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Edge motion and the distinguishing index ☆

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ABSTRACT

The *distinguishing index* $D'(G)$ of a graph G is the least number d such that G has an edge colouring with d colours that is preserved only by the trivial automorphism. We investigate the edge motion of a graph with respect to its automorphisms and compare it with the vertex motion. We prove an analog of the Motion Lemma of Russell and Sundaram, and we use it to determine the distinguishing index of powers of complete graphs and of cycles with respect to the Cartesian, direct and strong product.

1. Introduction

The *distinguishing index* $D'(G)$ of a graph G is the least number d such that G has an edge colouring with d colours that is preserved only by the trivial automorphism. This notion was introduced by Kalinowski and Pilśniak [10] as an analog of the well-known *distinguishing number* $D(G)$ of a graph G defined by Albertson and Collins [2] for vertex colourings. Symmetry breaking (in various ways) has interesting applications to numerous problems of theoretical computer science, for instance to the leader election problem and self-stabilizing algorithms (cf. [5,7,9]).

Obviously, the distinguishing index is not defined for K_2, thus from now on, we assume that K_2 is not a connected component of any graph being considered. There are graphs G with $D'(G) = D(G)$. Easy examples are paths and cycles: $D'(P_n) = D'(C_p) = 2$, for any $n \geq 3$ and any $p \geq 6$, and $D'(C_3) = D'(C_4) = D'(C_5) = 3$. It is also possible that $D'(G) > D(G)$, and a class of trees satisfying this inequality was found in [10]. However, very often $D'(G) < D(G)$. For example, $D'(K_n) = D'(K_{p,p}) = 2$, for any $n \geq 7$ and for any $p \geq 4$ (see [10]) while $D(K_n) = n$ and $D(K_{p,p}) = p + 1$.

A general sharp upper bound for $D'(G)$ was proved in [10].

Theorem 1. [10] *If G is a finite connected graph of order $n \geq 3$, then*

$$D'(G) \leq D(G) + 1.$$

Moreover, if $\Delta(G)$ is the maximum degree of G, then

$$D'(G) \leq \Delta(G)$$

unless G is a C_3, C_4 or C_5.

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In [10], all trees T with $D'(T) = \Delta(T)$ were characterized.

Given an automorphism φ of a graph $G = (V, E)$, let V_φ denote the set of all vertices φ moves:

$$V_\varphi = \{v \in V : \varphi(v) \neq v\}.$$

The *motion of an automorphism* φ of a graph G is the number $m(\varphi) = |V_\varphi|$, and the *motion of a graph* G is defined as

$$m(G) = \min\{m(\varphi) : \varphi \in \mathrm{Aut}(G) \setminus \{\mathrm{id}\}\}.$$

Russell and Sundaram [12] proved that the distinguishing number of a graph is small when every (nontrivial) automorphism of G moves many vertices. The precise statement of this powerful result follows.

Theorem 2 *(Russell–Sundaram Motion Lemma [12]). For any graph G and any positive integer d the inequality*

$$d^{\frac{m(G)}{2}} \geq |\mathrm{Aut}(G)|$$

implies $D(G) \leq d$.

In this paper we investigate the edge motion of finite graphs and prove an analogous result. In Section 2 we discuss a relationship between the vertex motion and the edge one of a connected graph. In particular, we obtain an interesting comparison of these two invariants for trees.

In Section 3 we prove the analog of the Motion Lemma of Russell and Sundaram for the motion of edges. We adopt the method of proof of the Motion Lemma [12], but we include this short proof for the sake of completeness. We observe that all graphs with minimum degree at least three which satisfy the hypothesis of the Motion Lemma of Russell and Sundaram for a certain d also satisfy the hypothesis of the Edge Motion Lemma with the same d, by Theorem 4 in Section 2. In such cases, we can similarly infer that $D'(G) \leq d$. But there exist graphs satisfying the hypothesis of the Edge Motion Lemma only (e.g., the Cartesian square of a complete graph of order $n \geq 4$, as we show in the next section). Therefore, in Section 4 we consider mainly such graphs, for which the distinguishing number cannot be determined by use the Motion Lemma. Thus we show how to apply the Edge Motion Lemma to determine the distinguishing index of powers of complete graphs and of cycles with respect to three standard graph products: the Cartesian, direct and strong ones.

