 3.4, and Lemma 3.5 when applied to $C = F(x)$ we obtain the following extension of [9, Theorem 2.3] (we do not demand $F(x)$ to be closed), and of [18, Theorem 3.3]:

Theorem 3.6 *Let X be a complete set in a metric space (Y, d). If $F\colon X \to 2^Y$ is an α-contraction, each $x \in X$ has a nearest point in $F(x)$, and such a point belongs to $\tilde{I}_X(x)$, then for each $\epsilon > 0$ (15) is satisfied, and consequently, F has a fixed point.*

For the inwardness condition we have a "strict" result:

Theorem 3.7 *Let X be a complete set in a normed space Y. If $F\colon X \to 2^Y$ is an α-contraction, each $x \in X$ has a nearest point in $F(x)$, and such a point belongs to $I_X(x)$, then (15) is satisfied for $\epsilon = 0$, and F has a fixed point.*

Proof. For $y = x + \lambda(z - x)$ we have $d(y, z) = d(y, x) - d(z, x)$. Hence it follows (see (2)) that

$$d(F(z), z) \le D(F(z), F(x)) + d(F(x), z) \le$$
$$\alpha d(z, x) + d(y, z) = \alpha d(z, x) + d(y, x) - d(z, x),$$

and for $d(y, x) = d(F(x), x)$ we get (15) with $\epsilon = 0$. Now, we can apply Theorem 3.3 for $\delta = (1 - \alpha)d$ (see Lemma 1.3). $\square$

Let us present another "tool" theorem.

Theorem 3.8 *Let X be a set in a complete metric space (Y, d). Assume that a mapping $F\colon X \to 2^Y$ has closed graph, and δ is a metric on $X \cup F(X)$ equivalent to d. If*

$$\begin{aligned}&\text{for each } x \in X \setminus F(x),\ t \in F(x) \text{ there exist } z \in X \setminus \{x\},\ v \in F(x)\\ &\text{such that } \delta(x,z) \le d(x,t) - d(z,v), \text{ and } \delta(t,v) \le k(d(x,t) - d(z,v))\end{aligned} \tag{16}$$

holds for a $k > 0$, then F has a fixed point.

Proof. Assume $x_0 \notin F(x_0)$, and let $A \neq \emptyset$ be a maximal set in $G = graph(F)$ such that for $(x,t), (z,v) \in A$

$$\delta(x,z) \le \mid d(x,t) - d(z,v) \mid, \quad \delta(t,v) \le k \mid d(x,t) - d(z,v) \mid$$

are satisfied. Let us adopt $\gamma = \inf\{d(x,t)\colon (x,t) \in A\}$, and suppose that $\gamma < d(x,t)$, $(x,t) \in A$. Then there exists a sequence $((x_n, t_n))_{n\in\mathbb{N}}$ such that $(d(x_n, t_n))_{n\in\mathbb{N}}$ decreases to γ. For each $m < n$ we have

$$\delta(x_m, x_n) \le \delta(x_m, x_{m+1}) + \cdots + \delta(x_{n-1}, x_n) \le d(x_m, t_m) - d(x_n, t_n),$$

and therefore, $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence. Similarly, $(t_n)_{n\in\mathbb{N}}$ is also a Cauchy sequence. Then for $x = \lim_{n\to\infty} x_n$, $t = \lim_{n\to\infty} t_n$, $(x,t) \in G$ holds as G is closed. For any $(z,v) \in A$, and large n we have

$$\delta(x,z) \le \delta(x,x_n) + \delta(x_n,z) \le \delta(x,x_n) + d(z,v) - d(x_n,t_n).$$

Both metrics are equivalent on $X \cup F(X)$, and consequently, $\delta(x,z) \le d(z,v) - d(x,t)$. In a similar way we state that $\delta(t,v) \le k(d(z,v) - d(x,t))$. Therefore, $(x,t) \in A$, and $d(x,t) = \gamma$. Suppose $x \notin F(x)$, and let $(z,v) \neq (x,t)$ be as in (16), *i.e.*

$$d(z,v) \le d(x,t) - \delta(x,z) < d(x,t) = \gamma.$$

On the other hand, for any $(y,s) \in A$ we have

$$\begin{aligned}&\delta(y,z) \le \delta(y,x) + \delta(x,z) \le d(y,s) - d(x,t) + d(x,t) - d(z,v) =\\ &d(y,s) - d(z,v), \qquad \delta(s,v) \le k(d(y,s) - d(z,v)).\end{aligned}$$

Consequently, $(z,v) \in A$ holds, and therefore, $\gamma \le d(z,v)$ which contradicts $d(z,v) < \gamma$. $\square$

Now, we are ready to prove an analog of Theorem 3.4.

