(i) $$f(x) = \frac{(x+3)^3}{(x+2)^2}$$

5

$$D_f = \mathbb{R} \setminus \{-2\}$$

$$\lim_{x \to -2^-} \frac{(x+3)^3}{(x+2)^2} = \left[\frac{1}{0^+}\right] = +\infty$$

$$\lim_{x \to -2^+} \frac{(x+3)^3}{(x+2)^2} = \left[\frac{1}{0^+}\right] = +\infty$$

Istnieje as. pionowa $x = -2$

$$\lim_{x \to +\infty} \frac{f(x)}{x} = \lim_{x \to +\infty} \frac{(x+3)^3}{x(x+2)^2} = \lim_{x \to +\infty} \frac{x^3\left(1+\frac{3}{x}\right)^3}{x^3\left(1+\frac{2}{x}\right)^2} = 1 = a$$

$$\lim_{x \to +\infty} \left[f(x) - ax\right] = \lim_{x \to +\infty} \frac{(x+3)^3}{(x+2)^2} - 1 \cdot x =$$

$$= \lim_{x \to +\infty} \frac{(x+3)^3 - x \cdot (x+2)^2}{(x+2)^2} =$$

$$= \cancel{\lim_{x \to +\infty}} \; \cancel{x^3 + 9x^2 + 27x}$$

$$= \lim_{x \to +\infty} \frac{x^3 + 9x^2 + 27x + 27 - x(x^2+4x+4)}{(x+2)^2} =$$

$$= \lim_{x \to +\infty} \frac{x^3 + 9x^2 + 27x + 27 - x^3 - 4x^2 - 4x}{(x+2)^2} =$$

$$= \lim_{x \to +\infty} \frac{5x^2 + 23x + 27}{(x+2)^2} =$$

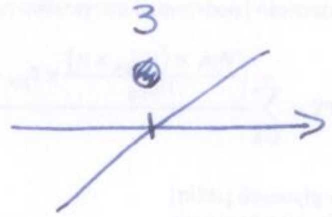
$$= \lim_{x \to +\infty} \frac{x^2\left(5 + \frac{23}{x} + \frac{27}{x^2}\right)}{x^2\left(1+\frac{2}{x}\right)^2} = 5 = b$$

Istnieje asymptota ukośna $y = x + 5$

(h) $f(x) = \dfrac{x^2 - 6x + 13}{x - 3}$ ⑤

$D_f = \mathbb{R} \setminus \{3\}$

$$\lim_{x \to 3^-} \frac{x^2 - 6x + 13}{x - 3} = \left[\frac{4}{0^-}\right] = -\infty$$



$$\lim_{x \to 3^+} \frac{x^2 - 6x + 13}{x - 3} = \left[\frac{4}{0^+}\right] = +\infty$$

Posiada asymptotę pionową, $x = 3$.

$$\lim_{x \to +\infty} \frac{f(x)}{x} = \lim_{x \to +\infty} \frac{x^2 - 6x + 13}{x(x-3)} = \lim_{x \to +\infty} \frac{x^2 - 6x + 13}{x^2 - 3x} =$$

$$= \lim_{x \to +\infty} \frac{x^2\left(1 + \frac{6}{x} + \frac{13}{x^2}\right)}{x^2\left(1 - \frac{3}{x}\right)} = 1 = a$$

$$\lim_{x \to +\infty} \left[f(x) - ax\right] = \lim_{x \to +\infty} \frac{x^2 - 6x + 13}{x - 3} - 1 \cdot x =$$

$$= \lim_{x \to +\infty} \frac{x^2 - 6x + 13}{x - 3} - \frac{x \cdot (x-3)}{x - 3} =$$

$$= \lim_{x \to +\infty} \frac{x^2 - 6x + 13 - x^2 + 3x}{x - 3} =$$

$$= \lim_{x \to +\infty} \frac{-3x + 13}{x - 3} = \lim_{x \to +\infty} \frac{x\left(-3 + \frac{13}{x}\right)}{x\left(1 - \frac{3}{x}\right)} = -3$$

