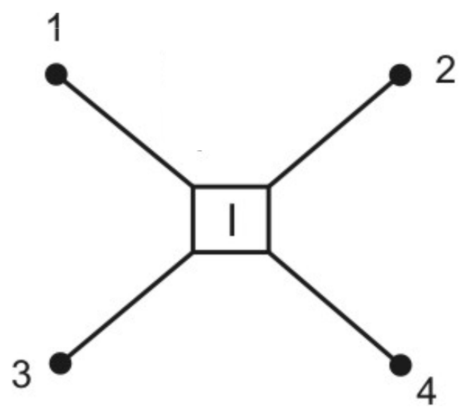
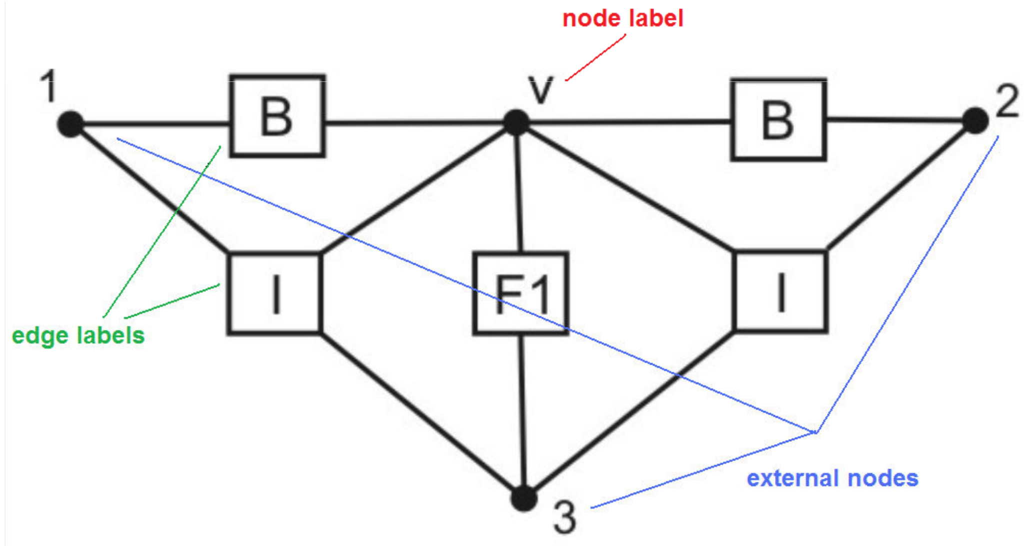
Hypergraphs
and Hypergraph Grammars
on the example of
hp-adaptive Finite Element Method

HYPERGRAPHS

- A hyperedge is an edge with sequences of nodes assigned to it.
- Each hypergraph is composed of a set of nodes and a set of hyperedges with sequences of nodes assigned to them.
- For each hypergraph a sequence of its external nodes is specified.

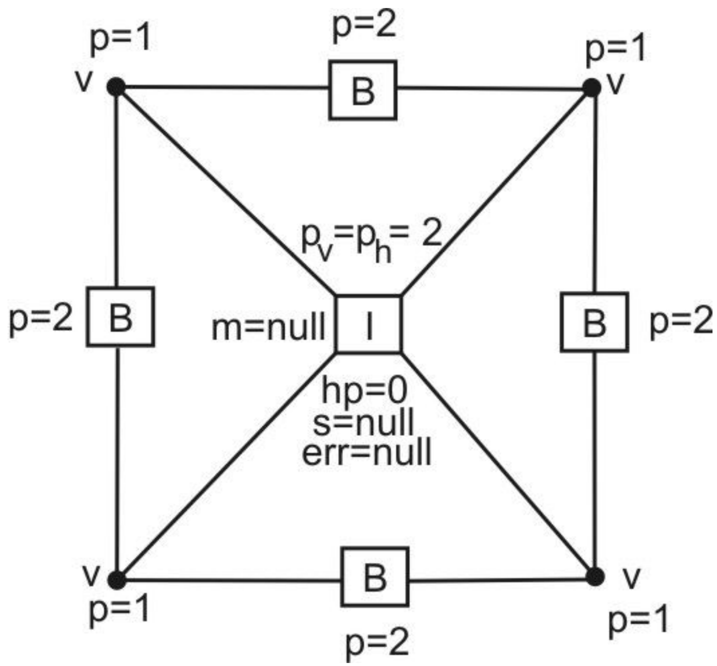




A hyperedge and an exemplary hypergraph

HYPERGRAPHS

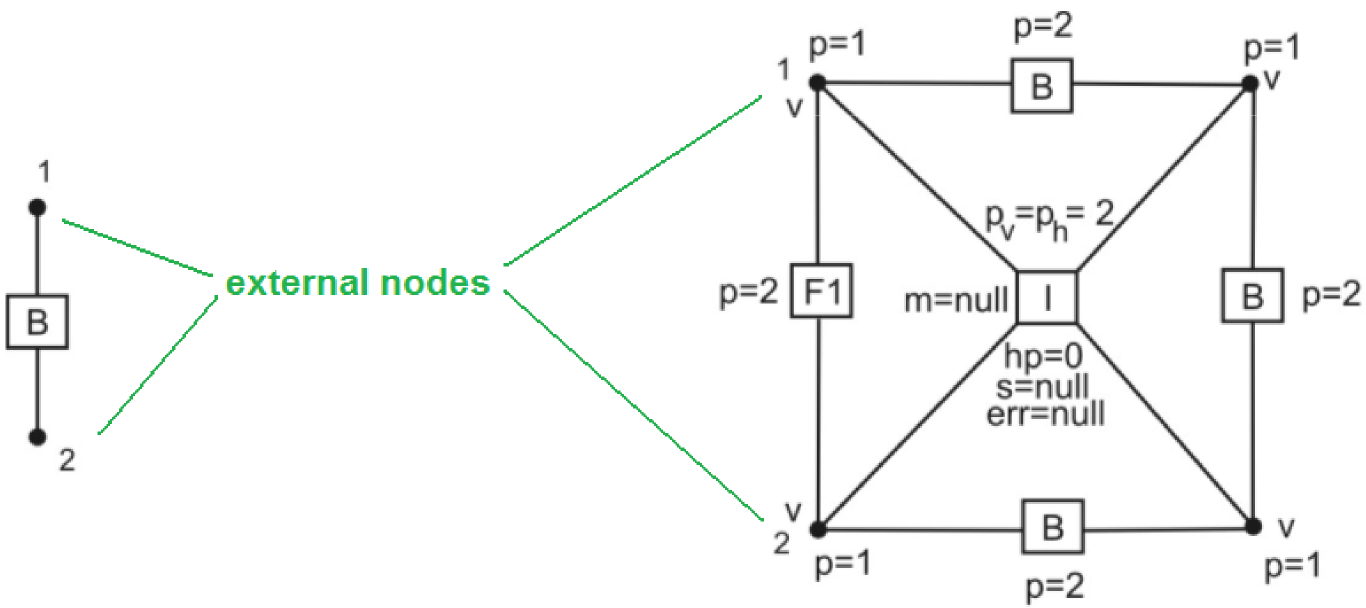
- To nodes and hyperedges the sets of attributes are assigned.