Theorem 3.9 *Let X be a closed set in a complete metric space (Y, d). If $F\colon X \to 2^Y$ is a closed valued α-contraction, $\alpha + \epsilon < 1$, and*

$$\begin{array}{l} \textit{for an } \epsilon_1 \in (0, \epsilon), \textit{ and each } x \in X \setminus F(x),\ t \in F(x) \textit{ there exists} \\ \textit{a } z \in X \setminus \{x\} \textit{ such that } (1 - \epsilon_1) d(x, z) \le d(x, t) - d(z, t) \end{array} \tag{17}$$

holds, then the following is satisfied:

$$\begin{array}{l} \textit{for each } x \in X \setminus F(x),\ t \in F(x) \textit{ there exist } z \in X \setminus \{x\},\ v \in F(x) \\ \textit{such that } (1 - \alpha - \epsilon) \max\{d(x, z), d(t, v)/(\alpha + \epsilon)\} \le d(x, t) - d(z, v), \end{array} \tag{18}$$

and F has a fixed point.

Proof. For each $\epsilon_1 \in (0, \epsilon)$, $x, z \in X$, and $t \in F(x)$ there exists a $v \in F(z)$ such that

$$d(t, v) \le (\alpha + \epsilon - \epsilon_1) d(x, z) < (\alpha + \epsilon) d(x, z) \tag{19}$$

holds, as F is an α-contraction. On the other hand, in view of (17), and (19) we have

$$\begin{array}{l} d(z, v) + (1 - \epsilon_1) d(x, z) \le d(z, v) + d(x, t) - d(z, t) \le \\ d(t, v) + d(x, t) \le (\alpha + \epsilon - \epsilon_1) d(x, z) + d(x, t). \end{array}$$

Hence it follows that

$$(1 - \alpha - \epsilon) d(x, z) \le d(x, t) - d(z, v).$$

Now, it is clear that (16) is satisfied for $\delta = (1 - \alpha - \epsilon) d$, $k = \alpha + \epsilon$ (*i.e.* (18) is valid), and we may apply Theorem 3.8, as the graph of a closed valued α-contraction on a closed set is closed. □

From Lemma 3.5, and Theorem 3.9 we obtain the following refinement of Maciejewski's Theorem 2.1 [9, p. 1210], and of the well known theorem of Lim for inward contractions [8, Theorem 1]:

Theorem 3.10 *Let X be a closed set in a complete metric space (Y, d). If $F\colon X \to 2^Y$ is a closed valued α-contraction such that $F(x) \subset \tilde{I}_X(x)$, $x \in X$, then for each $\epsilon > 0$ with $\alpha + \epsilon < 1$ condition* (18) *is satisfied, and F has a fixed point.*

The proofs of Theorems 3.8, 3.9 yield

Corollary 3.11 *Let X be a closed set in a complete metric space (Y,d). If $F\colon X\to 2^Y$ is a closed valued α-contraction such that $F(x)\subset \tilde{I}_X(x)$, $x\in X$, then for each $\epsilon>0$ with $\alpha+\epsilon<1$ there exist convergent sequences $(x_n)_{n\in\mathbb{N}}$, $(t_n)_{n\in\mathbb{N}}$ such that $t_n\in F(x_n)$, $n\in\mathbb{N}$, $x=\lim_{n\to\infty}x_n\in F(x)$, $t=\lim_{n\to\infty}t_n\in F(x)$, and*

$$(1-\alpha-\epsilon)d(x_n,x_{n+1})\le d(x_n,t_n)-d(x_{n+1},t_{n+1}),\quad n\in\mathbb{N}.$$

Theorem 3.9 is also a consequence of the next "tool" theorem.

Theorem 3.12 *Let X be a set in a complete metric space (Y,d). Assume that $F\colon X\to 2^Y$ is a mapping, and δ is a complete metric on $G=graph(F)$. If*

$$\begin{aligned}&\text{for each } x\in X\setminus F(x),\ t\in F(x) \text{ there exist } z\in X\setminus\{x\},\\ &v\in F(x) \text{ such that } \delta((x,t),(z,v))\le d(x,t)-d(z,v)\end{aligned}\tag{20}$$

holds, then F has a fixed point.

