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The Sierpiński domination number*

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Abstract

Let G and H be graphs and let $f\colon V(G) \to V(H)$ be a function. The Sierpiński product of G and H with respect to f, denoted by $G \otimes_f H$, is defined as the graph on the vertex set $V(G) \times V(H)$, consisting of $|V(G)|$ copies of H; for every edge gg' of G there is an edge between copies gH and $g'H$ of H associated with the vertices g and g' of G, respectively, of the form $(g, f(g'))(g', f(g))$. In this paper, we define the Sierpiński domination number as the minimum of $\gamma(G \otimes_f H)$ over all functions $f\colon V(G) \to V(H)$. The upper Sierpiński domination number is defined analogously as the corresponding maximum.

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After establishing general upper and lower bounds, we determine the upper Sierpiński domination number of the Sierpiński product of two cycles, and determine the lower Sierpiński domination number of the Sierpiński product of two cycles in half of the cases and in the other half cases restrict it to two values.

Keywords: Sierpiński graph, Sierpiński product, domination number, Sierpiński domination number.

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1 Introduction

Sierpiński graphs represent a very interesting and widely studied family of graphs. They were introduced in 1997 in the paper [15], where the primary motivation for their introduction was the intrinsic link to the Tower of Hanoi problem, for the latter problem see the book [11]. Intensive research of Sierpiński graphs led to a review article [12] in which state of the art up to 2017 is summarized and unified approach to Sierpiński-type graph families is also proposed. Later research on Sierpiński graphs includes [2, 3, 6, 19, 23].

Sierpiński graphs have a fractal structure, the basic graphs of which are complete graphs. In 2011, Gravier, Kovše, and Parreau [7] introduced a generalization in such a way that any graph can act as a fundamental graph, and called the resulting graphs generalized Sierpiński graphs. We refer to the papers [1, 4, 5, 13, 14, 16, 17, 20, 21, 22, 24] for investigations of generalized Sierpiński graphs in the last few years.

An interesting generalization of Sierpiński graphs in the other direction has recently been proposed by Kovič, Pisanski, Zemljič, and Žitnik in [18]. Namely, in the spirit of classical graph products, where the vertex set of a product graph is the Cartesian product of the vertex sets of the factors, they introduced the Sierpiński product of graphs as follows. Let G and H be graphs and let $f\colon V(G) \to V(H)$ be an arbitrary function. The *Sierpiński product of graphs G and H with respect to f*, denoted by $G \otimes_f H$, is defined as the graph on the vertex set $V(G) \times V(H)$ with edges of two types:

- *type*-1 *edge*: $(g,h)(g,h')$ is an edge of $G \otimes_f H$ for every vertex $g \in V(G)$ and every edge $hh' \in E(H)$,
- *type*-2 *edge*: $(g, f(g'))(g', f(g))$ is an edge of $G \otimes_f H$ for every edge $gg' \in E(G)$.

We observe that the edges of type-1 induce $n(G) = |V(G)|$ copies of the graph H in the Sierpiński product $G \otimes_f H$. For each vertex $g \in V(G)$, we let gH be the copy of H corresponding to the vertex g. A type-2 edge joins vertices from different copies of H in $G \otimes_f H$, and is called a *connecting edges* of $G \otimes_f H$. A vertex incident with a connecting edge is called a *connecting vertex*. We observe that two different copies of H in $G \otimes_f H$ are joined by at most one edge. A copy of the graph H corresponding to a vertex of the graph G in the Sierpiński product $G \otimes_f H$ is called an *H-layer*.

Let G and H be graphs and H^G be the family of functions from $V(G)$ to $V(H)$. We introduce new types of domination, the *Sierpiński domination number*, denoted by $\gamma_S(G,H)$, as the minimum over all functions f from H^G of the domination number of the Sierpiński product with respect to f, and *upper Sierpiński domination number*, denoted

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by $\Gamma_S(G,H)$, as the maximum over all functions $f \in H^G$ of domination number of the Sierpiński product with respect to f. That is,

$$\gamma_{\rm S}(G,H) \coloneqq \min_{f\in H^G}\{\gamma(G\otimes_f H)\}$$

and

$$\Gamma_{\rm S}(G,H) \coloneqq \max_{f\in H^G}\{\gamma(G\otimes_f H)\}\,.$$

In this paper, we initiate the study of Sierpiński domination in graphs. In Section 1.1 we present the graph theory notation and terminology we follow. In Section 2 we discuss general lower and upper bounds on the (upper) Sierpiński domination number. Our main contribution in this introductory paper is to determine the upper Sierpiński domination number of the Sierpiński product of two cycles, and to determine the lower Sierpiński domination number of the Sierpiński product of two cycles in half of the cases and in the other half cases restrict it to two values.

1.1 Notation and terminology

We generally follow the graph theory notation and terminology in the books [8, 9, 10] on domination in graphs. Specifically, let G be a graph with vertex set $V(G)$ and edge set $E(G)$, and of order $n(G) = |V(G)|$ and size $m(G) = |E(G)|$. For a subset S of vertices of a graph G, we denote by $G - S$ the graph obtained from G by deleting the vertices in S and all edges incident with vertices in S. If $S = \{v\}$, then we simply write $G - v$ rather than $G - \{v\}$. The subgraph induced by the set S is denoted by $G[S]$. We denote the path, cycle and complete graph on n vertices by P_n, C_n, and K_n, respectively. For $k \geq 1$ an integer, we use the notation $[k] = \{1,\ldots,k\}$ and $[k]_0 = \{0,1,\ldots,k\}$. We generally label vertices of the considered graphs by elements of $[n]$. In this case, the mod function over the set $[n]$ is to be understood in a natural way, more formally, we apply the following operation for $t \geq 1$: $t \bmod^* n = (t-1) \bmod n + 1$.

A vertex *dominates* itself and its neighbors, where two vertices are neighbors in a graph if they are adjacent. A *dominating set* of a graph G is a set S of vertices of G such that every vertex in G is dominated by a vertex in S. The *domination number*, $\gamma(G)$, of G is the minimum cardinality of a dominating set of G. A dominating set of cardinality $\gamma(G)$ is called a *γ-set of G*. A thorough treatise on dominating sets can be found in [8, 9].

If S is a set of vertices in a graph G, then we will use the notation $G|S$ to denote that the vertices in the set S are assumed to be dominated and hence $\gamma(G|S)$ is the minimum number of vertices in the graph G needed to dominate $V(G) \setminus S$. We note that it could be that a vertex in S is still a member of a such a minimum dominating set no matter that we do not need to dominate the vertices in S themselves. If $S = \{x\}$, then we simply denote $G|S$ by $G|x$ rather than $G|\{x\}$.