An exemplary attributed hypergraph

HYPERGRAPHS

A **hypergraph of type *k*** is a hypergraph with *k* external nodes,



Exemplary hypergraphs with type 2

HYPERGRAPHS

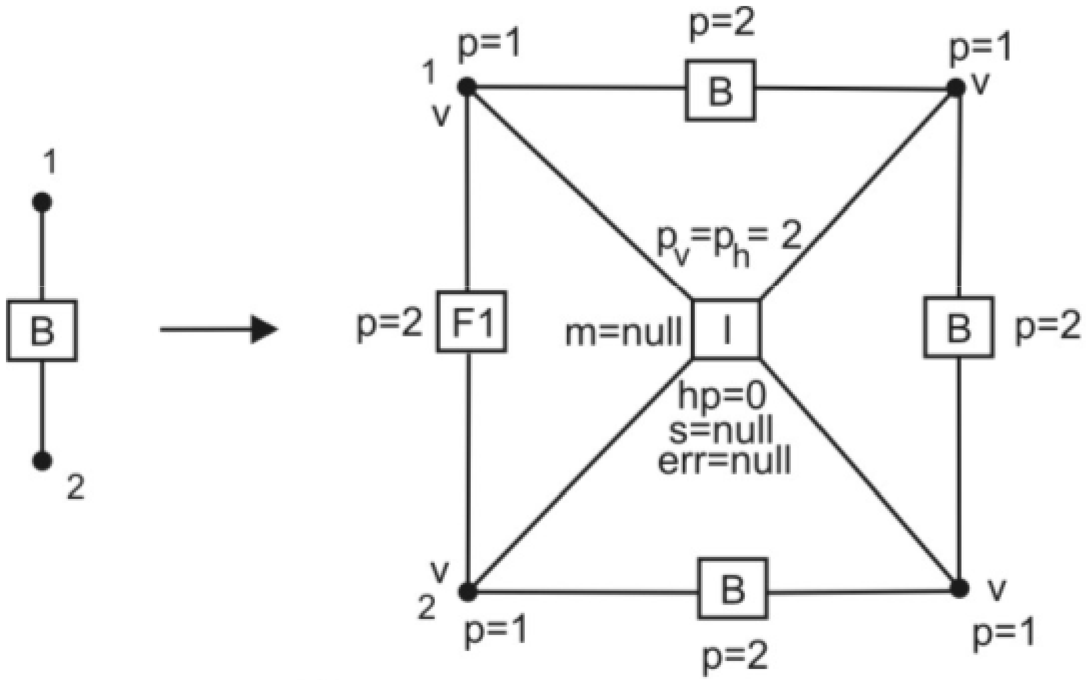
Let ***C*** be a fixed **alphabet of labels** for nodes and hyperedges.
Let ***A*** be a **set of** hypergraph **attributes**.

Definition 1.
An undirected **attributed labelled hypergraph** over C and A is a system
$\boldsymbol{G = (V, E, t, l, at)}$,
where:
• V is a finite set of nodes,
• E is a finite set of hyperedges,
• $t: E \rightarrow V^*$ is a mapping assigning sequences of target nodes to hyperedges of E,
• $l: V \cup E \rightarrow C$ is a node and hyperedge labelling function,
• $at: V \cup E \rightarrow 2^A$ is a node and hyperedge attributing function.

HYPERGRAPH PRODUCTION

The hypergraph production:

p=(L,R)

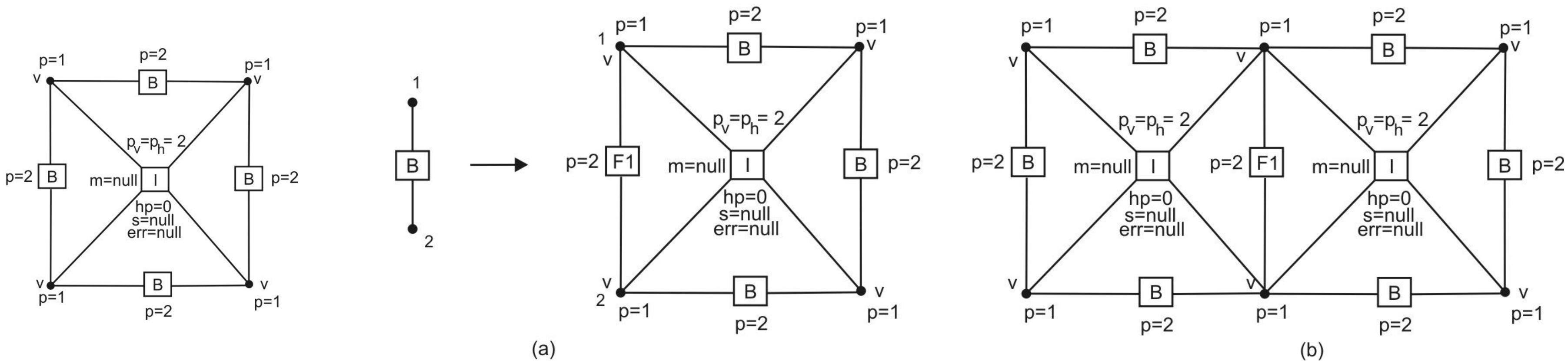


Exemplary hypergraph production:

Left-hand side graph and right-hand side graph should be of the same type

HYPERGRAPH GRAMMAR

The **application of a production** ***p* = (*L*, *R*)** to a hypergraph ***H*** consist of
replacing a subhypergraph of *H* isomorphic with *L* by a hypergraph *R* and replacing nodes of the removed subhypergraph isomorphic with external nodes of *L* by the corresponding external nodes of *R*.



An initial graph; A hypergraph production; Final hypergraph

A production *p=(L,R)* can be applied to a hypergraph *H* if *H* contains a subhypergraph isomorphic with *L*.

HYPERGRAPH GRAMMAR FOR FINITE ELEMENT METHOD

- mesh generation process,
- mesh refinements,
- translation of the mesh into the sequence of matrices for the solver algorithm

are expressed by graph grammar productions.

