paszynsk@agh.edu.pl

Lagrange polynomials

Brak plików do pobrania.

In this chapter we derive the basis functions called the classical Lagrange polynomials. 1st order Lagrange polynomials are equivalent to 1st order B-spline functions, they have exactly the same formulas.

Second order Lagrange polynomials can be derived from the general formula for B-splines using a knot vector generating second-order B-splines in which all internal knots have been repeated p times, so

$[0 \quad 0 \quad 0 \quad 1 \quad 1 \quad 2 \quad 2 \quad 3 \quad 3 \quad 4 \quad 4 \quad 5 \quad 5 \quad 5]$

The vector of knots defined in this way will generate basis functions equivalent to the second degree Lagrange polynomials (quadratic polynomials) used in the traditional finite element method. In order to illustrate the basis functions resulting from the various knot vectors, we recommend the attached MATLAB code.

What do such basis functions look like? We need to generate step-by-step all zero-degree polynomials, and first- and second-order polynomials, using the Cox-de-Boor formula. Let's start with zero degree polynomials. We have now

$\xi_1 = \xi_2 = \xi_3 = 0, \xi_4 = \xi_5 = 1, \xi_6 = \xi_7 = 2, \xi_8 = \xi_9 = 3, \xi_{10} = \xi_{11} = 4 \text{ oraz } \xi_{12} = \xi_{13} = \xi_{14} = 5$

$B_{1,0} = 1 \text{ for } x \in [\xi_1, \xi_2 = [0,0] = \{0\}$, 0 in other points,,

$B_{2,0} = 1 \text{ for } x \in [\xi_2, \xi_3] = [0,0] = \{0\}$, 0 in other points,,

$B_{3,0} = 1 \text{ for } x \in [\xi_3, \xi_4] = [0,1]$, 0 in other points,,

$B_{4,0} = 1 \text{ for } x \in [\xi_4, \xi_5] = [1,1] = \{1\}$, 0 in other points,,

$B_{5,0} = 1 \text{ for } x \in [\xi_5, \xi_6] = [1,2]$, 0 in other points,,

$B_{6,0} = 1 \text{ for } x \in [\xi_6, \xi_7] = [2,2] = \{2\}$, 0 in other points,,

$B_{7,0} = 1 \text{ for } x \in [\xi_7, \xi_8] = [2,3]$, 0 in other points,,

$B_{8,0} = 1 \text{ for } x \in [\xi_8, \xi_9] = [3,3] = \{3\}$, 0 in other points,,

$B_{9,0} = 1 \text{ for } x \in [\xi_{10}, \xi_{11}] = [3,4]$, 0 in other points,,

$B_{10,0} = 1 \text{ for } x \in [\xi_{11}, \xi_{12}] = [4,4] = \{4\}$, 0 in other points,,

$B_{11,0} = 1 \text{ for } x \in [\xi_{12}, \xi_{13}] = [4,5]$, 0 in other points,,

$B_{12,0} = 1 \text{ for } x \in [\xi_{13}, \xi_{14}] = [5,5] = \{5\}$, 0 in other points,,

$B_{13,0} = 1 \text{ for } x \in [\xi_{14}, \xi_{15}] = [5,5] = \{5\}$, 0 in other points,.

Similarly, we need to generate the first order basis functions for a new knot vector. We remember the pattern for $p = 1$

$B_{i,1}(\xi) = \frac{\xi-\xi_i}{\xi_{i+1}-\xi_i} B_{i,0}(\xi) + \frac{\xi_{i+2}-\xi}{\xi_{i+2}-\xi_{i+1}} B_{i+1,0}(\xi)$

where we insert the next knots: $B_{1,1}(\xi) = \frac{\xi-\xi_1}{\xi_2-\xi_1} B_{1,0}(\xi) + \frac{\xi_3-\xi}{\xi_3-\xi_2} B_{2,0}(\xi) = \frac{\xi-0}{0-0} B_{1,0}(\xi) + \frac{0-\xi}{0-0} B_{2,0}(\xi) = 0,$

$B_{2,1}(\xi) = \frac{\xi-\xi_2}{\xi_3-\xi_2} B_{2,0}(\xi) + \frac{\xi_4-\xi}{\xi_4-\xi_3} B_{3,0}(\xi) = \frac{\xi-0}{0-0} B_{2,0}(\xi) + \frac{1-\xi}{1-0} B_{3,0}(\xi) = 1-\xi \text{ for } \xi \in [0,1],$