2 General lower and upper bounds

We present in this section general lower and upper bounds on the (upper) Sierpiński domination number.

Theorem 2.1. *If G and H are graphs, then*

$$n(G)\gamma(H) - m(G) \leq \gamma_{\rm S}(G,H) \leq \Gamma_{\rm S}(G,H) \leq n(G)\gamma(H)\,.$$

Proof. Let $G \otimes_f H$ be an arbitrary Sierpiński product of graphs G and H and let X be a γ-set of $G \otimes_f H$. Assuming for a moment that all the connecting edges are removed from $G \otimes_f H$, we obtain $n(G)$ disjoint copies of H for which we clearly need $n(G)\gamma(H)$ vertices in a minimum dominating set. Consider now an arbitrary connecting edge $e = (g, f(g'))(g', f(g))$ of $G \otimes_f H$. If no end-vertex of e lies in X, then clearly $\gamma(G \otimes_f H - e) = \gamma(G \otimes_f H)$. Similarly, if both end-vertices of e lie in X, then $\gamma(G \otimes_f H - e) = \gamma(G \otimes_f H)$. Hence the only situation in which e has an effect on $\gamma(G \otimes_f H)$ is when $(g, f(g')) \in X$ and $(g', f(g)) \notin X$ (or the other way around). But in this case, the effect of the presence of the edge e is that because $(g, f(g'))$ dominates one vertex of $g'H$, the edge e might reduce the domination number by 1. That is, each connecting edge can drop the domination number of $G \otimes_f H$ by at most 1, which proves the left inequality. The other two inequalities are clear. □

To show that the lower bound of Theorem 2.1 is achieved, we show later in Theorem 3.10 that for $n \geq 3$ and $k \geq 1$, if we take $G = C_n$ and $H = C_{3k+1}$ where $n \equiv 0 \pmod 4$, then $\gamma_{\rm S}(G, H) = kn = n(G)\gamma(H) - m(G)$. The upper bound of Theorem 2.1 is obtained, for example, for the Sierpiński product of two complete graphs. More generally, to achieve equality in the upper bound of Theorem 2.1 we require the graph H to have the following property.

Theorem 2.2. *The equality in $\Gamma_{\rm S}(G, H) \leq n(G)\gamma(H)$ is achieved if and only if there exists a vertex $x \in V(H)$ such that $\gamma(H|x) = \gamma(H)$.*

Proof. Suppose that H has a vertex x that satisfies $\gamma(H|x) = \gamma(H)$. In this case, we consider the Sierpiński product $G \otimes_f H$ with the function $f\colon V(G) \to V(H)$ defined by $f(v) = x$ for every vertex $v \in V(G)$. Consequently each connecting edge in the product is of the form $(g, x)(g', x)$. Thus, if X is a γ-set of $G \otimes_f H$, then $|X \cap V(gH)| = \gamma(H)$ because the only vertex of gH that can be dominated from outside gH is (g, x), but we have assume that $\gamma(H|x) = \gamma(H)$. Therefore, $\Gamma_S(G, H) = n(G)\gamma(H)$.

For the other implication suppose that $\gamma(H|x) < \gamma(H)$ for every vertex $x \in V(H)$. For an arbitrary edge $g_1g_2 \in E(G)$, if D corresponds to a γ-set of the product $G \otimes_f H$, then $|D \cap V((G \otimes_f H)[V(g_1H) \cup V(g_2H)])| \leq 2\gamma(H) - 1$. Consequently, $\Gamma_S(G, H) < n(G)\gamma(H)$. □

To conclude this section we describe large classes of graphs for which the second and the third inequality of Theorem 2.1 are both equality.

Proposition 2.3. *If G and H are graphs such that $\Delta(G) < n(H)$ and $\gamma(H) = 1$, then $\Gamma_{\rm S}(G, H) = \gamma_{\rm S}(G, H) = n(G)$.*

Proof. Let G and H be graphs such that $\Delta(G) < n(H)$ and $\gamma(H) = 1$. Thus, by Theorem 2.1, $\gamma_{\rm S}(G, H) \leq \Gamma_{\rm S}(G, H) \leq n(G)$ and it is straightforward that the Sierpiński product of graphs G and H can be dominated by taking one dominating vertex from each H-layer to the dominating set. It remains to show that the inequality $\gamma_{\rm S}(G, H) \geq n(G)$ also holds. Suppose that $\gamma_{\rm S}(G, H) \leq n(G) - 1$. Let D be a dominating set of $G \otimes_f H$, where f is such that it minimizes the domination number. Therefore there is an H-layer, denote it by H', of $G \otimes_f H$ such that $D \cap V(H') = \emptyset$. Since there are at most $\Delta(G)$ connecting edges incident with vertices from each H-layer and $\Delta(G) < n(H)$,then all the vertices from an H-layer cannot be dominated by the vertices from the neighboring layers. Therefore we have $D \cap V(H'') \neq \emptyset$ for each H''-layer of $G \otimes_f H$. Thus $\gamma_{\rm S}(G, H) \geq n(G)$ and the result follows. □

3 The Sierpiński domination number of cycles

Let us recall firstly the domination number of a path and a cycle.

Fact 3.1. *For $n \geq 3$, $\gamma(P_n) = \gamma(C_n) = \lceil \frac{n}{3} \rceil$.*

In this section, we shall prove the following results.

Theorem 3.2. *For $n \geq 3$, $k \geq 1$, and $p \in [2]_0$,*

$$\gamma_{\rm S}(C_n, C_{3k+p}) \in \begin{cases} \{kn\}; & p = 0, \\ \{kn, kn+1\}; & p = 1, \\ \{kn + \left\lfloor \frac{n}{2} \right\rfloor, kn + \left\lfloor \frac{n}{2} \right\rfloor + 1\}; & p = 2. \end{cases}$$

Moreover, if $n \equiv 0 \mod 4$, then $\gamma_{\rm S}(C_n, C_{3k+1}) = kn$ and $\gamma_{\rm S}(C_n, C_{3k+2}) = kn + \left\lfloor \frac{n}{2} \right\rfloor$.

Theorem 3.3. *For $n \geq 3$, $k \geq 1$, and $p \in [2]_0$,*

$$\Gamma_{\rm S}(C_n, C_{3k+p}) = \begin{cases} kn; & p = 0, \\ kn + \left\lceil \frac{n}{3} \right\rceil; & p = 1, \\ (k+1)n; & p = 2. \end{cases}$$

In order to prove Theorems 3.2 and 3.3, we consider three cases, depending on the value of p.