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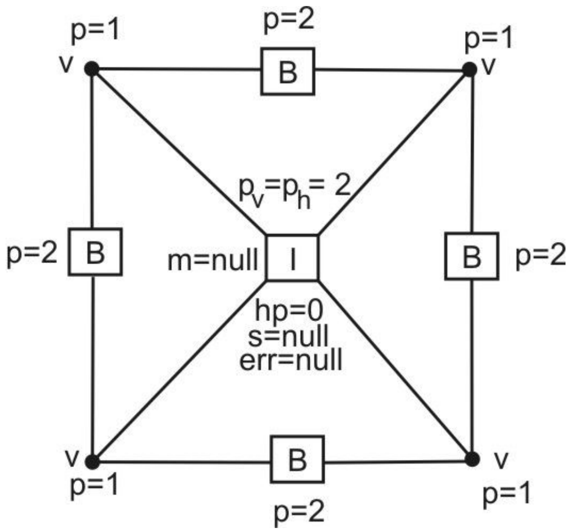
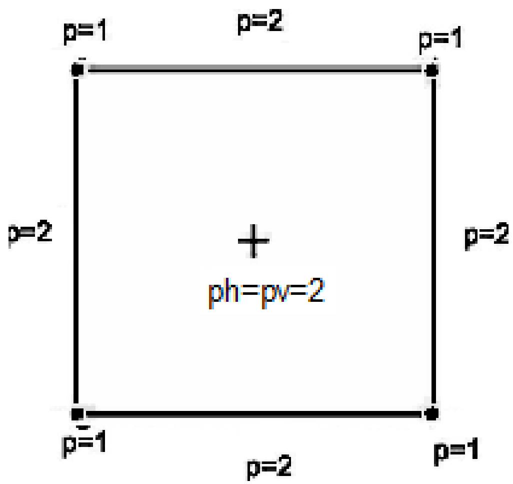
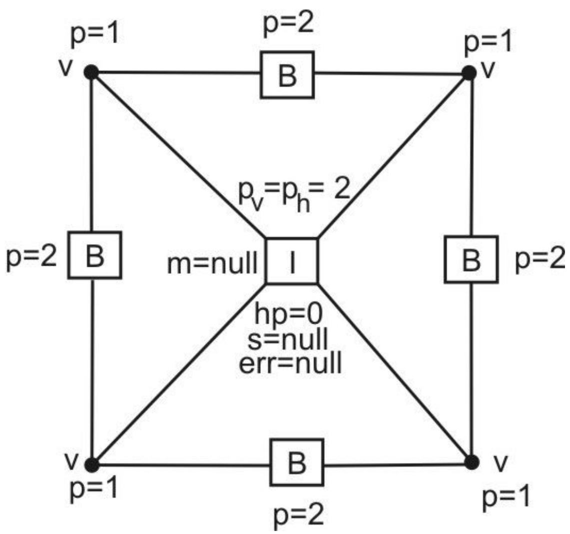




The hypergraph corresponding to one element mesh

HYPERGRAPH GRAMMAR FOR FINITE ELEMENT METHOD

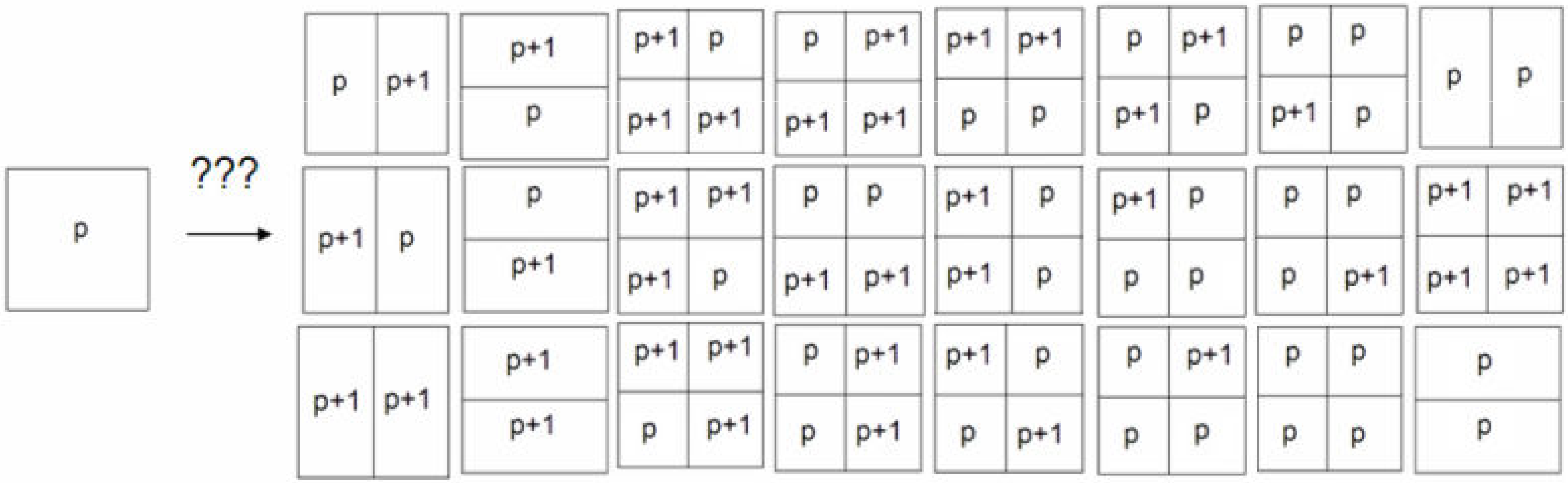
- attribute ***p*** - the **polynomial order of approximation**.
- attributes ***ph*** and ***pv*** **- polynomial order of approximation** in the **horizontal** and **vertical direction**, respectively.
- attributes ***m*****,** ***s*** and ***err*** - the **matrix, the solution** and **the vector of error** for the corresponding element.



The hypergraph corresponding to one element mesh

HYPERGRAPH GRAMMAR FOR FINITE ELEMENT METHOD

- attribute ***hp*** - the **kind of *hp*-adaptation** (values from *{0,…,24}*). The attribute value *hp=0* means no adaptation.
- The meaning of other values of attribute *hp* is presented