The distinguishing index and the edge motion for certain infinite graphs has been investigated in [3].

2. Edge motion compared with vertex motion

Every automorphism $\varphi : V \to V$ of a graph $G = (V, E)$ induces a permutation $\varphi^* : E \to E$ defined as $\varphi^*(uv) = \varphi(u)\varphi(v)$ for every edge $uv \in E$. Let E_φ be the set of edges φ^* moves, i.e.,

$$E_\varphi = \{e \in E : \varphi^*(e) \neq e\}.$$

The number $m^*(\varphi) = |E_\varphi|$ is called the *edge motion of an automorphism* φ, and the *edge motion of a graph* G is defined as

$$m^*(G) = \min\{m^*(\varphi) : \varphi \in \mathrm{Aut}(G) \setminus \{\mathrm{id}\}\}.$$

We set $m^*(G) = 0$ when $\mathrm{Aut}(G) = \{\mathrm{id}\}$. For example, $m^*(K_n) = 2n - 4$ while $m(K_n) = 2$, $m^*(C_{2k}) = m(C_{2k}) = 2k - 2$, $m^*(P_{2k+1}) = m(P_{2k+1}) = 2k$, but $m^*(P_{2k}) = 2k - 2 = m(P_{2k}) - 2$.

Let us compare the motion and the edge motion of graphs in general. For trees, we have the following relationship which is fully covered by paths depending on their parity. Let us recall the well-known fact that every tree has either a central vertex or a central edge, and it is fixed by every of its automorphisms.

Proposition 3. *If T is a tree, then either $m^*(T) = m(T)$ or $m^*(T) = m(T) - 2$. Moreover, the latter case holds if and only if $\mathrm{Aut}(T) \neq \{\mathrm{id}\}$ and T has a central edge e_0 and every non-trivial automorphism of T switches the end vertices of e_0.*

Proof. First assume that T has a central vertex v_0. To prove that $m^*(T) = m(T)$ in this case it suffices to show that for any $\varphi \in \mathrm{Aut}(T)$ there is a bijection between the set V_φ of vertices and the set E_φ of edges that are moved by φ. For any vertex $v \neq v_0$, let v^- denote its neighbour situated on the path between v and v_0. Clearly, $vv^- \in E_\varphi$ if and only if $v \in V_\varphi$. As v_0 is fixed by φ, the correspondence

$$V_\varphi \ni v \mapsto vv^- \in E_\varphi$$

is one-to-one.

Now, let T contain a central edge $e_0 = u_0v_0$. If the vertices u_0, v_0 are fixed by a certain nontrivial automorphism φ, then $m^*(\varphi) = m(\varphi)$, and consequently $m^*(T) = m(T)$ due to the same arguments as in the case of a central vertex. If $\varphi(u_0) = v_0$, and thus $\varphi(v_0) = u_0$, for every $\varphi \in \mathrm{Aut}(T) \setminus \{\mathrm{id}\}$, then $V_\varphi = V$ and $E_\varphi = E \setminus \{e_0\}$. Consequently, $m^*(T) = m(T) - 2$. □

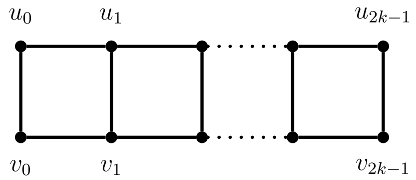

Fig. 1. A ladder L_{2k} with $\delta(L_{2k}) = 2$ and $m^*(L_{2k}) = m(L_{2k}) - 2$.

The class of trees T with $m^*(T) = m(T) - 2$ can be also characterized as follows. A tree T belongs to this class if and only if T has a central edge $e_0 = u_0v_0$, the components T_{u_0} and T_{v_0} of $T - e_0$ are isomorphic, and every non-trivial automorphism of the tree T_{u_0} moves u_0 (i.e., T_{u_0} is asymmetric). Observe also that $D'(T) \leq 2$ for every tree with $m^*(T) = m(T) - 2$. For, a usage of a distinct colour on exactly one edge different from the central one breaks all non-trivial automorphisms.