Proof. Once again we use the method from the proof of Theorem 3.2. For simplicity of notations let us adopt $p=(x,t)$, $p_n=(x_n,t_n)$, $q=(z,v)$, and (not quite correctly) $d(p)=d(x,t)$, $d(p_n)=d(x_n,t_n)$, $d(q)=d(z,v)$. If $x_0\notin F(x_0)$, then there exists a maximal set $A\subset G$, $A\ne\emptyset$ such that all $p,q\in A$ satisfy

$$\delta(p,q)\le\mid d(p)-d(q)\mid .$$

Let us adopt $\gamma=\inf\{d(q)\colon q\in A\}$, and suppose that $\gamma<d(q)$, for each $q\in A$. Then there exists a sequence $(p_n)_{n\in\mathbb{N}}$ in G such that $(\varphi(p_n))_{n\in\mathbb{N}}$ decreases to γ. Hence it follows that

$$\delta(p_m,p_n)\le d(p_m)-d(p_n),\quad m<n,$$

i.e. $(p_n)_{n\in\mathbb{N}}$ is a Cauchy sequence, and there exists a $p=\lim_{n\to\infty}p_n\in G$ as δ is complete. For any $q\in A$, and large n we have

$$\delta(p,q)\le\delta(p,p_n)+\delta(p_n,q)\le\delta(p,p_n)+d(q)-d(p_n),$$

and $\delta(p,q)\le d(q)-d(p)$, which means $p\in A$, and $d(p)=\gamma$ (implies p is unique in A). Suppose $x\notin F(x)$. Then for $q=(z,v)$ as in (20) from $z\ne x$ it follows that $d(q)<\gamma$, and for any $r=(y,s)\in A$ we obtain

$$\delta(r,q)\le\delta(r,p)+\delta(p,q)\le d(r)-d(p)+d(p)-d(q)=d(r)-d(q),$$

which means $q \in A$, and $\gamma \leq d(q)$ - a contradiction. $\square$

The previous theorem could be used to prove Theorem 3.10. The respective complete metric δ on $graph(F)$ is defined by the left side of the inequality in condition (18).

Let us go back to condition (15), and Theorem 3.6. As regards the "nonexpansive" case ($\alpha + \epsilon = 1$), (15) means

$$\begin{aligned}&\text{for each } x \in X \setminus F(x) \text{ there exists a } z \in X \setminus \{x\}\\ &\text{such that } d(z, F(z)) \leq d(x, F(x)),\end{aligned}$$

and it would be rather an ambitious task to prove anything for such a condition ;). More interesting is its sharper form:

$$\begin{aligned}&\text{for each } x \in X \setminus F(x) \text{ there exists a } z \in X\\ &\text{such that } d(z, F(z)) < d(x, F(x)).\end{aligned} \tag{21}$$

It is known (see [7, proof of Theorem 8]) that if X is a bounded closed convex set in a Banach space Y, and $F\colon X \to 2^Y$ is a nonexpansive weakly inward mapping, then there exists a $(x_n)_{n\in\mathbb{N}}$ such that $\lim_{n\to\infty} d(x_n, F(x_n)) = 0$. Consequently, for such a case, and $d(x, F(x)) = 0$ iff $x \in F(x)$ (21) holds. Therefore, the following is also an "inward" theorem (*cp.* Theorems 1.5, 1.7).

Theorem 3.13 *Let X be a compact set in a metric space (Y, d), and let $F\colon X \to 2^Y$ be a mapping satisfying (21), and such that $d(F(\cdot), \cdot)$ is lower semicontinuous. Then F has a fixed point.*

Proof. From the lower semicontinuity of $d(\cdot, F(\cdot))$ it follows that for each $\lambda \geq 0$ the set $X_\lambda = \{x \in X\colon d(x, F(x)) \leq \lambda\}$ is compact. Suppose $\alpha = inf\{\lambda\colon X_\lambda \neq \emptyset\} > 0$. X_α is clearly nonempty, as it is an intersection of a decreasing family of nonempty compact sets. For any $x \in X_\alpha$ there exists a $z \in X$ such that $d(z, F(z)) < d(x, F(x))$ (see (21)) - a contradiction. Thus X_0 consisting of fixed points of F is nonempty. $\square$

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