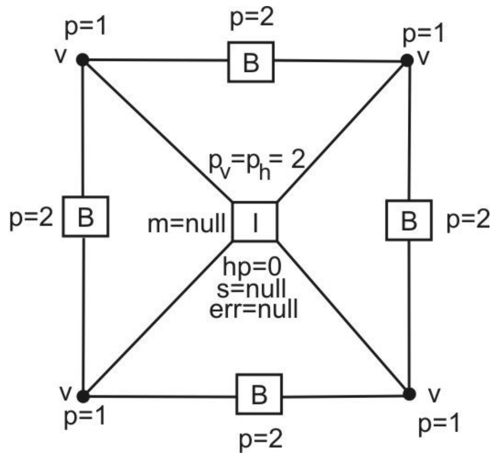


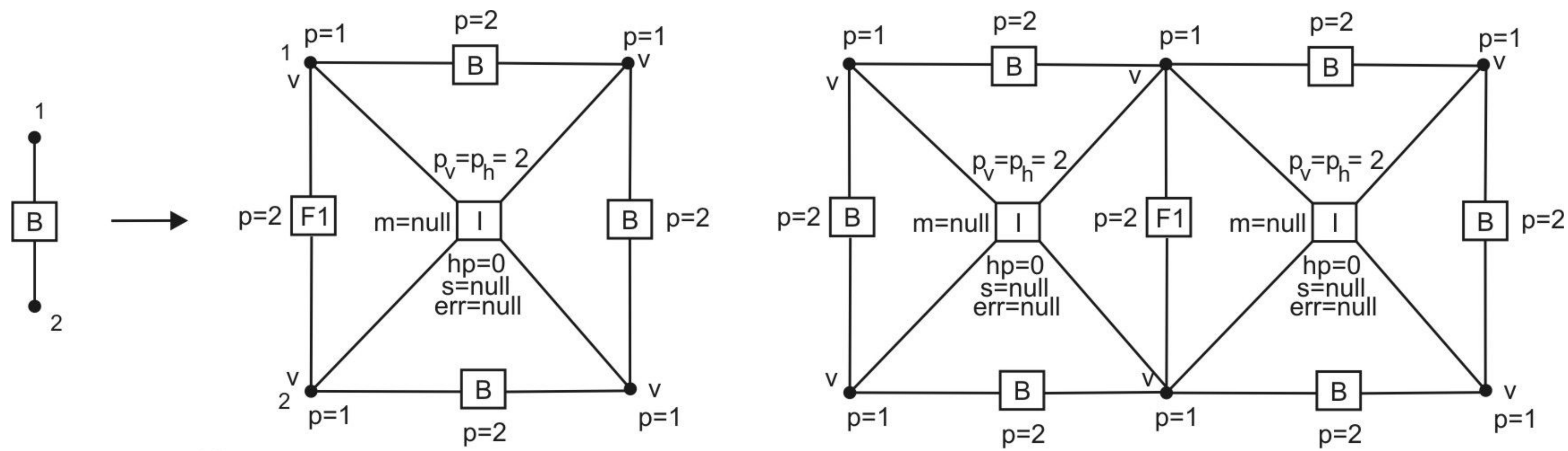
HYPERGRAPH GRAMMAR FOR FEM

The hypergraph grammar for hp-adaptive finite element method is composed of four groups of productions:

1. productions describing the **generation of the initial mesh**
2. productions for calculating the **matrix, the solution and error** for each finite element, as well as **maximum relative error**
3. productions for **virtual hp-adaptation**
4. productions performing **hp-adaptation**

HYPERGRAPH GRAMMAR PRODUCTION FOR GENERATION OF THE INITIAL MESH

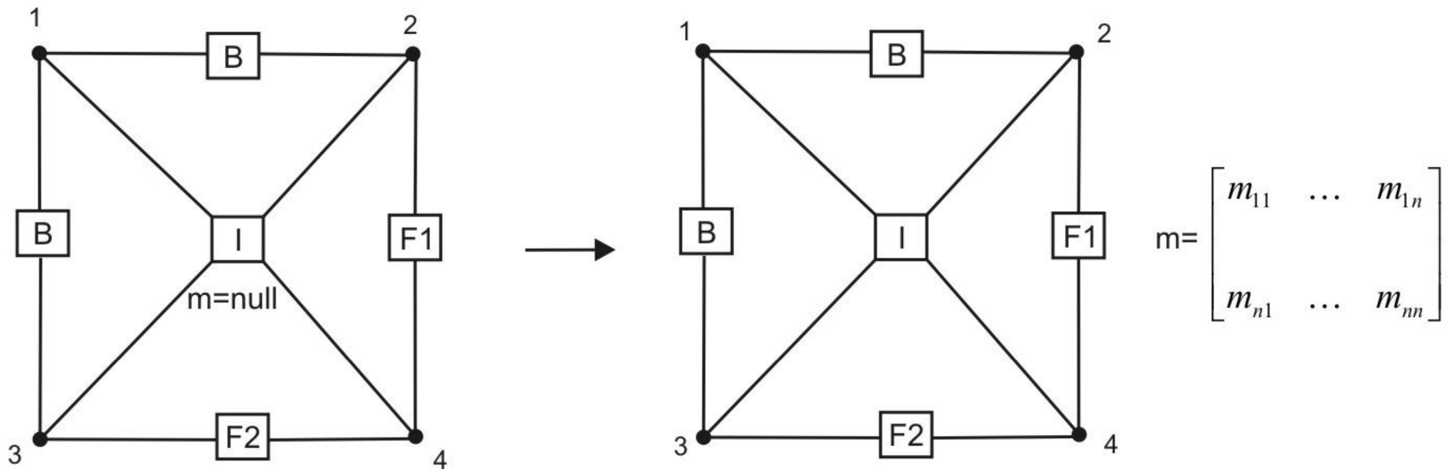




(a)

(b)

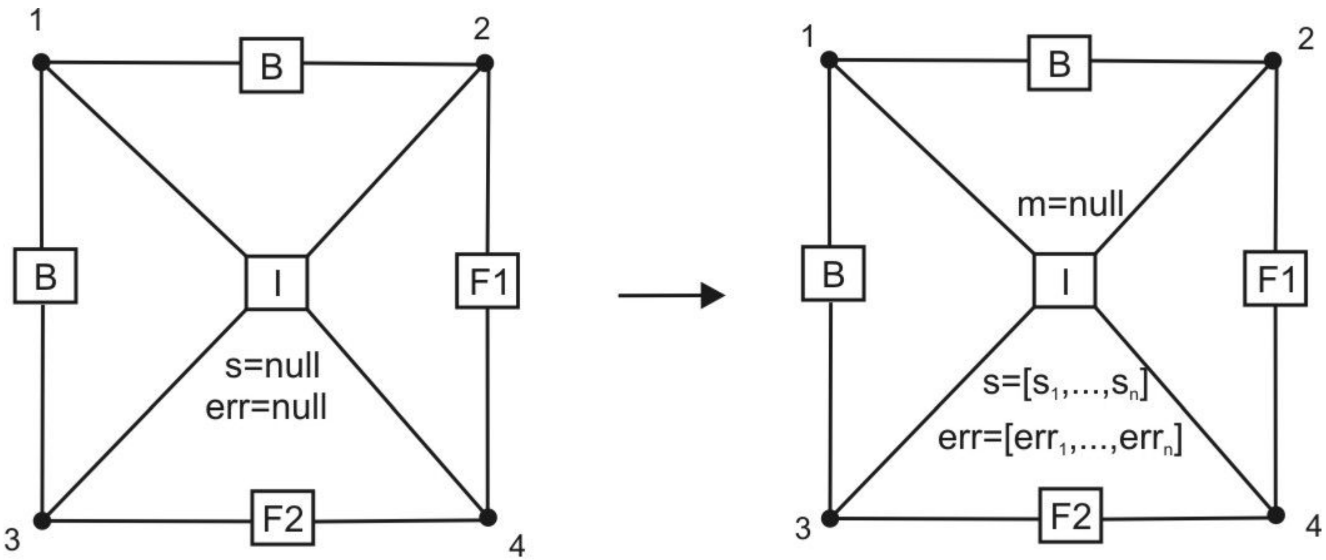
HYPERGRAPH GRAMMAR PRODUCTION FOR CALCULATING THE MATRIX



The production is applied to all elements

(the applicability predicate is m==null)

