Recursive definition of 1D B-splines

J.A. Cottrel, T.J.R. Hughes, Y. Bazilevs, *Isogeometric Analysis. Toward Integration of CAD and FEA*, Wiley, (2009).

$$N_{i,0}(\xi) = \begin{cases} 1 & \text{if } \xi_i \le \xi < \xi_{i+1}, \\ 0 & \text{otherwise} \end{cases}$$

$$N_{i,p}(\xi) = \frac{\xi - \xi_i}{\xi_{i+p} - \xi_i} N_{i,p-1}(\xi) + \frac{\xi_{i+p+1} - \xi}{\xi_{i+p+1} - \xi_{i+1}} N_{i+1,p-1}(\xi)$$

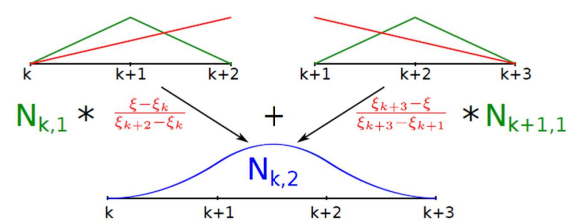

Figure: Recursive formulae for B-spline basis functions and its explanation

Representation of B-splines by knot vectors

J.A. Cottrel, T.J.R. Hughes, Y. Bazilevs, *Isogeometric Analysis. Toward Integration of CAD and FEA*, Wiley, (2009).

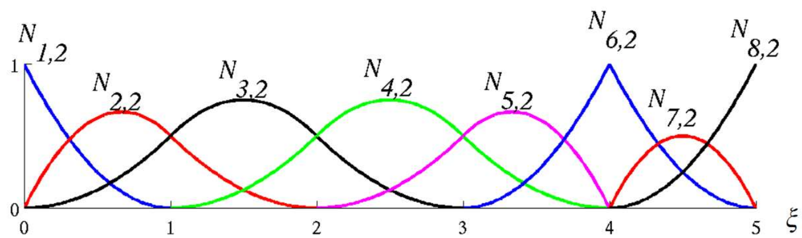

Figure: B-spline basis functions represented by knot vector $\{0,0,0,1,2,3,4,4,5,5,5\}$

Tensor product definition of 2D B-spline basis functions

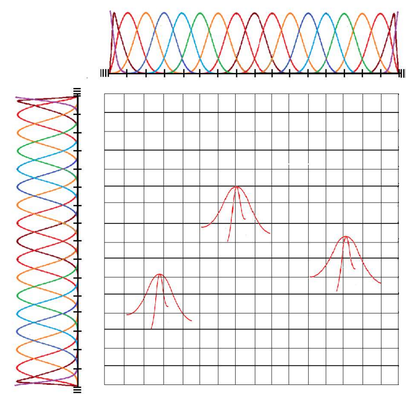

Figure: Tensor products B-splines basis funcitions

- 1D B-splines basis $B_1^x(x), \ldots, B_{N_x}^x(x)$, $B_1^y(y), \ldots, B_{N_y}^y(y)$,
- 2D B-splines basis $B_{i,j}(x,y) = B_i^x(x) * B_j^y(y)$

Projection with isogeometric finite element method

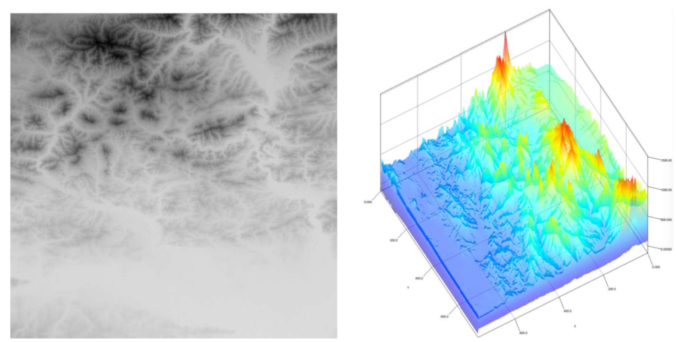

Figure: 2D terrain bitmap and its continuous B-spline approximation

- We want to approximate a BITMAP(x,y) with a linear combination of B-splines
  $u(x,y) \approx BITMAP(x,y)$
  where $u(x,y) = \sum_{i,j} u_{i,j} B_i^x(x) B_j^y(y)$
- How to construct a system of linear equations to get the coefficients $u_{i,j}$?