3.1 The cycle C_n and cycles C_{3k+1}

To determine $\Gamma_S(C_n, C_{3k+1})$, we prove a slightly more general result. For this purpose, we define a class of graphs $\mathcal{H}_k$ as follows.

Definition 3.4. For $k \geq 1$, let $\mathcal{H}_k$ be the class of all graphs H that have the following properties.

(a) $\gamma(H) = k + 1$ and $\gamma(H - v) = k$ for every vertex $v \in V(H)$.

(b) If $x, y \in V(H)$, then there exists a γ-set of H that contains x and y, where $x = y$ is allowed.

We show, for example, that for every $k \geq 1$, the cycle C_{3k+1} belongs to the class $\mathcal{H}_k$.

Proposition 3.5. *For $k \geq 1$, the class $\mathcal{H}_k$ of graphs contains the cycle C_{3k+1}.*

Proof. For $k \geq 1$, let $H \cong C_{3k+1}$. Since $\gamma(C_n) = \gamma(P_n) = \lceil n/3 \rceil$, property (a) in Definition 3.4 holds. To prove that property (b) in Definition 3.4 holds, let $x, y \in V(H)$. Since H is vertex-transitive, every specified vertex belongs to some γ-set of H. In particular, if $x = y$, then property (b) is immediate. Hence, we may assume that $x \neq y$. Let H be the cycle $v_1 v_2 \ldots v_{3k+1} v_1$, where renaming vertices if necessary, we may assume that $x = v_1$. Let $y = v_i$, and so $i \in [3k+1] \setminus \{1\}$.

Let $H' = H - N[\{x, y\}]$, that is, H' is obtained from H by removing x and y, and removing all neighbors of x and y. If H' is connected, then H' is a path $P_{3(k-2)+j}$ for

some j where $j \in [3]$. In this case, $\gamma(H') = k-1$. If H' is disconnected, then H' is the disjoint union of two paths P_{k_1} and P_{k_2}, where $k_1 + k_2 = 3(k-2)+1$. Thus renaming k_1 and k_2 if necessary, we may assume that either $k_1 = 3j_1$ and $k_2 = 3j_2+1$ where $j_1 \geq 1$, $j_2 \geq 0$, and $j_1 + j_2 = k-2$ or $k_1 = 3j_1+2$ and $k_2 = 3j_2+2$ where $j_1, j_2 \geq 0$ and $j_1 + j_2 = k-3$. In both cases, $\gamma(H') = \lceil k_1/3 \rceil + \lceil k_2/3 \rceil = k-1$. Letting D' be a γ-set of H', the set $D = D' \cup \{x, y\}$ is a dominating set of H of cardinality $k+1 = \gamma(H)$, implying that D is a γ-set of H that contains both x and y. Hence, property (b) holds. □

For $n \geq 3$ an integer, a *circulant graph* $C_n\langle L\rangle$ with a given list $L \subseteq \{1, \ldots, \lfloor \frac{1}{2}n \rfloor\}$ is a graph on n vertices in which the ith vertex is adjacent to the $(i+j)$th and $(i-j)$th vertices for each j in the list L and where addition is taken modulo n. For example, for $n = 3k+1$ where $k \geq 1$ and $L = \{1\}$, the circulant graph $C_n\langle L\rangle$ is the cycle C_{3k+1}, which, by Proposition 3.5, belongs to the class $\mathcal{H}_k$. More generally, for $n = k(2p+1)+1$ where $k \geq 1$, $p \geq 1$, and $L = [p]$, the circulant graph $C_n\langle L\rangle$ belongs to the class $\mathcal{H}_k$. We omit the relatively straightforward proof. These examples of circulant graphs serve to illustrate that for each $k \geq 1$, one can construct infinitely many graphs in the class $\mathcal{H}_k$. We determine next the upper Sierpiński domination number $\Gamma_S(C_n, H)$ of a cycle C_n and a graph H in the family $\mathcal{H}_k$.

Theorem 3.6. *For $n \geq 3$ and $k \geq 1$, if $H \in \mathcal{H}_k$, then*

$$\Gamma_S(C_n, H) = kn + \left\lceil \frac{n}{3} \right\rceil.$$

Proof. For $n \geq 3$ and $k \geq 1$, let $G \cong C_n$ and let $H \in \mathcal{H}_k$. Let G be the cycle given by $g_1g_2 \ldots g_ng_1$. In what follows, we adopt the following notation. For each $i \in [n]$, we denote the copy g_iH of H corresponding to the vertex g_i simply by H_i. We proceed further with two claims. The first claim establishes a lower bound on $\Gamma_S(C_n, H)$, and the second claim establishes an upper bound on $\Gamma_S(C_n, H)$.

Claim 3.7. $\Gamma_S(C_n, H) \geq kn + \left\lceil \frac{n}{3} \right\rceil$.

Proof. Let $f\colon V(G) \to V(H)$ be a constant function, that is, we select $h \in V(H)$ and for every vertex $g \in V(G)$, we set $f(g) = h$. Let D_G be a γ-set of G. Thus, $|D_G| = \gamma(C_n) = \lceil n/3 \rceil$. By property (a) in Definition 3.4, for every vertex $g \in V(G)$, there exists a γ-set of gH that contains the vertex $(g, f(g)) = (g, h)$. If $g \in D_G$, let D_g be a γ-set of gH that contains the vertex $(g, f(g)) = (g, h)$, and so $|D_g| = \gamma(H) = k+1$. If $g \in V(G) \setminus D_G$, let D_g be a γ-set of $gH - (g, f(g)) = gH - (g, h)$, and so in this case $|D_g| = \gamma(H - h) = \gamma(H) - 1 = k$. Let

$$D = \bigcup_{g \in V(G)} D_g.$$

The set D is a dominating set of $G \otimes_f H$, and so

$$\gamma(G \otimes_f H) \leq |D| = \gamma(G)(k+1) + (n - \gamma(G))k = kn + \gamma(G) = kn + \left\lceil \frac{n}{3} \right\rceil. \quad (3.1)$$