HYPERGRAPH GRAMMAR PRODUCTION FOR CALCULATING ERROR AND SOLUTION VECTOR

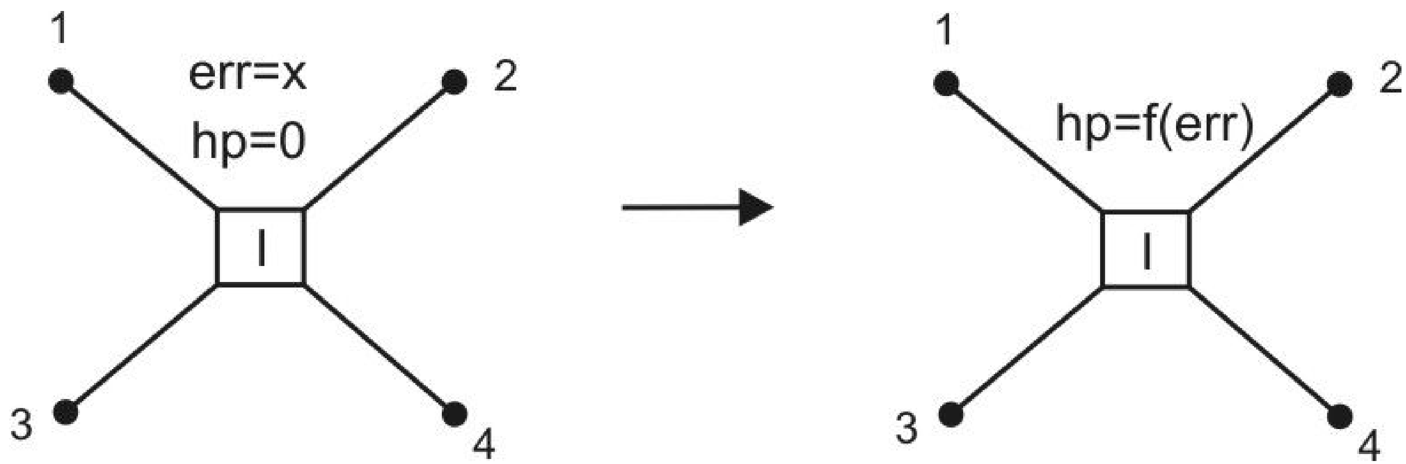


$$m=\begin{bmatrix} m_{11} & \cdots & m_{1n} \\ \\ m_{n1} & \cdots & m_{nn} \end{bmatrix}$$

The production is applied to all elements

(the applicability predicate is: s==null;err=null; m!=null)

HYPERGRAPH GRAMMAR PRODUCTION FOR VIRTUAL HP-ADAPTATION



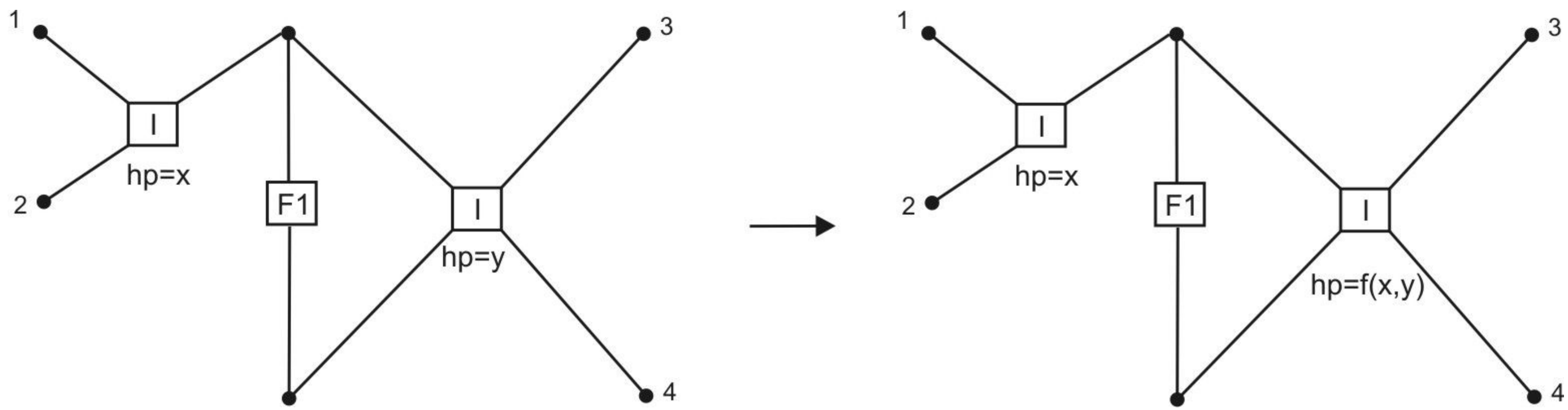
The production is applied to all elements

(the applicability predicate is: hp==null; err!=null;)

The **decision about the kind of the adaptation** is based on the error vector *err* of the element and is denoted by **hp=*f(err)***.

HYPERGRAPH GRAMMAR PRODUCTION FOR PROPAGATING OF THE ADAPTATION

In order to follow the **1-irregularity rule** (a finite element can be broken only once without breaking the adjacent large elements), after performing virtual hp-adaptation for each element, several **additional refinements** on large adjacent elements may be required.



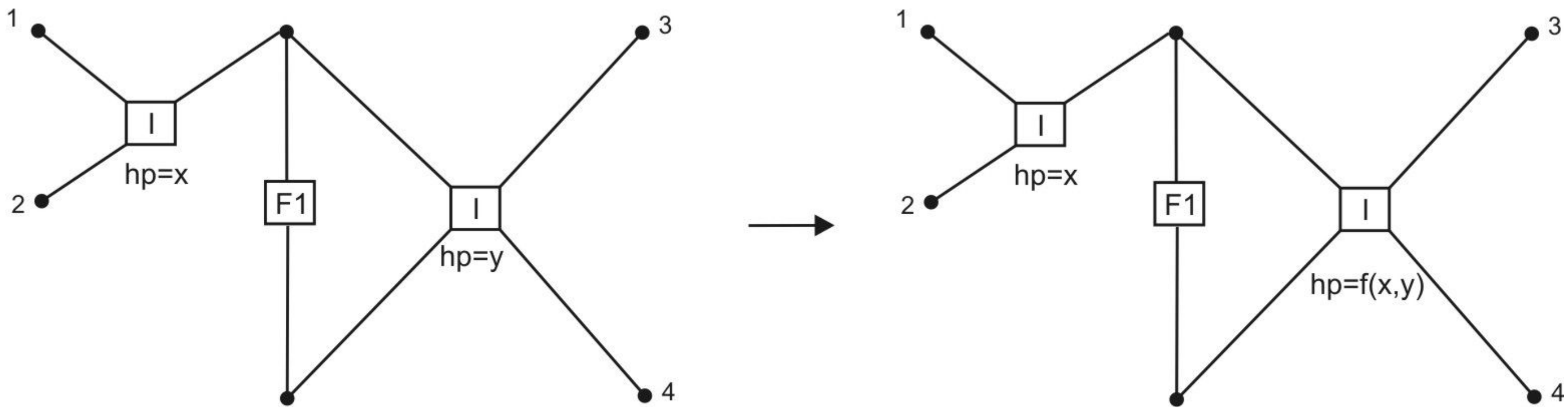
A hypergraph production for propagating the adaptation