In general, we have the following lower bound for the edge motion of a graph.

Theorem 4. *Let G be a connected graph of order $n \geq 3$ with minimum degree δ. Then*

$$m^*(G) \geq \begin{cases} \frac{1}{2}(\delta - 1)m(G), & \text{if } \delta \geq 3, \\ m(G) - 2, & \text{if } \delta \leq 2. \end{cases}$$

Moreover, if $\delta \geq 3$, then $m^*(G) = \frac{1}{2}(\delta - 1)m(G)$ if and only if G is regular, $m(G) = |V(G)|$ and there exists an automorphism $\varphi \in \mathrm{Aut}(G)$ that has a decomposition into cycles of length two such that every vertex v of G is adjacent to $\varphi(v)$.

Proof. First assume that $\delta \geq 3$. Let $m^*(G) = m^*(\varphi)$ for some $\varphi \in \mathrm{Aut}(G)$. If $v \in V_\varphi$, then every incident edge uv belongs to E_φ unless $\varphi(u) = v$ and $\varphi(v) = u$, i.e., (u, v) is a cycle of length two in a decomposition of the permutation φ into cycles. Hence the number of edges incident to a given $v \in V_\varphi$ that belong to E_φ equals either $\deg(v)$ or $\deg(v) - 1$. If both ends of such an edge belong to V_φ, then we cannot count it twice, therefore

$$m^*(G) = |E_\varphi| \geq \frac{1}{2}(\delta - 1)|V_\varphi| = \frac{1}{2}(\delta - 1)m(\varphi) \geq \frac{1}{2}(\delta - 1)m(G).$$

Furthermore, it is not difficult to see that $m^*(G) = \frac{1}{2}(\delta - 1)m(G)$ if and only if both inequalities in the line above hold. Equivalently, $\deg(v) = \delta$ for each $v \in V$, there is no edge between V_φ and $V \setminus V_\varphi$ (hence $V_\varphi = V$), and for each $v \in V$ there is an edge vu with $u = \varphi(v)$, for every $\varphi \in \mathrm{Aut}(G)$.

Now, we show that $m^*(G) \geq m(G) - 2$ for any connected graph G distinct from K_2. Let $m^*(G) = m^*(\varphi)$. Consider a subgraph H_φ induced by the edges of E_φ. Clearly, the set V_φ is contained in the set of vertices of H_φ. Hence $m^*(G) = |E(H_\varphi)|$, and $|V(H_\varphi)| \geq m(\varphi) \geq m(G)$. If H_φ is connected then we are done since $|E(H_\varphi)| \geq |V(H_\varphi)| - 1$. Then suppose that H_φ has at least two connected components $H^1, \ldots, H^s$. By the minimality of the edge motion of φ, there exists a vertex v in H^1 such that $\varphi(v)$ does not belong to H^1, say $\varphi(v) \in V(H^2)$. The shortest path between v and $\varphi(v)$ contains an edge $e = u^1u^2$ that is fixed by φ. It is easily seen that $\varphi(u^1) = u^2$ and $\varphi(u^2) = u^1$. Consequently, $s = 2$. Therefore $|E(H_\varphi)| \geq |V(H_\varphi)| - 2$, whence $m^*(G) \geq m(G) - 2$. □

By Proposition 3, there exist trees with $m^*(T) = m(T) - 2$, hence the lower bound in Theorem 4 is sharp for $\delta = 1$. For $\delta = 2$, every ladder L_{2k} of odd length attains the lower bound for the edge motion. Indeed, L_{2k} has exactly three non-trivial automorphisms, namely two reflections, "vertical" and "horizontal", and their composition. Thus $m(L_{2k}) = 4k$ and $m^*(L_{2k}) = 4k - 2$ (see Fig. 1).