For the fixed vertex h chosen earlier, we note that the set of vertices (g, h) for all $g \in V(G)$ induces a subgraph of $G \otimes_f H$ that is isomorphic to $G \cong C_n$. We denote this

copy of G by Gh. Among all γ-sets of $G \otimes_f H$, let D^* be chosen to contain as many vertices of Gh as possible. Let $D_g^* = D^* \cap V(gH)$ for every $g \in V(G)$. Further let $D_G^* = \{(g,h) \in D^* \colon g \in V(G)\}$, that is, D_G^* is the restriction of D^* to the copy of G. If a vertex $(g,h) \notin D_G^*$ and (g,h) is not dominated by D_G^*, then D_g^* is a γ-set of gH by the minimality of the set D^*. However in this case, we could replace the set D_g^* be a γ-set of gH that contains the vertex (g,h) to produce a new γ-set of $G \otimes_f H$ that contains more vertices from the copy of G than does D^*, a contradiction. Hence, the set D_G^* is a dominating set in the copy of G, and so $|D_g^*(G)| \geq \gamma(G)$. By the minimality of the set D^* and by property (a) in Definition 3.4, for each vertex $g \in V(G)$, we have $|D_g^*| = \gamma(H) = k+1$ if the vertex $(g,h) \in D_G^*$ and $|D_g^*| = \gamma(H-h) = k$ if the vertex $(g,h) \notin D_G^*$. Therefore,

$$\begin{aligned}\gamma(G \otimes_f H) = |D^*| &= |D_G^*|(k+1) + (n - |D_G^*|)k = kn + |D_G^*| \\ &\geq kn + \gamma(G) = kn + \left\lceil \frac{n}{3} \right\rceil. \end{aligned} \tag{3.2}$$

By inequalities (3.1) and (3.2), we have

$$\gamma(G \otimes_f H) = kn + \left\lceil \frac{n}{3} \right\rceil. \tag{3.3}$$

By equation (3.3), we have $\Gamma_{\rm S}(C_n, H) \geq \gamma(G \otimes_f H) = kn + \lceil n/3 \rceil$. This completes the proof of Claim 3.7. (□)

Claim 3.8. $\Gamma_{\rm S}(C_n, H) \leq kn + \left\lceil \frac{n}{3} \right\rceil$.

Proof. Let $f \colon V(G) \to V(H)$ be an arbitrary function. Let H_i be the ith copy of H corresponding to the vertex g_i of G for all $i \in [n]$. Let D be the dominating set of $G \otimes_f H$ constructed as follows. Let $x_i y_{i+1}$ be the connecting edge from H_i to H_{i+1} for all $i \in [n]$, where addition is taken modulo n. Thus, the vertex $x_i \in V(H_i)$ is adjacent to the vertex $y_{i+1} \in V(H_{i+1})$ in the graph $G \otimes_f H$, that is, $x_i = (g_i, f(g_{i+1}))$ and $y_{i+1} = (g_{i+1}, f(g_i))$. We note that possibly $x_i = y_i$. By property (b) in Definition 3.4, there exists a γ-set of H_i that contains both x_i and y_i. For $i \in [n]$, we define the sets $D_{i,1}$, $D_{i,2}$, and $D_{i,3}$ as follows. Let $D_{i,1}$ be a γ-set of $H_i - x_i$. Let $D_{i,2}$ be a γ-set of H_i that contains both x_i and y_i. Let $D_{i,3}$ be a γ-set of $H_i - y_i$. We note that $|D_{i,1}| = |D_{i,3}| = k$ and $|D_{i,2}| = k+1$. For $i \in [n]$, we define the set D_i as follows.

$$D_i = \begin{cases} D_{i,1}; & i \equiv 1 \pmod 3 \text{ and } i \neq n, \\ D_{i,2}; & i \equiv 2 \pmod 3 \text{ or } i \equiv 1 \pmod 3 \text{ and } i = n, \\ D_{i,3}; & i \equiv 0 \pmod 3. \end{cases}$$

For example, the set D_1 dominates all vertices of $H_1 - x_1$. The set D_2 contains the vertex y_2, which is adjacent to the vertex x_1 of H_1, and contains the vertex x_2, which is adjacent to the vertex y_3 of H_3, implying that D_2 dominates the vertex x_1 of H_1, all vertices of H_2, and the vertex y_3 of H_3. The set D_3 dominates all vertices of $H_3 - y_3$. Thus, $D_1 \cup D_2 \cup D_3$ dominates all vertices in $V(H_1) \cup V(H_2) \cup V(H_3)$ in the Sierpiński product $G \otimes_f H$. Moreover, $|D_1| + |D_2| + |D_3| = k + (k+1) + k = 3k+1$. More generally, the set $D_{3j-2} \cup D_{3j-1} \cup D_{3j}$ dominates all vertices in $V(H_{3j-2}) \cup V(H_{3j-1}) \cup V(H_{3j})$ in

the Sierpiński product $G \otimes_f H$ for all $j \in \{1, \ldots, \lfloor n/3 \rfloor\}$. Moreover, $|D_{3j-2}| + |D_{3j-1}| + |D_{3j}| = k + (k+1) + k = 3k+1$. If $n \equiv 1 \pmod 3$, then the set D_n is a γ-set of H_n, and in this case $|D_n| = k+1$. If $n \equiv 2 \pmod 3$, then the set $D_{n-1} \cup D_n$ dominates all vertices in $V(H_{n-1}) \cup V(H_n)$, and in this case $|D_{n-1}| + |D_n| = k + (k+1) = 2k+1$. The set

$$D = \bigcup_{i=1}^{n} D_i$$

is therefore a dominating set of $G \otimes_f H$, implying that

$$\gamma(G \otimes_f H) \le |D| = \sum_{i=1}^{n} |D_i| = kn + \left\lceil \frac{n}{3} \right\rceil .$$

This completes the proof of Claim 3.8. (□)

The proof of Theorem 3.6 follows as an immediate consequence of Claims 3.7 and 3.8. □

As a consequence of Proposition 3.5, we have the following special case of Theorem 3.6.

Corollary 3.9. *For $n \ge 3$ and $k \ge 1$,*

$$\Gamma_{\mathrm{S}}(C_n, C_{3k+1}) = kn + \left\lceil \frac{n}{3} \right\rceil .$$

We consider next the Sierpiński domination number of C_n and C_{3k+1}, and show that $\gamma_{\mathrm{S}}(C_n, C_{3k+1}) = kn$ if $n \equiv 0 \pmod 4$ and $\gamma_{\mathrm{S}}(C_n, C_{3k+1}) \in \{kn, kn+1\}$, otherwise.

Theorem 3.10. *For $n \ge 3$ and $k \ge 1$,*

$$\gamma_{\mathrm{S}}(C_n, C_{3k+1}) \in \{kn, kn+1\}.$$

Moreover, if $n \equiv 0 \mod 4$, then $\gamma_{\mathrm{S}}(C_n, C_{3k+1}) = kn$.

Proof. For $n \ge 3$ and $k \ge 1$, let $G = C_n$ and let $H = C_{3k+1}$. Let G be the cycle given by $g_1 g_2 \ldots g_n g_1$. We adopt our notation employed in our earlier proofs. For notational convenience, we let $V(H) = \{1, 2, \ldots, 3k+1\}$ where vertices i and $i+1$ are consecutive on the cycle H for all $i \in [3k+1]$ (and where addition is taken modulo $3k+1$, and so vertex 1 and vertex $3k+1$ are adjacent).