Projection with isogeometric finite element method

- We choose several "test functions" v, and use them to average the BITMAP at test functions supports
  $\int u(x,y)v(x,y)dx = \int BITMAP(x,y)v(x,y)dx$
- Each selection of $v = B_i^x(x)B_j^y(y)$ leads to one equation
  $\int u(x,y)B_i^x(x) * B_j^y(y)dx = \int BITMAP(x,y)B_i^x(x)B_j^y(y)dx$

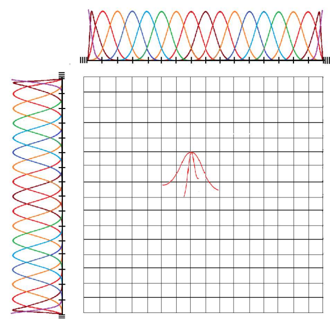

Figure: One exemplary selection of $v = B_i^x(x)B_j^y(y)$

Projection with isogeometric finite element method

We end up with several equations
(one equation per one testing B-spline in 2D)

$$\int u(x,y)B_1^x(x)B_1^y(y)dx = \int BITMAP(x,y)B_1^x(x)B_1^y(y)dx$$
$$\int u(x,y)B_1^x(x)B_2^y(y)dx = \int BITMAP(x,y)B_1^x(x)B_2^y(y)dx$$
$$\vdots$$
$$\int u(x,y)B_k^x(x)B_l^y(y)dx = \int BITMAP(x,y)B_k^x(x)B_l^y(y)dx$$
$$\vdots$$
$$\int u(x,y)B_{N_x}^x(x)B_{N_y-1}^y(y)dx = \int BITMAP(x,y)B_{N_x}^x(x)B_{N_y-1}^y(y)dx$$
$$\int u(x,y)B_{N_x}^x(x)B_{N_y}^y(y)dx = \int BITMAP(x,y)B_{N_x}^x(x)B_{N_y}^y(y)dx$$

Projection with isogeometric finite element method

We approximate $u(x,y) \approx \sum_{i,j} u_{i,j} B^x_i(x) B^y_j(y)$

$$\int u(x,y) B^x_1(x) B^y_1(y) dx = \int BITMAP(x,y) B^x_1(x) B^y_1(y) dx$$
$$\int u(x,y) B^x_1(x) B^y_2(y) dx = \int BITMAP(x,y) B^x_1(x) B^y_2(y) dx$$
$$\vdots$$
$$\int u(x,y) B^x_k(x) B^y_l(y) dx = \int BITMAP(x,y) B^x_k(x) B^y_l(y) dx$$
$$\vdots$$
$$\int u(x,y) B^x_{N_x}(x) B^y_{N_y-1}(y) dx = \int BITMAP(x,y) B^x_{N_x}(x) B^y_{N_y-1}(y) dx$$
$$\int u(x,y) B^x_{N_x}(x) B^y_{N_y}(y) dx = \int BITMAP(x,y) B^x_{N_x}(x) B^y_{N_y}(y) dx$$