HYPERGRAPH GRAMMAR PRODUCTION FOR PROPAGATING OF THE ADAPTATION

The function *f(x,y)* decides about the kind of the adaptation:

•If the value for *hp* attribute for bigger element equals *0 (*no adaptation) the new value for *hp* attribute for the bigger element is the same as for the smaller element (*f(x,y)=x*).

•In other case *f(x,y)* works like logical *OR,*
for example if *x* denotes h-adaptation in one direction and *y* in other direction, then *f(x,y)* denotes h-adaptation in both directions.



A hypergraph production for propagating the adaptation

HYPERGRAPH GRAMMAR PRODUCTIONS FOR HP-ADAPTATION

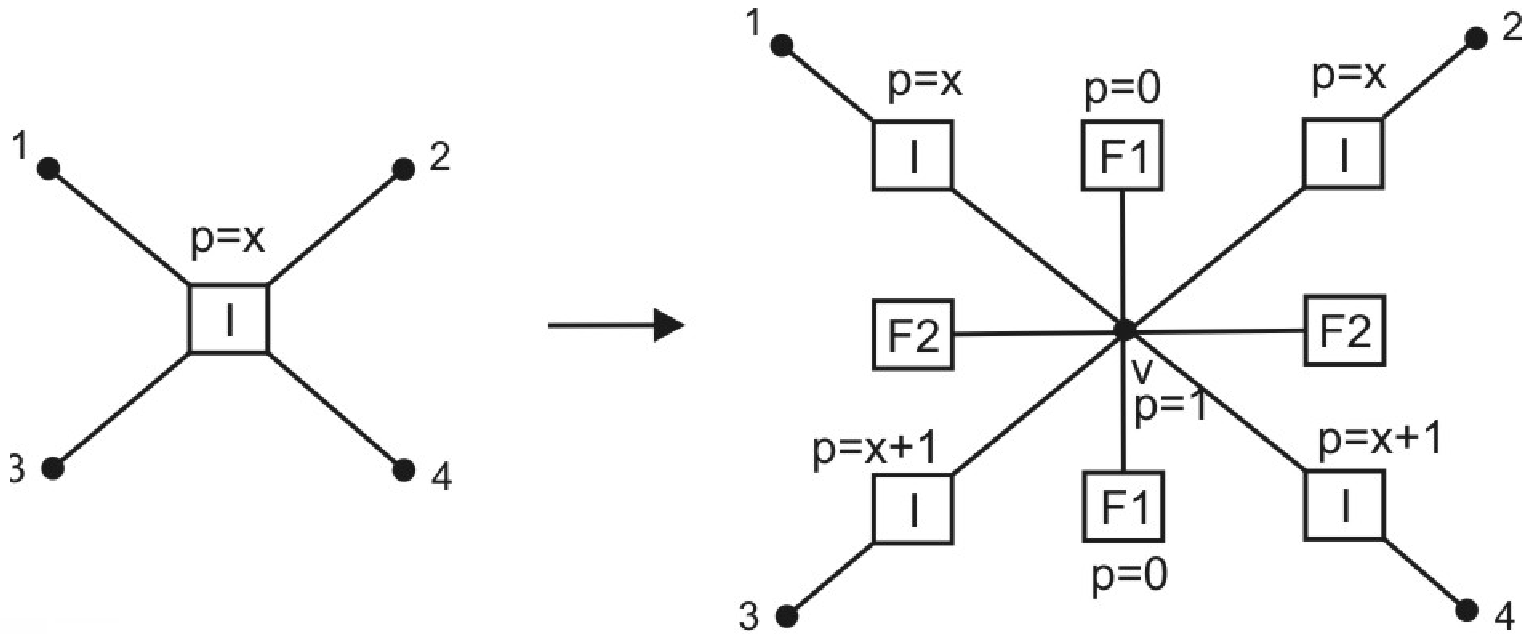
The **hp-adaptation** is modeled by:

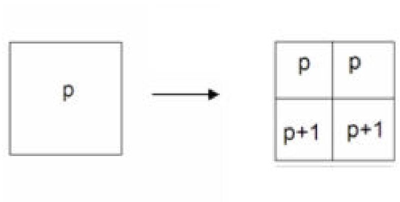
•breaking interiors of elements and assigning values for attribute *p* for newly created interiors according the value of attribute *hp* of the parental interior,

•breaking the edges,

•performing the minimum rule in order to calculate the polynomial order of approximation for edges (the value of attribute *p*).