$B_{3,1}(\xi) = \frac{\xi-\xi_3}{\xi_4-\xi_3} B_{3,0}(\xi) + \frac{\xi_5-\xi}{\xi_5-\xi_4} B_{4,0}(\xi) = \frac{\xi-0}{1-0} B_{3,0}(\xi) + \frac{1-\xi}{1-1} B_{4,0}(\xi) = \xi \text{ for } \xi \in [0,1],$

$B_{4,1}(\xi) = \frac{\xi-\xi_4}{\xi_5-\xi_4} B_{4,0}(\xi) + \frac{\xi_6-\xi}{\xi_6-\xi_5} B_{5,0}(\xi) = \frac{\xi-1}{1-1} B_{4,0}(\xi) + \frac{2-\xi}{2-1} B_{5,0}(\xi) = 2-\xi \text{ for } \xi \in [1,2],$

$B_{5,1}(\xi) = \frac{\xi-\xi_5}{\xi_6-\xi_5} B_{5,0}(\xi) + \frac{\xi_7-\xi}{\xi_7-\xi_6} B_{6,0}(\xi) = \frac{\xi-1}{2-1} B_{5,0}(\xi) + \frac{2-\xi}{2-2} B_{6,0}(\xi) = \xi-1 \text{ for } \xi \in [1,2],$

$B_{6,1}(\xi) = \frac{\xi-\xi_6}{\xi_7-\xi_6} B_{6,0}(\xi) + \frac{\xi_8-\xi}{\xi_8-\xi_7} B_{7,0}(\xi) = \frac{\xi-2}{2-2} B_{6,0}(\xi) + \frac{3-\xi}{3-2} B_{7,0}(\xi) = 3-\xi \text{ for } \xi \in [2,3],$

$B_{7,1}(\xi) = \frac{\xi-\xi_7}{\xi_8-\xi_7} B_{7,0}(\xi) + \frac{\xi_9-\xi}{\xi_9-\xi_8} B_{8,0}(\xi) = \frac{\xi-2}{3-2} B_{7,0}(\xi) + \frac{3-\xi}{3-3} B_{9,0}(\xi) = \xi-2 \text{ for } \xi \in [2,3],$

$B_{8,1}(\xi) = \frac{\xi-\xi_8}{\xi_9-\xi_8} B_{8,0}(\xi) + \frac{\xi_{10}-\xi}{\xi_{10}-\xi_9} B_{9,0}(\xi) = \frac{\xi-3}{3-3} B_{8,0}(\xi) + \frac{4-\xi}{4-3} B_{9,0}(\xi) = 4-\xi \text{ for } \xi \in [3,4],$

$B_{9,1}(\xi) = \frac{\xi-\xi_9}{\xi_{10}-\xi_9} B_{9,0}(\xi) + \frac{\xi_{11}-\xi}{\xi_{11}-\xi_{10}} B_{10,0}(\xi) = \frac{\xi-4}{4-3} B_{9,0}(\xi) + \frac{4-\xi}{4-4} B_{10,0}(\xi) = \xi-3 \text{ for } \xi \in [3,4],$

$B_{10,1}(\xi) = \frac{\xi-\xi_{10}}{\xi_{11}-\xi_{10}} B_{10,0}(\xi) + \frac{\xi_{12}-\xi}{\xi_{12}-\xi_{11}} B_{11,0}(\xi) = \frac{\xi-4}{4-4} B_{10,0}(\xi) + \frac{5-\xi}{5-4} B_{11,0}(\xi) 5-\xi \text{ for } \xi \in [4,5],$

$B_{11,1}(\xi) = \frac{\xi-\xi_{11}}{\xi_{12}-\xi_{11}} B_{11,0}(\xi) + \frac{\xi_{13}-\xi}{\xi_{13}-\xi_{12}} B_{12,0}(\xi) = \frac{\xi-4}{5-4} B_{11,0}(\xi) + \frac{5-\xi}{5-5} B_{12,0}(\xi) = \xi-4 \text{ for } \xi \in [4,5],$

$B_{12,1}(\xi) = \frac{\xi-\xi_{12}}{\xi_{13}-\xi_{12}} B_{12,0}(\xi) + \frac{\xi_{14}-\xi}{\xi_{14}-\xi_{13}} B_{13,0}(\xi) = \frac{\xi-5}{5-5} B_{11,0}(\xi) + \frac{5-\xi}{5-5} B_{12,0}(\xi) = 0.$