There is also an obvious upper bound for the edge motion:

$$m^*(G) \leq \Delta(G)m(G).$$

Indeed, if $m(G) = m(\varphi)$, then each vertex of V_φ gives at most $\Delta(G)$ edges moved by φ. This bound is also sharp since the equality is achieved for $K_{2k} - M$, a complete graph of even order with a perfect matching deleted, namely $m^*(K_{2k} - M) = 2(n - 2) = m(K_{2k} - M)\Delta(K_{2k} - M)$.

3. The Edge Motion Lemma

In this section we prove the following analog of the Motion Lemma of Russell and Sundaram.

Theorem 5 *(Edge Motion Lemma). For any graph G and any positive integer d the inequality*

$$d^{\frac{m^*(G)}{2}} \geq |\mathrm{Aut}(G)|$$

implies $D'(G) \leq d$.

The proof is analogous to that of the Motion Lemma [12]. To prepare for it, we define the edge cycle norm of an automorphism φ. Let $\varphi^* = (O_1, \ldots, O_s)$ be a decomposition into cycles of the permutation φ^* induced by φ (see Section 2). We define the *edge cycle norm of* φ as the number

$$c^*(\varphi) = \sum_{i=1}^{s}(|O_i| - 1),$$

where $|O_i|$ stands for the length of a cycle $|O_i|$. The *edge cycle norm of a graph* G is the number

$$c^*(G) = \min\{c^*(\varphi) : \varphi \in \mathrm{Aut}(G) \setminus \{\mathrm{id}\}\}.$$

Clearly $c^*(\varphi) \geq m^*(\varphi)/2$, and the equality holds if $|O_i| \leq 2$ for all i. Thus $c^*(G) \geq m^*(G)/2$.

We will now prove a more general result, and Theorem 5 will be an immediate consequence of it.

Theorem 6 *(Edge Cycle Norm Lemma). The distinguishing index of a graph G is at most d if*

$$d^{-c^*(G)} \geq |\mathrm{Aut}(G)|.$$

Proof. Consider a random colouring f of edges of G with the probability distribution given by selecting the colour of each edge independently and uniformly in the set $\{1, \ldots, d\}$. Fix an automorphism $\varphi \neq \mathrm{id}$ and consider the undesirable event that the random colouring f is preserved by φ, that is, every cycle O_i in a decomposition of φ^* into cycles is monochromatic. Clearly, the probability that a given cycle O_i is monochromatic equals $\left(\frac{1}{d}\right)^{|O_i|-1}$. Hence

$$\mathrm{Prob}_f[\forall e:\ f(e) = f(\varphi^*(e))] = \left(\frac{1}{d}\right)^{c^*(\varphi^*)} \leq \left(\frac{1}{d}\right)^{c^*(G)}.$$

Collecting together these undesirable events for all $\varphi \in \mathrm{Aut}(G)$, we have

$$\mathrm{Prob}_f[\exists \varphi \neq \mathrm{id}\ \forall e:\ f(e) = f(\varphi^*(e))] \leq \sum_{\varphi \in \mathrm{Aut}(G)\setminus\{\mathrm{id}\}} \left(\frac{1}{d}\right)^{c^*(\varphi^*)} \leq |\mathrm{Aut}(G)| \left(\frac{1}{d}\right)^{c^*(G)}.$$

By hypothesis, the left-hand side of this inequality is less than one, thus there exists a colouring f such that for all nontrivial φ there is an edge e such that $f(e) \neq f(\varphi^*(e))$. Hence $D'(G) \leq d$. □

Observe that all graphs with minimum degree at least three which satisfy the hypothesis of the Motion Lemma of Russell and Sundaram for a certain d also satisfy the hypothesis of the Edge Motion Lemma with the same d, by Theorem 4. Then similarly we can infer that $D'(G) \leq d$. But there exist graphs satisfying the hypothesis of the Edge Motion Lemma but not that of the Motion Lemma (e.g., the Cartesian square of a complete graph of order $n \geq 4$, as we shall show in the next section).

4. Application to products

In this section we show how to apply the Edge Motion Lemma to determine the distinguishing index of powers of complete graphs and of cycles with respect to three standard graph products: the Cartesian, direct and strong ones.