Istnieje asymptota ukośna, dana wzorem

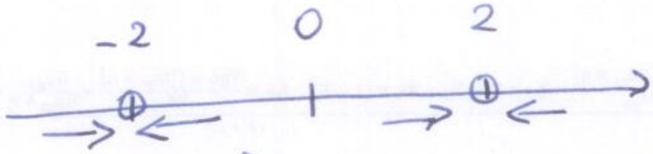
$y = x - 3$

(c)

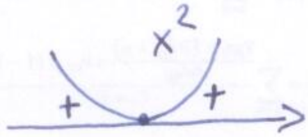
$$f(x) = \frac{x^2}{x^2-4} = \frac{x^2}{(x-2)(x+2)}$$

$$\mathbb{D}_f = \mathbb{R} \setminus \{-2; 2\}$$

As. pionowe mogą być w pobliżu $x=-2$ lub $x=2$



$$\lim_{x\to -2^-} \frac{x^2}{x^2-4} = \left[\frac{4}{0^+}\right] = +\infty$$



$$\lim_{x\to -2^+} \frac{x^2}{x^2-4} = \left[\frac{4}{0^+}\right] = +\infty$$

$$\lim_{x\to +2^-} \frac{x^2}{x^2-4} = \left[\frac{4}{0^+}\right] = +\infty$$

$$\lim_{x\to +2^+} \frac{x^2}{x^2-4} = \left[\frac{4}{0^+}\right] = +\infty$$

Istnieją dwie asymptoty pionowe $x=-2 \wedge x=2$

$$\lim_{x\to +\infty} \frac{f(x)}{x} = \lim_{x\to+\infty} \frac{x^2}{x(x^2-4)} = \lim_{x\to+\infty} \frac{x}{x^2-4} =$$

$$= \lim_{x\to+\infty} \frac{x}{x(x-\frac{4}{x})} = \lim_{x\to+\infty} \frac{1}{x - \underbrace{\frac{4}{x}}_{\to 0}} = 0$$

$$\lim_{x\to+\infty} f(x) - a\cdot x = \lim_{x\to+\infty} f(x) = \lim_{x\to+\infty} \frac{x^2}{x^2-4} =$$

$$= \lim_{x\to+\infty} \frac{x^2}{x^2(1-\underbrace{\frac{4}{x^2}}_{\to 0})} = 1$$

Istnieje asymptota pozioma $y=1$.

6

(a) $y = \dfrac{2x-3}{x+1}$

$$y' = \frac{2(x+1) - 1\cdot(2x-3)}{(x+1)^2} = \frac{2x+2-2x+3}{(x+1)^2} = \frac{5}{(x+1)^2}$$

$(2x-3)' = 2$

$(x+1)' = 1$

(c) $y = \dfrac{x^2}{x^2-4}$

$$y' = \frac{2x\,(x^2-4) - 2x\,(x^2)}{(x^2-4)^2} = \frac{2x\,(-4)}{(x^2-4)^2} = \frac{-8x}{(x^2-4)^2}$$

(i) $y = \sqrt{x^2-4x+3}$

$f(a) = \sqrt{a}$

$g(b) = b^2 - 4b + 3$

$y = f(g(\cancel{x}))$ $\qquad y' = f'(g(\cancel{x}))\cdot g'(x)$

$(\sqrt{x})' = \dfrac{1}{2\sqrt{x}}$, qdyż $\sqrt{x} = x^{\frac{1}{2}}$

$(x^2-4x+3)' = 2x-4$

$$y' = \frac{1}{2\sqrt{x^2-4x+3}}\cdot(2x-4) = \frac{\cancel{2}(x-2)}{\cancel{2}\sqrt{x^2-4x+3}}$$

$$= \frac{x-2}{\sqrt{x^2-4x+3}}$$

(h)

$$y = \sqrt{\frac{1-x}{1+x}}$$

$$f(a) = \sqrt{a}$$

$$g(b) = \frac{1-b}{1+b} \qquad y = f(g(x))$$

$$y' = f'(g(x)) \cdot g'(x)$$

$$\cancel{f'} \quad (\sqrt{x})' = \frac{1}{2\sqrt{x}}$$

$$\left(\frac{1-x}{1+x}\right)' = \frac{(-1)(1+x) - 1(1-x)}{(1+x)^2} = \frac{(-1)(2)}{(1+x)^2} = \frac{-2}{(1+x)^2}$$

$$(1-x)' = ((-1)\cdot x + 1)' = -\underline{1}$$

$$y' = \frac{1}{\cancel{2}\cdot\sqrt{\frac{1-x}{1+x}}} \cdot -\frac{\cancel{2}}{(1+x)^2} = -\sqrt{\frac{1+x}{1-x}} \cdot \frac{1}{(1+x)^2}$$