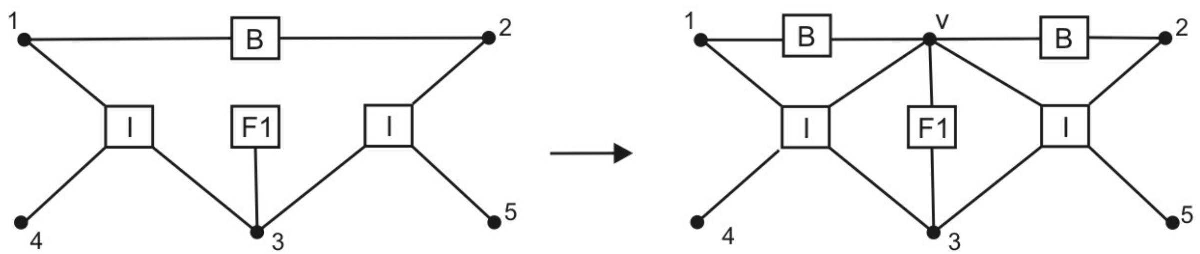
HYPERGRAPH GRAMMAR PRODUCTION
FOR HP-ADAPTATION – BREAKING INTERIORS





Production for performing hp-adaptation, where *hp* equals to *12*

HYPERGRAPH GRAMMAR PRODUCTION
FOR HP-ADAPTATION – BREAKING OF BOUNDARY EDGES

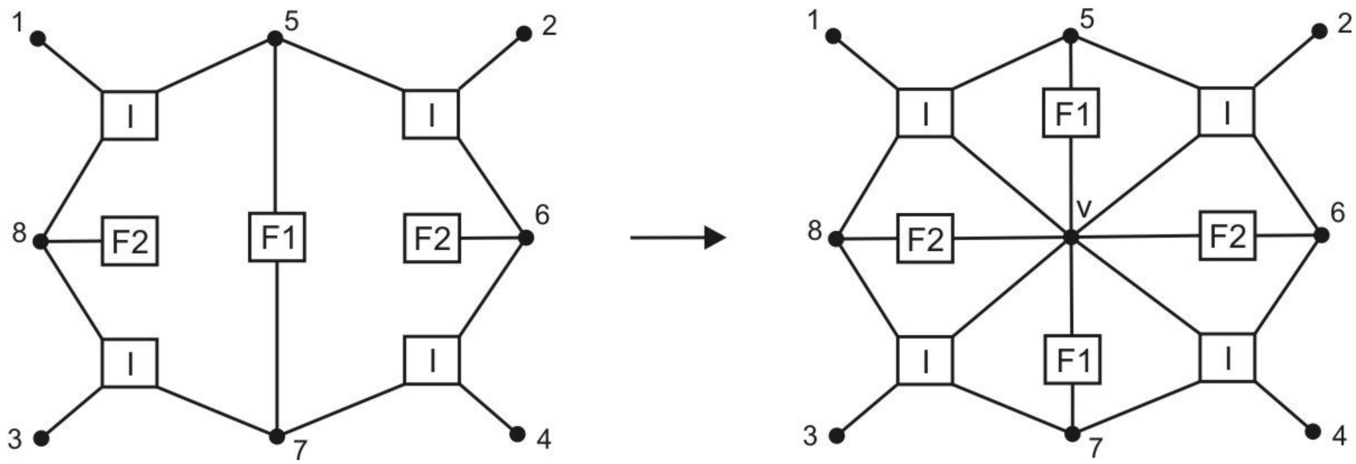


After breaking interiors of elements, all **boundary edges adjacent to one broken element have to be broken.**

In the similar way all edges adjacent to two broken elements **have to be broken.**

Hypergraph productions for breaking boundary edges

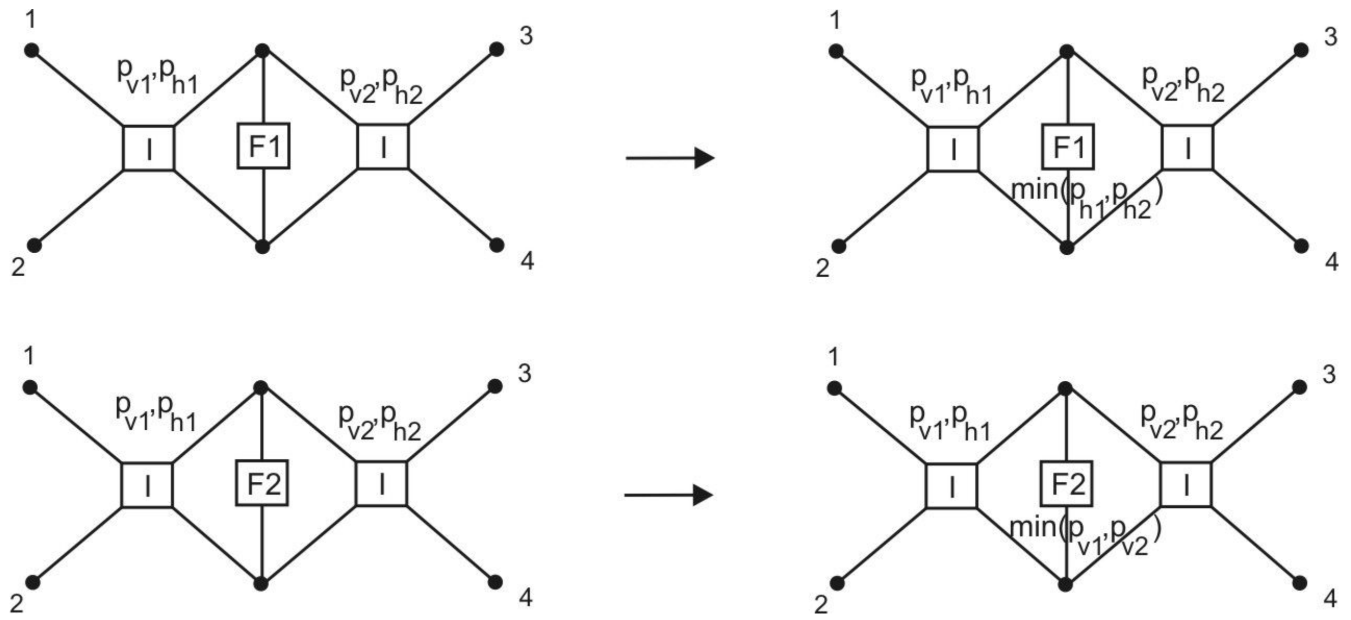
HYPERGRAPH GRAMMAR PRODUCTION
FOR HP-ADAPTATION – BREAKING OF SHARED EDGES



After breaking interiors of elements, **all edges adjacent to two broken elements** have to be broken.