We can now generate all second order B spline functions using the formula again for $p=2$, assuming that the subsequent knots inserted into the denominator must be different, and if they are not different, then the given term is changed to zero. The elements that disappear are marked red again. In the final stage of the derivation, we insert the formulas calculated before $B_{1,1}(\xi) = 0$, $B_{2,1} = 1-\xi$ for $\xi \in [0,1]$, $B_{3,1} = \xi$ for $\xi \in [0,1]$, $B_{4,1} = 2-\xi$ for $\xi \in [1,2]$, $B_{5,1} = \xi - 1$ for $\xi \in [1,2]$, $B_{6,1} = 3-\xi$ for $\xi \in [2,3]$, $B_{7,1} = \xi-2$ for $\xi \in [2,3]$, $B_{8,1} = 4-\xi$ for $\xi \in [3,4]$, $B_{9,1} = \xi-3$ for $\xi \in [3,4]$, $B_{10,1} = 5-\xi$ for $\xi \in [4,5]$, $B_{11,1} = \xi-4$ for $\xi \in [4,5]$, $B_{12,1} = 0$.

We get

$B_{1,2}(\xi) = \frac{\xi-\xi_1}{\xi_3-\xi_1} B_{1,1}(\xi) + \frac{\xi_4-\xi}{\xi_4-\xi_2} B_{2,1}(\xi) = \frac{\xi-0}{0-0} B_{1,1}(\xi) + \frac{1-\xi}{1-0} B_{2,1}(\xi) = (1-\xi)^2 \text{ for } \xi \in [0,1].$

$B_{2,2}(\xi) = \frac{\xi-\xi_2}{\xi_4-\xi_2} B_{2,1}(\xi) + \frac{\xi_5-\xi}{\xi_5-\xi_3} B_{3,1}(\xi) = \frac{\xi-0}{1-0} B_{2,1}(\xi) + \frac{1-\xi}{1-0} B_{3,1}(\xi) = \; = \frac{\xi-0}{1-0}[1-\xi \text{ for } \xi \in [0,1]] + \frac{1-\xi}{1-0}[\xi \text{ for } \xi \in [0,1]] = \xi(1-\xi) + (1-\xi)\xi \text{ for } \xi \in [0,1] = 2\xi(1-\xi) \text{ for } \xi \in [0,$

.

$B_{3,2}(\xi) = \frac{\xi-\xi_3}{\xi_5-\xi_3} B_{3,1}(\xi) + \frac{\xi_6-\xi}{\xi_6-\xi_4} B_{4,1}(\xi) = \frac{\xi-0}{1-0} B_{3,1}(\xi) + \frac{2-\xi}{2-1} B_{4,1}(\xi) = \; = \frac{\xi-0}{1-0}[\xi \text{ for } \xi \in [0,1]] + \frac{1-\xi}{1-0}[2-\xi \text{ for } \xi \in [1,2]] = \xi^2 \text{ for } \xi \in [0,1] + (1-\xi)(2-\xi) \text{ for } \xi \in [1,2]$

.

$B_{4,2}(\xi) = \frac{\xi-\xi_4}{\xi_6-\xi_4} B_{4,1}(\xi) + \frac{\xi_7-\xi}{\xi_7-\xi_5} B_{5,1}(\xi) = \frac{\xi-1}{2-1} B_{4,1}(\xi) + \frac{2-\xi}{2-1} B_{5,1}(\xi) = \; = \frac{\xi-1}{2-1}[2-\xi \text{ for } \xi \in [1,2]] + \frac{2-\xi}{2-1}[\xi-1 \text{ for } \xi \in [1,2]] = \; = (\xi-1)(2-\xi) + (2-\xi)(\xi-1) \text{ for } \xi \in [1,2] = 2($

.

$$B_{5,2}(\xi) = \frac{\xi-\xi_5}{\xi_7-\xi_5}B_{5,1}(\xi) + \frac{\xi_8-\xi}{\xi_8-\xi_6}B_{6,1}(\xi) = \frac{\xi-1}{2-1}B_{5,1}(\xi) + \frac{3-\xi}{3-2}B_{6,1}(\xi) = \ = \frac{\xi-1}{2-1}[\xi-1 \text{ for } \xi \in [1,2]] + \frac{3-\xi}{3-2}[3-\xi \text{ for } \xi \in [2,3]] = (\xi-1)^2 \text{ for } \xi \in [1,2] + (3-\xi)^2 \text{ for } \xi \in [2,3]$$

.

$$B_{6,2}(\xi) = \frac{\xi-\xi_6}{\xi_8-\xi_6}B_{6,1}(\xi) + \frac{\xi_9-\xi}{\xi_9-\xi_7}B_{7,1}(\xi) = \frac{\xi-2}{3-2}B_{6,1}(\xi) + \frac{3-\xi}{3-2}B_{7,1}(\xi) = \ = \frac{\xi-2}{3-2}[3-\xi \text{ for } \xi \in [2,3]] + \frac{3-\xi}{3-2}[\xi-2 \text{ for } \xi \in [2,3]] = \ = (\xi-2)(3-\xi) + (3-\xi)(\xi-2) \text{ for } \xi \in [2,3] = 2($$