As before, we denote the copy $g_i H$ of H corresponding to the vertex g_i simply by H_i for each $i \in [n]$. Thus, $H_i = C_{3k+1}$ is the cycle $(g_i, 1), (g_i, 2), \ldots, (g_i, 3k+1), (g_i, 1)$ for all $i \in [n]$. Recall that we denote the connecting edge from H_i to H_{i+1} by $x_i y_{i+1}$ for all $i \in [n]$, where $x_i \in V(H_i)$, $y_{i+1} \in V(H_{i+1})$, and addition is taken modulo n. Thus, $y_i = (g_i, f(g_{i-1}))$ and $x_i = (g_i, f(g_{i+1}))$ for all $i \in [n]$.

By Proposition 3.5, the graph H belongs to the class $\mathcal{H}_k$. Thus, $\gamma(H) = k+1$ and $\gamma(H - v) = k$ for every vertex $v \in V(H)$. Furthermore, if $x, y \in V(H)$ where $x = y$ is allowed, then there exists a γ-set of H that contains x and y.

By the elementary lower bound on the Sierpiński domination number given in Theorem 2.1, $\gamma_{\mathrm{S}}(G, H) \ge n(G)\gamma(H) - m(G) = kn$, noting that here $n(G) = m(G) = n$ and $\gamma(H) = k+1$. It follows that $\gamma_{\mathrm{S}}(C_n, H) \ge kn$.

To complete the proof we are going to prove that

$$\gamma_{\rm S}(C_n, H) \le kn + \left\lceil \frac{n}{4} \right\rceil - \left\lfloor \frac{n}{4} \right\rfloor .$$

Let $f\colon V(G) \to V(H)$ be the function defined by

$$f(g_i) = \begin{cases} 1; & i \bmod 4 \in \{1,2\}, \\ 3; & \text{otherwise.} \end{cases}$$

for all $i \in [n]$ where addition is taken modulo n. Adopting our earlier notation, recall that $y_i = (g_i, f(g_{i-1}))$ and $x_i = (g_i, f(g_{i+1}))$ for all $i \in [n]$. Let $n = 4\ell + j$ where $j \in [3]_0 = \{0,1,2,3\}$. We note that $f(g_{4i-3}) = f(g_{4i-2}) = 1$ and $f(g_{4i-1}) = f(g_{4i}) = 3$ for all $i \in [\ell]$. Let D_i be the unique γ-set of $H_i - y_i \cong P_{3k}$ which consists of all vertices at distance 2 modulo 3 from y_i in the graph H_i for all $i \in [n]$, and let

$$D = \bigcup_{i=1}^{n} D_i.$$

We note that $|D_i| = k$ for all $i \in [n]$, and so $|D| = kn$. For all $i \in \{2,3,\ldots,\ell-1\}$, the following four properties hold.

(P1) $y_{4i-3} = (g_{4i-3}, 3)$ and $x_{4i-3} = (g_{4i-3}, 1)$.

(P2) $y_{4i-2} = (g_{4i-2}, 1)$ and $x_{4i-2} = (g_{4i-2}, 3)$.

(P3) $y_{4i-1} = (g_{4i-1}, 1)$ and $x_{4i-1} = (g_{4i-1}, 3)$.

(P4) $y_{4i} = (g_{4i}, 3)$ and $x_{4i} = (g_{4i}, 1)$.

Hence for all $i \in \{2,3,\ldots,\ell-1\}$, the vertices x_i and y_i are at distance 2 in H_i, implying that $x_i \in D_i$. We consider four cases to determine which properties hold for the boundary conditions (that is for $i \in \{1,\ell\}$) and finally to set the upper bound on the domination number in each case.

Case 1. $n \equiv 0 \,(\mathrm{mod}\, 4)$*, that is* $n = 4\ell$.
In this case, properties (P1) and (P4) also hold for $i = 1$ and $i = \ell$, respectively. Thus, $y_1 = (g_1, 3)$ and $x_1 = (g_1, 1)$, and $y_{4\ell} = (g_{4\ell}, 3)$ and $x_{4\ell} = (g_{4\ell}, 1)$, implying that $x_1, x_{4\ell} \in D$. The set D is therefore a dominating set of $G \otimes_f H$, and so $\gamma(G \otimes_f H) \le |D| = kn = kn + \lceil n/4 \rceil - \lfloor n/4 \rfloor$.

Case 2. $n \equiv 1 \,(\mathrm{mod}\, 4)$*, that is* $n = 4\ell + 1$.
In this case, $y_1 = x_1 = (g_1, 1)$, and $y_{4\ell+1} = (g_{4\ell+1}, 3)$ and $x_{4\ell+1} = (g_{4\ell+1}, 1)$. In particular, property (P4) also holds for $i = \ell$, and so $x_{4\ell+1} \in D$. The set $D \cup \{x_1\}$ is therefore a dominating set of $G \otimes_f H$, and so $\gamma(G \otimes_f H) \le |D| + 1 = kn + 1 = kn + \lceil n/4 \rceil - \lfloor n/4 \rfloor$.

Case 3. $n \equiv 2 \,(\mathrm{mod}\, 4)$*, that is* $n = 4\ell + 2$.
In this case, $y_1 = x_1 = (g_1, 1)$, and $y_{4\ell+2} = x_{4\ell+2} = (g_{4\ell+2}, 1)$. We note that neither x_1 nor $x_{4\ell+2}$ belong to the set D. The set $D \cup \{x_1\}$ is a dominating set of $G \otimes_f H$, and so $\gamma(G \otimes_f H) \le |D| + 1 = kn + 1 = kn + \lceil n/4 \rceil - \lfloor n/4 \rfloor$.

Case 4. $n \equiv 3 \,(\mathrm{mod}\, 4)$*, that is* $n = 4\ell + 3$.

In this case, $y_1 = (g_1, 3)$ and $x_1 = (g_1, 1)$, and $y_{4\ell+3} = x_{4\ell+3} = (g_{4\ell+3}, 1)$. In particular, property (P1) also holds for $i = 1$, and so $x_1 \in D$. However, $x_{4\ell+3} \notin D$. The set $D \cup \{x_{4\ell+3}\}$ is therefore a dominating set of $G \otimes_f H$, and so $\gamma(G \otimes_f H) \le |D| + 1 = kn + 1 = kn + \lceil n/4 \rceil - \lfloor n/4 \rfloor$.