4.1. Cartesian powers

Recall that the Cartesian product of two graphs G and H, denoted by $G\Box H$, is the graph with the vertex set $V(G\Box H) = V(G) \times V(H)$, and a vertex (u, v) is adjacent to a vertex (w, z) if either $u = w$ and $vz \in E(H)$, or $v = z$ and $uw \in E(G)$. Denote $G^2 = G\Box G$ and recursively $G^r = G^{r-1}\Box G$. Imrich and Miller independently proved the following theorem.

Theorem 7. [8,11] *If G is connected and $G = G_1\Box G_2\Box \cdots \Box G_r$ is its prime decomposition, then every automorphism of G is generated by automorphisms of the factors and transpositions of isomorphic factors.*

For additional results on the Cartesian and other products consult [6].

Consider the r-th Cartesian power K_n^r of a complete graph of order n. Albertson [1] determined the numbers $|\mathrm{Aut}(K_n^r)|$ and $m(K_n^r)$.

Theorem 8. [1] $|\mathrm{Aut}(K_n^r)| = r!(n!)^r$ *and* $m(K_n^r) = 2n^{r-1}$.

Then by a straightforward application of the Motion Lemma of Russell and Sundaram, he proved the following.

Theorem 9. [1] *$D(K_n^r) = 2$ whenever $r \geq 4$. Moreover, if in addition $n \geq 5$, then $D(K_n^r) = 2$ whenever $r \geq 3$.*

We now determine the distinguishing index of Cartesian powers of a complete graph. For such graphs, we can infer more from the Edge Motion Lemma than from the Motion Lemma.

Theorem 10. *For every $n \geq 3$ and $r \geq 2$, except for $n = 3$ and $r = 2$, the distinguishing index $D'(K_n^r)$ of the r-th power of the complete graph of order n is 2.*

Proof. Clearly, $D'(K_n^r) \geq 2$ as K_n^r has non-trivial automorphisms. For $r \geq 3$, we even need not evaluate the edge motion of K_n^r for it suffices to apply Theorem 4. Indeed, $\delta(K_n^r) = r(n-1)$ hence $m^*(K_n^r) \geq n^{r-1}(r(n-1)-1)$. And to derive the conclusion from the Edge Motion Lemma, it suffices to prove by induction that

$$2^{\frac{n^{r-1}(r(n-1)-1)}{2}} \geq r!(n!)^r,$$

for every $n \geq 3$ and $r \geq 3$.

Let $r = 2$ and $n \geq 4$. To prove that $m^*(K_n^2) = n(3n-5)$ we need a definition of a layer of the Cartesian square of a graph [6]. Given $x_0, y_0 \in V(G)$, a *vertical layer* L_{x_0} of G^2 is a subgraph L_{x_0} induced by the set $\{(x_0, y) : y \in V(G)\}$, and a *horizontal layer* of G^2 is a subgraph L^{y_0} induced by the set $\{(x, y_0) : x \in V(G)\}$. Let $\varphi \in \mathrm{Aut}(K_n^2)$ be a transposition of two layers. Then clearly $m^*(\varphi) = 2\binom{n}{2} + 2n(n-2) = n(3n-5)$. On the other hand, it follows from Theorem 7 that the image of any layer under any $\varphi \in \mathrm{Aut}(K_n^2)$ is also a layer. Without loss of generality, assume that φ moves a vertical layer L_{x^0}. Suppose first that the image $\varphi(L_{x^0})$ is also a vertical layer L_{x_1}. Thus, L_{x_1} is also moved by φ. If φ permutes $t \geq 3$ vertical layers, then it is easily seen that φ moves at least $tn(n-2) \geq t\binom{n}{2} + tn(n-2) = n(3n-5) \geq 2\binom{n}{2} + 2n(n-2) = n(3n-5)$ edges. Suppose now that the image $\varphi(L_{x^0})$ is horizontal layer L^{y_0}. Then φ also moves L^{y_0}, and thus it moves all edges incident to the vertices of both layers L_{x_0} and L^{y_0} except, possibly, the edges between these two layers. Hence $m^*(\varphi) = 2\binom{n}{2} + 2n(n-2) = n(3n-5)$.

