



A NOTE ON SOME FIXED POINT THEOREMS

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Abstract

In this note, a precise proof of a far extension of the Meir-Keeler theorem is presented, and it is shown that some celebrated fixed point theorems can be easily derived from our result.

The main aim of this note is to present a complete and short proof of a fixed point theorem that is a far extension of the well-known Meir-Keeler theorem [5] proved for metric spaces. Our theorem was announced in [9], and it also extends the celebrated results of Theorem 1.5.1 of Matkowski in [3], and of Ćirić [1]. In addition, a short reasoning is given to show that those results can be easily derived from the general theorem of ours.

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In [8, Definition 6.1], the notion of a felt metric was introduced. It is more convenient not to specify the topology of a felt metric space (X, p) , as in particular, p can be a metric, a dislocated metric [2] or a partial metric [4, Definition 3.1] while the respective topologies can differ (see also [6]). Therefore, the below definition is a slight modification of the original one.

Definition 1. A mapping $p : X \times X \rightarrow [0, \infty)$ is a *felt metric* for X if the following system of conditions is satisfied:

$$p(x, y) = 0 \text{ yields } x = y, \quad x, y \in X, \quad (1a)$$

$$p(x, y) = p(y, x), \quad x, y \in X, \quad (1b)$$

for each $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$p(z, y) < \delta \text{ yields } |p(z, x) - p(y, x)| < \varepsilon, \quad x, y, z \in X. \quad (1c)$$

A felt metric space (X, p) is 0-complete, if for each sequence $(x_n)_{n \in \mathbb{N}}$ in X , such that $\lim_{m, n \rightarrow \infty} p(x_n, x_m) = 0$, there exists an $x \in X$ for which $\lim_{n \rightarrow \infty} p(x_n, x) = 0$; a self mapping f on X is 0-continuous at an $x \in X$, if $\lim_{n \rightarrow \infty} p(x_n, x) = 0$ yields $\lim_{n \rightarrow \infty} p(fx_n, fx) = 0$, for each sequence $(x_n)_{n \in \mathbb{N}}$ in X ; f is 0-continuous if it is 0-continuous at each $x \in X$.

Let us prove the following result which extends [8, Theorem 6.6] (see [9, Theorem 2]).

Theorem 2. Assume that p is a felt metric for X , f is a self mapping on X , $x_n = f^n x_0$, $n \in \mathbb{N}$, $\lim_{n \rightarrow \infty} p(x_{n+1}, x_n) = 0$, and let

for each $\alpha > 0$, there exists an $\varepsilon > 0$ such that

$$\alpha \leq p(y, x) < \alpha + \varepsilon \text{ yields } p(fy, fx) \leq \alpha, \quad x, y \in X \quad (2)$$

be satisfied. If (X, p) is 0-complete, then there exists an $x \in X$ such that $\lim_{n \rightarrow \infty} p(x_n, x) = p(x, x) = 0$ and $fx = x$.

Proof. In particular, for $\alpha = p(y, x) > 0$, condition (2) yields $p(fy, fx) \leq p(y, x)$, i.e., f is nonexpansive (for $p(y, x) > 0$). Therefore, (2) yields the following [8, condition (14)] for all $m, n \in \mathbb{N}$ (n_0 can be disregarded):

for each $\alpha > 0$, there exist $\varepsilon > 0$ and $n_0 \in \mathbb{N}$ such that for all

$$m, n > n_0, \alpha - \varepsilon < p(x_n, x_m) < \alpha + \varepsilon \text{ implies } p(x_{n+1}, x_{m+1}) \leq \alpha,$$

and [8, Lemma 6.3] applies. Consequently, $\lim_{m, n \rightarrow \infty} p(x_n, x_m) = 0$, and what is more, $\lim_{n \rightarrow \infty} p(x_n, x) = p(x, x) = 0$ holds for some $x \in X$, as (X, p) is 0-complete. Let us prove that x is a fixed point of f . Suppose $p(x, fx) = 2\beta > 0$. From (1c), it follows that