In all four cases, $\gamma(G \otimes_f H) \le kn + \lceil n/4 \rceil - \lfloor n/4 \rfloor$. □

3.2 The cycle C_n and cycles C_{3k+2}

In this section, we determine the Sierpiński domination number $\gamma_S(C_n, C_{3k+2})$ and the upper Sierpiński domination number $\Gamma_S(C_n, C_{3k+2})$.

Theorem 3.11. *For $n \ge 3$ and $k \ge 1$, we have $\Gamma_S(C_n, C_{3k+2}) = (k+1)n$.*

Proof. For $n \ge 3$ and $k \ge 1$, let $G \cong C_n$ and let $H \cong C_{3k+2}$. Let $f\colon V(G) \to V(H)$ be a constant function, that is, we select $h \in V(H)$ and for every vertex $g \in V(G)$, we set $f(g) = h$. For each vertex $g \in V(G)$, let H_g denote the copy of H associated with the vertex g. Let D be a dominating set of $G \otimes_f H$, and let $D_g = D \cap V(H_g)$, and so D_g is the restriction of D to the copy H_g of H. If the vertex (g, h) does not belong to D_g, then D_g dominates all vertices on the path $H_g - (g, h) \cong P_{3k+1}$, and so $|D_g| \ge \gamma(P_{3k+1}) = k + 1$. If the vertex (g, h) does belong to D_g, then D_g dominates all vertices on the cycle $H_g \cong C_{3k+2}$, and so $|D_g| \ge \gamma(C_{3k+2}) = k + 1$. In both cases, $|D_g| \ge k + 1$. Therefore,

$$\gamma(G \otimes_f H) = |D| = \sum_{g \in V(G)} |D_g| \ge (k+1)n,$$

implying that $\Gamma_S(C_n, C_{3k+2}) \ge (k+1)n$. By the upper bound in Theorem 2.1, we have $\Gamma_S(G, H) \le n(G)\gamma(H) = (k+1)n$, noting that in this case $\gamma(H) = \gamma(C_{3k+2}) = k + 1$. Consequently, $\Gamma_S(C_n, C_{3k+2}) = (k+1)n$. □

Theorem 3.12. *For $n \ge 3$ and $k \ge 1$,*

$$\gamma_S(C_n, C_{3k+2}) \in \{kn + \left\lfloor \frac{n}{2} \right\rfloor, kn + \left\lfloor \frac{n}{2} \right\rfloor + 1\}.$$

Moreover, if $n \equiv 0 \mod 4$, then $\gamma_S(C_n, C_{3k+2}) = kn + \left\lfloor \frac{n}{2} \right\rfloor$.

Proof. For $n \ge 3$ and $k \ge 1$, let $G \cong C_n$ and let $H \cong C_{3k+2}$. We adopt our notation employed in our earlier proofs. Thus, the cycle G is given by $g_1 g_2 \ldots g_n g_1$, and $V(H) = \{1, 2, \ldots, 3k+2\}$ where vertices i and $i+1$ are consecutive on the cycle H for all $i \in [3k+2]$ (and where addition is taken modulo $3k+2$, and so vertex 1 and vertex $3k+2$ are adjacent). As before, we denote the copy $g_i H$ of H corresponding to the vertex g_i simply by H_i for each $i \in [n]$. Thus, $H_i = C_{3k+2}$ is the cycle $(g_i, 1), (g_i, 2), \ldots, (g_i, 3k+2), (g_i, 1)$ for all $i \in [n]$.

We adopt our notation from the proof of Theorem 3.6. Thus, we denote the connecting edge from H_i to H_{i+1} by $x_i y_{i+1}$ for all $i \in [n]$, where $x_i \in V(H_i)$, $y_{i+1} \in V(H_{i+1})$, and addition is taken modulo n. Thus, $y_i = (g_i, f(g_{i-1}))$ and $x_i = (g_i, f(g_{i+1}))$ for all $i \in [n]$.

We proceed further with two claims. The first claim establishes a lower bound on $\gamma_S(G, H)$, and the second claim an upper bound on $\gamma_S(G, H)$. Combining these two bounds yields the desired result in the statement of the theorem.

Claim 3.13. $\gamma_{\rm S}(C_n, H) \ge kn + \left\lfloor \frac{n}{2} \right\rfloor$.

Proof. Let $f\colon V(G) \to V(H)$ be an arbitrary function. We show that

$$\gamma(G \otimes_f H) \ge kn + \left\lfloor \frac{n}{2} \right\rfloor . \tag{3.4}$$

Let D be a γ-set of $G \otimes_f H$ constructed, and let $D_i = D \cap V(H_i)$ for $i \in [n]$. If the vertex x_i is not dominated by D_i, then either $x_i \ne y_i$, in which case x_i is dominated by the vertex $y_{i+1} \in D$, or $x_i = y_i$, in which case x_i is dominated by the vertex $x_{i-1} \in D$ or the vertex $y_{i+1} \in D$. Analogously, if the vertex y_i is not dominated by D_i, then either $x_i \ne y_i$, in which case y_i is dominated by the vertex $x_{i-1} \in D$, or $x_i = y_i$, in which case y_i is dominated by the vertex $x_{i-1} \in D$ or the vertex $y_{i+1} \in D$. If a vertex is not dominated by D_i, then such a vertex is x_i or y_i, and we say that such a vertex is dominated from outside H_i.

Similarly as before, we proceed with a claim that delivers properties of sets D_i leading to the desired lower bound on the Sierpiński domination number.

Claim 3.14. *The following properties hold in the graph H_i.*

(a) *If $d(x_i, y_i) \equiv 1 \pmod 3$, then $|D_i| = k$. Further, both x_i and y_i are dominated from outside H_i.*

(b) *If $d(x_i, y_i) \not\equiv 1 \pmod 3$, then $|D_i| = k + 1$.*

Proof. Suppose that D_i contains a vertex w_i that dominates x_i. Possibly, $w_i = x_i$. In order to dominate the $3(k-1)+2$ vertices in H_i not dominated by w_i, at least k additional vertices are needed even if the vertex y_i is dominated outside the cycle H_i. Thus in this case, $|D_i| \ge k+1$, implying by the minimality of the set D that $|D_i| = k+1$. Analogously, if D_i contains a vertex that dominates y_i, then $|D_i| = k + 1$. Hence, if x_i or y_i (or both x_i and y_i) are dominated by D_i, then $|D_i| = k + 1$.