Again, we can apply the Edge Motion Lemma since

$$2^{\frac{m^*(K_n^2)}{2}} = 2^{\frac{n(3n-5)}{2}} \geq 2(n!)^2 = |\mathrm{Aut}(K_n^2)|,$$

for every $n \geq 4$, and this can be done by induction. □

Remark. As an immediate consequence of Theorem 9, Albertson proved in [1] that the distinguishing number of the r-th Cartesian power of any connected, prime graph G is 2, provided n and r satisfy the assumptions of Theorem 9. This clearly follows from Theorem 7 and the fact that $\mathrm{Aut}(G) \subseteq \mathrm{Aut}(K_n)$, hence $D(G) \leq D(K_n)$. However, this is certainly not the case for edge colourings because there are graphs G of order n with $D'(G) > D'(K_n)$. For results on the distinguishing index of the Cartesian powers of arbitrary connected graphs we refer the reader to [4]. In this paper, we only investigate the edge motion and its applications.

Let us now consider the Cartesian squares of cycles. It is well known that the automorphism group of C_n is the dihedral group of $2n$ elements. So it follows from Theorem 7 that

$$|\mathrm{Aut}(C_n^r)| = r!(2n)^r.$$

In particular, $|\mathrm{Aut}(C_n^2)| = 8n^2$. It is easy to see that $m(C_n^2) = n(n-2)$ for even n, and $m(C_n^2) = n(n-1)$ if n is odd. Using similar arguments as for the squares of complete graphs, one can show that $m^*(C_n^2) = n(n-2) + n^2 = 2n(n-1)$ for even n, and $m^*(C_n^2) = 2n^2 - 2n = 2n(n-1)$ for odd n. So by induction, for every $n \geq 4$

$$2^{\frac{m^*(C_n^2)}{2}} = 2^{n(n-1)} \geq 8n^2 = |\mathrm{Aut}(C_n^2)|,$$

that is, the assumptions of the Edge Motion Lemma are fulfilled. This justifies the following.

Proposition 11. *If $n \geq 4$, then $D'(C_n^2) = 2$.* □

4.2. Direct powers

The direct product of two graphs G and H, denoted by $G \times H$, is the graph with the vertex set $V(G \times H) = V(G) \times V(H)$ and a vertex (u, v) is adjacent to a vertex (w, z) if $vz \in E(H)$ and $uw \in E(G)$. Let $G^{\times,2} = G \times G$ be called the *direct square of a graph* G, and recursively let $G^{\times,r} = G^{\times,r-1} \times G$.

As above, we can apply the Edge Motion Lemma to evaluate the distinguishing index for the direct square of complete graphs and cycles. The direct product of bipartite graphs is not connected. As we are interested in distinguishing colourings of connected graphs, we shall consider only odd cycles. Connected non-bipartite graphs have unique prime factorization with respect to the direct product ([6]). If such a graph G has no pairs u, v of vertices with the same neighbourhood, then the structure of the automorphism group of G depends on those of its prime factors exactly in the same way as in the case of the Cartesian product (cf. Theorem 7), hence $|\mathrm{Aut}(K_n^{\times,r})| = r!(n!)^r$.

Observe that $\deg(v) = (n-1)^r$ for $v \in V(K_n^{\times,r})$. Moreover, analogously as for the Cartesian square, the edge motion of $K_n^{\times,2}$ is realized by an automorphism such that there exist exactly two vertices x, y in the first factor K_n which are transposed one onto another transposition and all other vertices are fixed. Hence $m^*(K_n^{\times,2}) = 2n(n-1)^2$.

Again by the Edge-Motion Lemma we obtain the value of the distinguishing index of the direct square of a complete graph. Indeed,

$$2^{\frac{m^*(K_n^{\times,2})}{2}} = 2^{n(n-1)^2} \geq 2(n!)^2 = |\mathrm{Aut}(K_n^{\times,2})|$$

for $n \geq 3$. Note that $D'(K_n^{\times,r}) \geq 2$ for any r since $K_n^{\times,r}$ is not asymmetric.