Projection with isogeometric finite element method

We approximate $u(x,y) \approx \sum_{i,j} u_{i,j} B^x_i(x) B^y_j(y)$ to get

$$\int \sum_{i,j} u_{i,j} B^x_i(x) B^y_j(y) B^x_1(x) * B^y_1(y) dx = \int BITMAP(x,y) B^x_1(x) * B^y_1(y) dx$$

$$\int \sum_{i,j} u_{i,j} B^x_i(x) B^y_j(y) B^x_1(x) * B^y_2(y) dx = \int BITMAP(x,y) B^x_1(x) * B^y_2(y) dx$$

$$\vdots$$

$$\int \sum_{i,j} u_{i,j} B^x_i(x) B^y_j(y) B^x_k(x) * B^y_l(y) dx = \int BITMAP(x,y) B^x_k(x) * B^y_l(y) dx$$

$$\vdots$$

$$\int \sum_{i,j} u_{i,j} B^x_i(x) B^y_j(y) B^x_{N_x}(x) * B^y_{N_y-1}(y) dx = \int BITMAP(x,y) B^x_{N_x}(x) * B^y_{N_y-1}(y) dx$$

$$\int \sum_{i,j} u_{i,j} B^x_i(x) B^y_j(y) B^x_{N_x}(x) * B^y_{N_y}(y) dx = \int BITMAP(x,y) B^x_{N_x}(x) * B^y_{N_y}(y) dx$$

Projection with isogeometric finite element method

We take the sum out

$$\sum_{i,j} u_{i,j} \int B^x_i(x) B^y_j(y) B^x_k(x) * B^y_l(y) dx = \int BITMAP(x,y) B^x_k(x) * B^y_l(y) dx$$

and we end up with a system of linear equations

$$\begin{bmatrix} \int B^x_{1,p}B^y_{1,p}B^x_{1,p}B^y_{1,p} & \int B^x_{1,p}B^y_{1,p}B^x_{2,p}B^y_{1,p} & \cdots & \int B^x_{1,p}B^y_{1,p}B^x_{N_x,p}B^y_{N_y,p} \\ \int B^x_{2,p}B^y_{1,p}B^x_{1,p}B^y_{1,p} & \int B^x_{2,p}B^y_{1,p}B^x_{2,p}B^y_{1,p} & \cdots & \int B^x_{2,p}B^y_{1,p}B^x_{N_x,p}B^y_{N_y,p} \\ \vdots & \vdots & \vdots & \vdots \\ \int B^x_{N_x,p}B^y_{N_y,p}B^x_{1,p}B^y_{1,p} & \int B^x_{N_x,p}B^y_{N_y,p}B^x_{2,p}B^y_{1,p} & \cdots & \int B^x_{N_x,p}B^y_{N_y,p}B^x_{N_x,p}B^y_{N_y,p} \end{bmatrix} \begin{bmatrix} u_{1,1} \\ u_{2,1} \\ \vdots \\ u_{N_x,N_y} \end{bmatrix}$$

$$= \begin{bmatrix} \int BITMAP(x,y) B^x_1(x) * B^y_1(y) dx \\ \int BITMAP(x,y) B^x_1(x) * B^y_2(y) dx \\ \vdots \\ \int BITMAP(x,y) B^x_{N_x}(x) * B^y_{N_y}(y) dx \end{bmatrix}$$

Projection with isogeometric finite element method

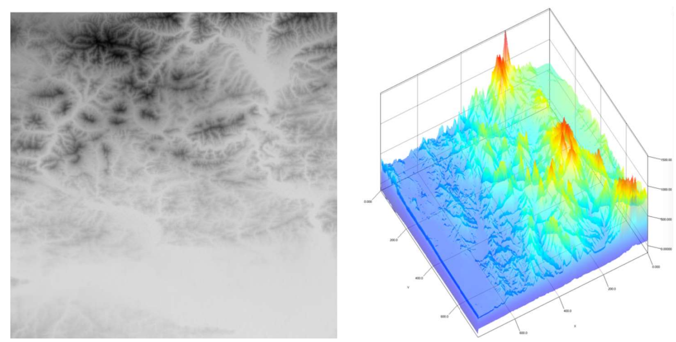

Figure: 2D terrain bitmap and its continuous B-spline approximation

We approximated a BITMAP(x,y) with a linear combination of B-splines $u(x,y) \approx BITMAP(x,y)$ where $u(x,y) = \sum_{i,j} u_{i,j} B_i^x(x) B_j^y(y)$