$$\lim_{n \rightarrow \infty} |p(x_{n+1}, fx) - p(x, fx)| = \lim_{n \rightarrow \infty} p(x_{n+1}, x) = 0,$$

and we obtain $\beta < p(x_{n+1}, fx)$ for large $n \in \mathbb{N}$. Let us consider an $n \in \mathbb{N}$ such that $0 < p(x_n, x) = \alpha < \beta$. Now, (2) yields $\beta < p(x_{n+1}, fx) \leq \alpha$, a contradiction. Consequently, $p(x, fx) = 0$, and $fx = x$ (see (1a)). \square

It is worth of noting that for $X = \{-1, 1\}$, $fx = -x$ and $p(y, x) = |y - x|$, condition (2) holds, but $p(x_{n+1}, x_n) = 2$, $n \in \mathbb{N}$.

The Banach theorem is an immediate consequence of Theorem 2, as for $0 \leq c < 1$, one can calculate the respective $\varepsilon > 0$ (see (2)) from the subsequent formula:

$$p(fy, fx) \leq cp(y, x) \leq c(\alpha + \varepsilon) \leq \alpha.$$

Meir and Keeler [5] used the following condition (p was a metric in their paper):

for each $\alpha > 0$, there exists an $\varepsilon > 0$ such that

$$\alpha \leq p(y, x) < \alpha + \varepsilon \text{ implies } p(fy, fx) < \alpha. \quad (3)$$

Clearly, (3) yields (2).

Theorem 1.5.1 of Matkowski in [3] and Ćirić [1] used two conditions. The first one is

$$p(fy, fx) < p(y, x), \quad x \neq y.$$

If p is a metric, then the above condition is equivalent to the following one:

$$p(fy, fx) > 0 \text{ yields } p(fy, fx) < p(y, x). \quad (4)$$

The second condition of Matkowski, and of Ćirić is

for each $\alpha > 0$, there exists an $\varepsilon > 0$ for which

$$\alpha < p(y, x) < \alpha + \varepsilon \text{ yields } p(fy, fx) \leq \alpha. \quad (5)$$

If (4) holds, then we may put $\alpha \leq p(y, x) < \alpha + \varepsilon$ in (5), and our condition (2) is obtained. Thus, (4) together with (5) yield (2). Consequently, Theorem 2 extends the results of Meir and Keeler, of Matkowski, and of Ćirić, also for the case of metric spaces. To derive those celebrated equivalent results from Theorem 2, it is sufficient to show that from condition (3) or from (4) with (5), it follows that $\lim_{n \rightarrow \infty} p(x_{n+1}, x_n) = 0$ (see the assumptions of Theorem 2). The following was proved shortly as Lemma 2.1 in [7].

Lemma 3. *Let $(a_n)_{n \in \mathbb{N}}$ be a nonnegative sequence such that*

$$a_{n+1} > 0 \text{ yields } a_{n+1} < a_n, \quad n \in \mathbb{N}. \quad (6)$$

Then $\lim_{n \rightarrow \infty} a_n = 0$ iff the following condition is satisfied:

for each $\alpha > 0$, there exists an $\varepsilon > 0$ such that

$$\alpha < a_n < \alpha + \varepsilon \text{ yields } a_{n+1} \leq \alpha, \quad n \in \mathbb{N}. \quad (7)$$

Let us put $y = x_{n+1}$, $x = x_n$ and $a_n = p(x_{n+1}, x_n)$. Then it is obvious that (4) yields (6), and (5) yields (7). As regards (3), for $a_n = \alpha$, we get $a_{n+1} < \alpha = a_n$, i.e., (6). In turn, for $\alpha < a_n < \alpha + \varepsilon$, (3) yields

$a_{n+1} < \alpha < a_n$, i.e., (7) is satisfied. Consequently, Theorem 2 contains the respective results of Meir and Keeler, of Matkowski, and of Ćirić.

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