(j)

$$y = \frac{x}{2} + \frac{2}{x} = \frac{1}{2}x + \frac{2}{x}$$

$$y' = \frac{1}{2} - \frac{2}{x^2}$$

$$\left(\frac{1}{2}x\right)' = \frac{1}{2}$$

$$\left(\frac{2}{x}\right)' = \frac{0\cdot x - 1\cdot 2}{x^2} = \frac{-2}{x^2}$$

(m) $y = \dfrac{x^2 - 3x + 2}{x^2 + 3x + 2}$

6

$$y' = \frac{(2x-3)(x^2+3x+2) - (2x+3)(x^2-3x+2)}{(x^2+3x+2)^2} =$$

$$= \frac{\cancel{2x^3} + 6x^2 + \cancel{4x} - 3x^2 - \cancel{9x} - 6 - \cancel{2x^3} + 6x^2 - \cancel{4x} - 3x^2 + \cancel{9x} - 6}{(x^2+3x+2)^2}$$

$$= \frac{6x^2 - 12}{(x^2+3x+2)^2} = \frac{6(x^2-2)}{(x^2+3x+2)^2}$$

(p) $y = \dfrac{(x+2)^4}{(x+1)^3}$

$$[(x+2)^4]' \underset{\uparrow}{=} 4(x+2)^3 \cdot (x+2)' = 4 \cdot (x+2)^3$$

fcja złożona

$$[(x+1)^3]' = 3(x+1)^2 \cdot (x+1)' = 3 \cdot (x+1)^2$$

$$y' = \frac{4(x+2)^3 \cdot (x+1)^3 - 3(x+1)^2(x+2)^4}{[(x+1)^3]^2} =$$

$$= \frac{(x+1)^2[4(x+1)(x+2)^3 - (x+2)^4]}{(x+1)^4} =$$

$$= \frac{(x+2)^3[4(x+1) - (x+2)]}{(x+1)^4} =$$

$$= \frac{(x+2)^3[4x+4-x-2]}{(x+1)^4} =$$

$$= \frac{(x+2)^3[3x+2]}{(x+1)^4} = \frac{(3x+2)(x+2)^3}{(x+1)^4}$$

(c)

$$y = \cancel{(x+3)^2(x-2)}$$

$$= x^2 \cdot (x^2-4)^3$$

$$(f \cdot g)' = f'g + \boxed{g'} \cdot \boxed{f}$$

$$f = x^2$$

$$g = (x^2-4)^3$$ { jest to fcja złożona }

$$y' = 2x \cdot (x^2-4)^3 + \boxed{3(x^2-4)^2 (x^2-4)'} \cdot \boxed{x^2} =$$

$$= 2x \cdot (x^2-4)^3 + 3(x^2-4)^2 \cdot 2x \cdot x^2 =$$

$$= 2x(x^2-4)^2 \left[x^2 - 4 + 3x^2 \right] =$$

$$= 2x(x^2-4)^2 (4x^2-4) =$$

$$= 8x(x^2-1)(x^2-4)^2 =$$

$$= 8x(x-1)(x+1)(x-2)^2(x+2)^2$$

(k)

$$y = \frac{x^2+2x+25}{(x+1)^2}$$

$$y' = \frac{(2x+2)(x+1)^2 - 2(x+1) \cdot \overbrace{(x+1)'}^{1} (x^2+2x+25)}{(x+1)^4}$$

$$= \frac{\cancel{(x+1)}^{1} \left[(2x+2)(x+1) - 2(x^2+2x+25) \right]}{(x+1)^{\cancel{4}\ 3}} =$$

$$= \frac{2(x+1)(x+1) - 2(x^2-2x+25)}{(x+1)^3} =$$

$$= \frac{2x^2+4x+2 - 2x^2-4x-50}{(x+1)^3} = \frac{-48}{(x+1)^3}$$