.

$$B_{7,2}(\xi) = \frac{\xi-\xi_7}{\xi_9-\xi_7}B_{7,1}(\xi) + \frac{\xi_{10}-\xi}{\xi_{10}-\xi_8}B_{8,1}(\xi) = \frac{\xi-2}{3-2}B_{7,1}(\xi) + \frac{4-\xi}{4-3}B_{8,1}(\xi) = \ = \frac{\xi-2}{3-2}[\xi-2 \text{ for } \xi \in [2,3]] + \frac{4-\xi}{4-3}[4-\xi \text{ for } \xi \in [3,4]] = (\xi-2)^2 \text{ for } \xi \in [2,3] + (4-\xi)^2 \text{ for } \xi \in [3,4]$$

.

$$B_{8,2}(\xi) = \frac{\xi-\xi_8}{\xi_{10}-\xi_8}B_{8,1}(\xi) + \frac{\xi_{11}-\xi}{\xi_{11}-\xi_9}B_{9,1}(\xi) = \frac{\xi-3}{4-3}B_{8,1}(\xi) + \frac{4-\xi}{4-3}B_{9,1}(\xi) = \ = \frac{\xi-3}{4-3}[4-\xi \text{ for } \xi \in [3,4]] + \frac{4-\xi}{4-3}[\xi-3 \text{ for } \xi \in [3,4]] = \ = (\xi-3)(4-\xi) + (4-\xi)(\xi-3) \text{ for } \xi \in [3,4] = 2$$

.

$$B_{9,2}(\xi) = \frac{\xi-\xi_9}{\xi_{11}-\xi_9}B_{9,1}(\xi) + \frac{\xi_{12}-\xi}{\xi_{12}-\xi_{10}}B_{10,1}(\xi) = \frac{\xi-3}{4-3}B_{9,1}(\xi) + \frac{6-\xi}{5-4}B_{9,1}(\xi) = \ = \frac{\xi-3}{4-3}[\xi-3 \text{ for } \xi \in [3,4]] + \frac{5-\xi}{5-4}[5-\xi \text{ for } \xi \in [4,5]] = (\xi-3)^2 \text{ for } \xi \in [3,4] + (5-\xi)^2 \text{ for } \xi \in [4,5]$$

.

$$B_{10,2}(\xi) = \frac{\xi-\xi_{10}}{\xi_{12}-\xi_{10}}B_{10,1}(\xi) + \frac{\xi_{13}-\xi}{\xi_{13}-\xi_{11}}B_{11,1}(\xi) = \frac{\xi-4}{5-4}B_{10,1}(\xi) + \frac{5-\xi}{5-4}B_{11,1}(\xi) = \ = \frac{\xi-4}{5-4}[5-\xi \text{ for } \xi \in [4,5]] + \frac{5-\xi}{5-4}[\xi-4 \text{ for } \xi \in [4,5]] = \ = (\xi-4)(5-\xi) + (5-\xi)(\xi-4) \text{ for } \xi \in [4,$$

.

$$B_{11,2}(\xi) = \frac{\xi-\xi_{11}}{\xi_{12}-\xi_{11}}B_{11,1}(\xi) + \frac{\xi_{14}-\xi}{\xi_{14}-\xi_{12}}B_{12,1}(\xi) = \frac{\xi-4}{5-4}B_{11,1}(\xi) + \frac{8-\xi}{5-5}B_{11,1}(\xi) = \ = \frac{\xi-4}{5-4}[\xi-4 \text{ for } \xi \in [4,5]] = (\xi-4)^2 \text{ for } \xi \in [4,5].$$

We just derived the basis functions spanned over the knot vector with repeated knots between each element. It can be mathematically proved that the functions derived here $B_{1,2},...,B_{11,2}$ they constitute a base equivalent to the so-called Lagrange polynomials. Thus, one can use the knot vector notation to derive the B-spline basis functions used in the isogeometric finite element method, or to derive the Lagrange equivalent basis used in the classical finite element method.

Utworzona przez admin. Ostatnia aktualizacja: Środa 07 z Październik, 2020 06:49:34 UTC przez paszynsk@agh.edu.pl. Autor: Maciej Paszynski

STATUS: W opracowaniu Zgłoś do recenzji Edytuj