Lemma 12. *If $n \geq 3$, then $D'(K_n^{\times,2}) = 2$.* □

For larger r, it is easy to observe that the motion of the r-th powers with respect to both products, the Cartesian and the direct ones, of complete graphs are equal since $\mathrm{Aut}(K_n^{\times,r}) = \mathrm{Aut}(K_n^r)$. For $n \geq 3$ and $r \geq 3$, it follows from Theorems 4 and 8 that

$$m^*(K_n^{\times,r}) \geq \frac{1}{2}((n-1)^2 - 1)m(K_n^{\times,r}) = n^{r-1}((n-1)^r - 1).$$

By an easy induction we have

$$2^{\frac{m^*(K_n^{\times,r})}{2}} = 2^{\frac{1}{2}n^{r-1}((n-1)^r-1)} \geq r!(n!)^r = |\mathrm{Aut}(K_n^{\times,r})|$$

and we can apply the Edge Motion Lemma.

Theorem 13. *$D'(K_n^{\times,r}) = 2$ for every $n \geq 3$ and $r \geq 2$.* □

The direct square of the odd cycle $C_n^{\times,2}$ is a 4-regular graph with the motion $m(C_n^{\times,2}) = m(C_n^2) = n(n-2)$ and with $8n^2$ automorphisms. Due to Theorem 4 we can bound its edge motion from below, and then we obtain the distinguishing index $D'(C_n^{\times,2})$ of the direct square of a cycle of odd length by the Edge Motion Lemma since

$$2^{\frac{m^*(C_n^{\times,2})}{2}} \geq 2^{\frac{3}{4}n(n-2)} \geq 8n^2 = |\mathrm{Aut}(C_n^{\times,2})|$$

for $n \geq 4$.

Proposition 14. *If n is odd and $n \geq 5$, then $D'(C_n^{\times,2}) = 2$.* □

4.3. Strong powers

The strong product of two graphs G and H, denoted by $G \boxtimes H$, is the graph with the vertex set is $V(G \boxtimes H) = V(G) \times V(H)$ and the edge set $E(G \boxtimes H) = E(G \Box H) \cup E(G \times H)$. The strong product of r copies of G will be denoted by $G^{\boxtimes,r}$.

Note that $K_n^{\boxtimes,r} = K_{n^r}$, hence $D'(K_n^{\boxtimes,r}) = 2$ for $n \geq 3$. So we shall only look at the strong powers of cycles.

Connected non-bipartite graphs without any pair u, v of vertices with the same closed neighbourhoods have unique prime factorization with respect to the strong product [6] and the structure of the automorphism group of G depends on that of its prime factors exactly as in the case of the Cartesian product. Thus, $|\mathrm{Aut}(C_n^{\boxtimes,2})| = 8n^2$ for odd $n \geq 3$, by Theorem 7. Furthermore, $m^*(C_n^{\boxtimes,2}) \geq \frac{7}{2}n(n-1)$ for odd n, by Theorem 4. Using an easy induction we prove that

$$2^{\frac{m^*(C_n^{\boxtimes,2})}{2}} \geq 2^{\frac{7}{4}n(n-1)} \geq 8n^2 = |\mathrm{Aut}(C_n^{\boxtimes,2})|$$

for $n \geq 3$. Applying the Edge Motion Lemma yields the proof of the following.

Proposition 15. *The distinguishing index of the strong square of an odd cycle is equal to two.* □

5. Concluding remarks

As the distinguishing index of the Cartesian powers of graphs has been studied in [4], then higher powers of cycles, and of other graphs, with respect to the direct and strong product can be considered. Note that even cycles are bipartite, and the direct product of bipartite graphs is disconnected (if the number of isomorphic components of a graph G is large enough, then the distinguishing index of G is greater than that of a single component).

Another open problem is a lower bound for the sizes of graphs G and H that implies that $D'(G \times H) \geq d$ (or $D'(G \boxtimes H) \geq d$, respectively) for a given integer d. A corresponding result for $D'(G \square H)$ was proved in [4].

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Further reading

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