Suppose that neither x_i nor y_i is dominated by D_i, implying that both x_i and y_i are dominated from outside the cycle H_i. Thus, D_i is a dominating set of $H_i' = H_i - x_i - y_i$. If $x_i = y_i$, then $H_i' = P_{3k+1}$, and by the minimality of D we have $|D_i| = \gamma(P_{3k+1}) = k+1$. Hence, we may assume that $x_i \ne y_i$. If x_i and y_i are adjacent, then $H_i' = P_{3k}$, and by the minimality of D we have $|D_i| = \gamma(P_{3k}) = k$. Suppose that x_i and y_i are not adjacent, and so H' is the disjoint union of two paths P_{k_1} and P_{k_2}, where $k_1 + k_2 = 3k$. If $k_1 = 3j_1 + 1$ and $k_2 = 3j_2 + 2$ (or if $k_1 = 3j_1 + 2$ and $k_2 = 3j_2 + 1$) for some integers j_i and j_2 where $j_1 + j_2 = k - 1$, then $|D_i| = \lceil k_1/3 \rceil + \lceil k_2/3 \rceil = (j_1 + 1) + (j_2 + 1) = k + 1$. If $k_1 = 3j_1$ and $k_2 = 3j_2$ where $j_1 + j_2 = k$, then $|D_i| = \lceil k_1/3 \rceil + \lceil k_2/3 \rceil = j_1 + j_2 = k$. Hence if neither x_i nor y_i is dominated by D_i, then either $d(x_i, y_i) \equiv 1 \pmod 3$, in which case $|D_i| = k$, or $d(x_i, y_i) \not\equiv 1 \pmod 3$, in which case $|D_i| = k+1$. This proves properties (a) and (b) of the claim. (□)

By Claim 3.14, if $|D_i| = k$ for some $i \in [n]$, then $|D_{i-1}| = |D_{i+1}| = k + 1$ where addition is taken modulo n. Furthermore in this case when $|D_i| = k$, the vertices x_i and y_i are distinct and are both dominated from outside H_i, implying that $y_{i+1} \in D_{i+1}$ and $x_{i-1} \in D_{i-1}$. This implies that if n is even, then $|D| \ge kn + n/2$, and if n is odd, then $|D| \ge kn + (n + 1)/2$. This proves inequality (3.4).

Claim 3.15. $\gamma_{\rm S}(C_n, H) \le kn + \left\lfloor \frac{n}{2} \right\rfloor + \left\lceil \frac{n}{4} \right\rceil - \left\lfloor \frac{n}{4} \right\rfloor$.

Proof. Let $f\colon V(G) \to V(H)$ be the function defined by

$$f(g_i) = \begin{cases} 1; & i \equiv 1 \,(\mathrm{mod}\ 4), \\ 2; & i \equiv 2 \,(\mathrm{mod}\ 4), \\ 3; & \text{otherwise.} \end{cases}$$

for all $i \in [n]$ where addition is taken modulo n. Adopting our earlier notation, recall that $y_i = (g_i, f(g_{i-1}))$ and $x_i = (g_i, f(g_{i+1}))$ for all $i \in [n]$. Let $n = 4\ell + j$ where $j \in [3]_0 = \{0, 1, 2, 3\}$. We note that $f(g_{4i-3}) = 1$, $f(g_{4i-2}) = 2$, and $f(g_{4i-1}) = f(g_{4i}) = 3$ for all $i \in [\ell]$.

Case 1. $n \equiv 0 \,(\mathrm{mod}\ 4)$.
Thus, $n = 4\ell$. We note that $y_{4i-3} = (g_{4i-3}, 3)$ and $x_{4i-3} = (g_{4i-3}, 2)$ for all $i \in [n]$, and so in the graph H_{4i-3} the vertices x_{4i-3} and y_{4i-3} are at distance 1. Moreover, $y_{4i-1} = (g_{4i-1}, 2)$ and $x_{4i-1} = (g_{4i-1}, 3)$ for all $i \in [n]$, and so in the graph H_{4i-1} the vertices x_{4i-1} and y_{4i-1} are at distance 1. This implies that $H_{4i-j} - \{x_{4i-j}, y_{4i-j}\} \cong C_{3k}$ for $j \in \{1, 3\}$. Let D_{4i-j} be a γ-set of $H_{4i-j} - \{x_{4i-j}, y_{4i-j}\}$ for $j \in \{1, 3\}$, and so $|D_{4i-j}| = k$.

We also note that $y_{4i-2} = (g_{4i-2}, 1)$ and $x_{4i-2} = (g_{4i-2}, 3)$ for all $i \in [n]$, and so in the graph H_{4i-2} the vertices x_{4i-2} and y_{4i-2} are at distance 2. Moreover, $y_{4i} = (g_{4i}, 3)$ and $x_{4i} = (g_{4i}, 1)$ for all $i \in [n]$, and so in the graph H_{4i} the vertices x_{4i} and y_{4i} are at distance 2. This implies that $H_{4i-j} - N[\{x_{4i-j}, y_{4i-j}\}] \cong C_{3(k-1)}$ for $j \in \{0, 2\}$. Let D_{4i-j} be a γ-set of H_{4i-j} that contains both vertices x_{4i-j} and y_{4i-j} for $j \in \{0, 2\}$, and so $|D_{4i-j}| = k + 1$. The set

$$D = \bigcup_{i=1}^{4\ell} D_i$$

is a dominating set of $G \otimes_f H$, and so $\gamma(G \otimes_f H) \le |D| = 4k\ell + 2\ell = kn + n/2$.

Case 2. $n \equiv 2 \,(\mathrm{mod}\ 4)$.
Thus, $n = 4\ell + 2$ and in this case, $f(g_{4\ell+1}) = 1$ and $f(g_{4\ell+2}) = 2$. We note that in the graph $H_{4\ell+1}$, the vertices $x_{4\ell+1}$ and $y_{4\ell+1}$ are at distance 1 and in the graph $H_{4\ell+2}$ we have $x_{4\ell+2} = y_{4\ell+2}$. For $i \in [4\ell]$, we define the set D_i exactly as in the previous case. Further, let $D_{4\ell+1}$ be a γ-set of $H_{4\ell+1} - \{x_{4\ell+1}, y_{4\ell+1}\} \cong C_{3k}$, and let $D_{4\ell+2}$ be a γ-set of $H_{4\ell+2}$ containing $x_{4\ell+2}$. We note that $|D_{4\ell+1}| = k$ and $|D_{4\ell+2}| = k + 1$. The set

$$D = \bigcup_{i=1}^{4\ell+2} D_i$$

is a dominating set of $G \otimes_f H$, and so $\gamma(G \otimes_f H) \le |D| = 4k\ell + 2k + 2\ell + 1 = kn + n/2$.