Hypergraph productions for breaking shared edges

HYPERGRAPH GRAMMAR PRODUCTION
FOR HP-ADAPTATION – PERFORMING THE MINIMUM RULE

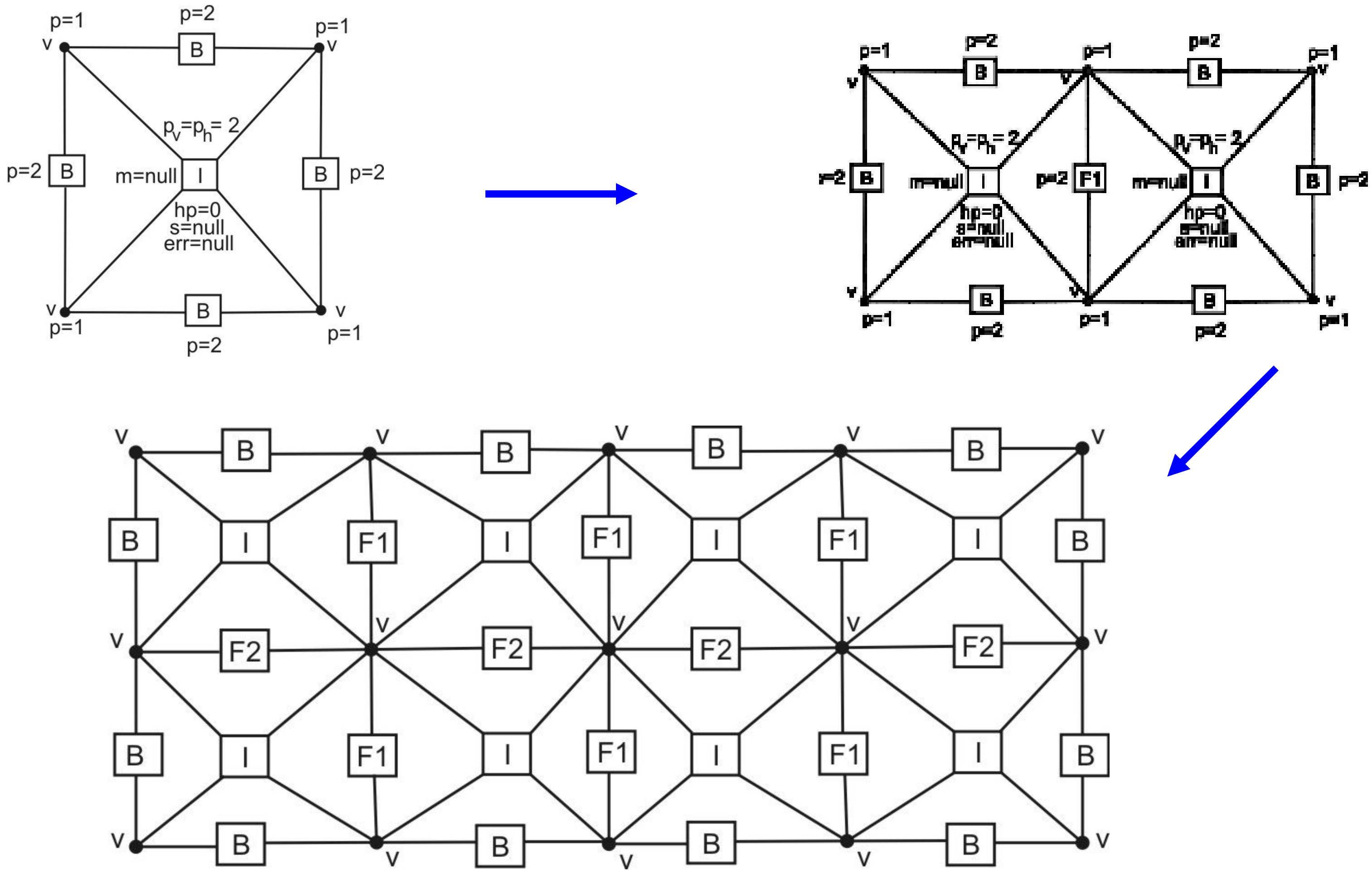


Minumum rule:

The value for attribute p for the edge is the minimum from the p values for adjacent interiors

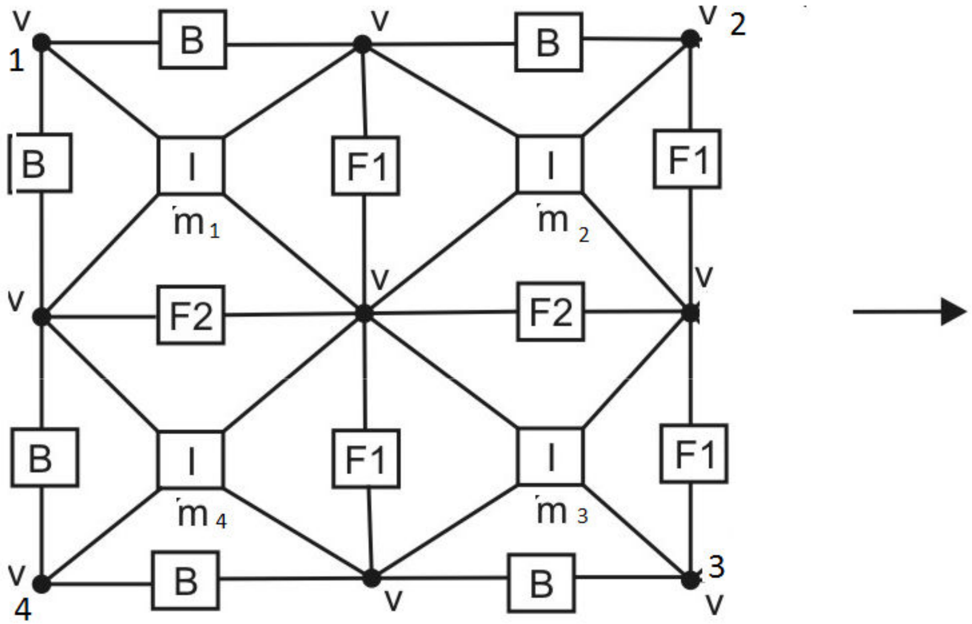
A hypergraph production for assigning appriopriate polynomial orders of approximation for edges

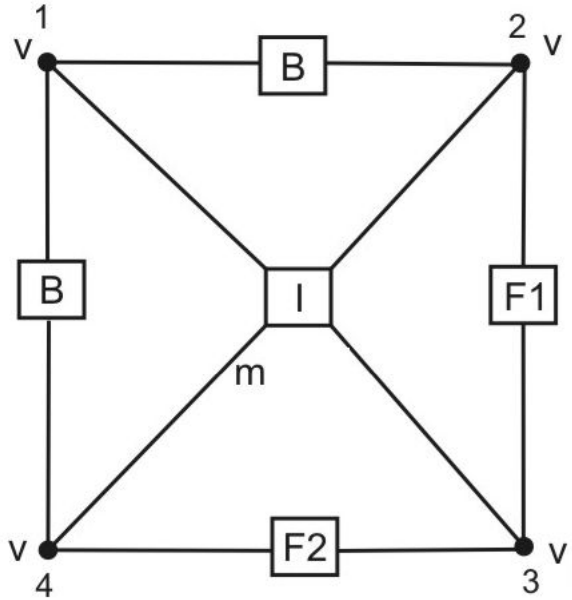
HYPERGRAPH GRAMMAR FOR HP-ADAPTATION



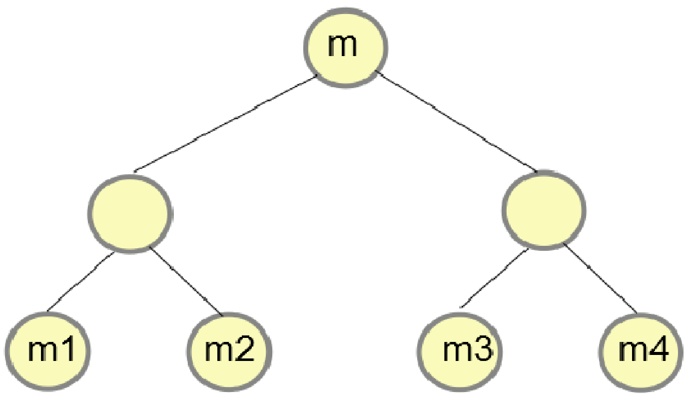
A hypergraph resulting after using production generating the mesh (once) and production breaking mesh elements to all hyperedges of the hypergraph

HYPERGRAPH PRODUCTION FOR SOLVER,
based on *h*-adaptation,





If during h-adaptation an element was broken, then during the solver algorithm matrices of „son" elements should be merged



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