Case 3. $n \equiv 1 \,(\mathrm{mod}\ 4)$.
Thus, $n = 4\ell + 1$, and in this case, $f(g_{4\ell+1}) = 1$. Thus, $y_{4\ell+1} = (g_{4\ell+1}, 3)$ and $x_{4\ell+1} = (g_{4\ell+1}, 1)$, and so in the graph $H_{4\ell+1}$, the vertices $x_{4\ell+1}$ and $y_{4\ell+1}$ are at distance 2. For $i \in [4\ell]$, we define the set D_i exactly as in the previous cases. Further, let $D_{4\ell+1}$ be a γ-set of $H_{4\ell+1}$ that contains the vertex $x_{4\ell+1}$. We note that $|D_{4\ell+1}| = k + 1$. The set

$$D = \bigcup_{i=1}^{4\ell+1} D_i$$

is a dominating set of $G \otimes_f H$, and so $\gamma(G \otimes_f H) \le |D| = 4k\ell+k+2\ell+1 = kn+(n+1)/2$.

Case 4. $n \equiv 3 \pmod 4$.

Thus, $n = 4\ell + 3$, and in this case, $f(g_{4\ell+1}) = 1$, $f(g_{4\ell+2}) = 2$, and $f(g_{4\ell+3}) = 3$. In particular, $y_{4\ell+3} = (g_{4\ell+3}, 2)$ and $x_{4\ell+3} = (g_{4\ell+3}, 1)$, and so in the graph $H_{4\ell+3}$, the vertices $x_{4\ell+3}$ and $y_{4\ell+3}$ are at distance 1. For $i \in [4\ell + 2]$, we define the set D_i exactly as in Case 2. Further, let $D_{4\ell+3}$ be a γ-set of $H_{4\ell+3}$ containing the vertex $x_{4\ell+3}$. We note that $|D_{4\ell+3}| = k + 1$. The set

$$D = \bigcup_{i=1}^{4\ell+3} D_i$$

is a dominating set of $G \otimes_f H$, and so $\gamma(G \otimes_f H) \le |D| = 4k\ell + 3k + 2\ell + 2 = kn + (n + 1)/2$. The desired result of the claim now follows from the four cases above. (□)

The proof of Theorem 3.12 follows as an immediate consequence of Claim 3.13 and Claim 3.15. □

3.3 The cycle C_n and cycles C_{3k}

In this section, we determine the Sierpiński domination number $\gamma_S(C_n, C_{3k})$ and the upper Sierpiński domination number $\Gamma_S(C_n, C_{3k})$.

Theorem 3.16. *For $n \ge 3$ and $k \ge 1$,*

$$\gamma_{\rm S}(C_n, C_{3k}) = \Gamma_{\rm S}(C_n, C_{3k}) = kn.$$

Proof. We adopt our notation from the earlier sections. Let $G \cong C_n$ be the cycle $g_1g_2 \ldots g_ng_1$, and let H_i be the ith copy of C_{3k} corresponding to the vertices g_i of G for $i \in [n]$. As before, we denote the connecting edge from H_i to H_{i+1} by x_iy_{i+1} for all $i \in [n]$.

Let $f\colon V(G) \to V(H)$ be an arbitrary function. Let D be a γ-set of $G \otimes_f H$, and let $D_i = D \cap V(H_i)$ for $i \in [n]$. We show that $|D_i| = k$ for all $i \in [n]$. If both vertices x_i and y_i are dominated by D_i, then D_i is a γ-set of $H_i \cong C_{3k}$, and so $|D_i| = k$. If exactly one of x_i and y_i is dominated by D_i, say x_i, then by the minimality of the set D, the set D_i is a γ-set of $H_i - y_i \cong P_{3k-1}$, and so $|D_i| = k$. Hence, we may assume that neither x_i nor y_i is dominated by D_i, for otherwise, $|D_i| = k$ and the desired bound follows.

With our assumption that neither x_i nor y_i is dominated by D_i, the set D_i is a γ-set of $H_i' = H_i - \{x_i, y_i\}$. If $x_i = y_i$, then $H_i' = P_{3k-1}$, and by the minimality of D we have $|D_i| = \gamma(P_{3k-1}) = k$. Hence, we may assume that $x_i \ne y_i$. If x_i and y_i are adjacent, then $H_i' = P_{3k-2}$, and by the minimality of D we have $|D_i| = \gamma(P_{3k-2}) = k$. Suppose that x_i and y_i are not adjacent, and so H' is the disjoint union of two paths P_{k_1} and P_{k_2}, where $k_1 + k_2 = 3k - 2$. If $k_1 = 3j_1 + 1$ and $k_2 = 3j_2$ for some integers j_i and j_2 where $j_1+j_2 = k-1$, then $|D_i| = \lceil k_1/3\rceil+\lceil k_2/3\rceil = (j_1+1)+j_2 = k$. Analogously, if $k_1 = 3j_1$ and $k_2 = 3j_2 + 1$, then $|D_i| = k$. If $k_1 = 3j_1 + 2$ and $k_2 = 3j_2 + 2$ for some integers j_i and j_2 where $j_1 + j_2 = k - 2$, then $|D_i| = \lceil k_1/3\rceil + \lceil k_2/3\rceil = (j_1 + 1) + (j_2 + 1) = k$. In all cases, $|D_i| = k$, implying that

$$\gamma(G \otimes_f H) = |D| = \sum_{i=1}^{n} |D_i| = kn.$$

Since $f\colon V(G) \to V(H)$ was chosen as an arbitrary function, and D as an arbitrary γ-set of $G \otimes_f H$, we deduce that $\gamma_{\rm S}(C_n, C_{3k}) = \Gamma_{\rm S}(C_n, C_{3k}) = \gamma(G \otimes_f H) = kn$. □

4 Concluding remarks

It seems to us that in the vast majority of cases where the lower Sierpiński domination number of the Sierpiński product of two cycles is specified to two values exactly, the larger of the two is the correct value. However, the following example, which surprised us, demonstrates that there are also cases where the exact value is the smaller of the two possible values.

Let $G \cong C_{18}$ with $V(G) = [18]$ and let $H \cong C_7$ with $V(H) = [7]$ and let the function $f\colon V(G) \to V(H)$ be defined as follows:

$$\begin{aligned} f(1) = f(4) = f(5) = f(18) &= 4,\\ f(2) = f(3) = f(6) = f(7) &= 2,\\ f(8) = f(9) &= 7,\\ f(10) = f(11) &= 5,\\ f(12) = f(13) &= 3,\\ f(14) = f(15) &= 1,\\ f(16) = f(17) &= 6. \end{aligned}$$

Then Theorem 3.2 asserts that $\gamma(G \otimes_f H) \in \{36, 37\}$ and it is straightforward to check that the exact value is $\gamma(G \otimes_f H) = 36$.

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