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ORIGINAL PAPER

Unfriendly Partition Conjecture Holds for Line Graphs

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Abstract

A majority edge-coloring of a graph without pendant edges is a coloring of its edges such that, for every vertex v and every color α, there are at most as many edges incident to v colored with α as with all other colors. We extend some known results for finite graphs to infinite graphs, also in the list setting. In particular, we prove that every infinite graph without pendant edges has a majority edge-coloring from lists of size 4. Another interesting result states that every infinite graph without vertices of finite odd degrees admits a majority edge-coloring from lists of size 2. As a consequence of our results, we prove that line graphs of any cardinality admit majority vertex-colorings from lists of size 2, thus confirming the Unfriendly Partition Conjecture for line graphs.

Keywords Infinite graphs · Unfriendly partition conjecture · Majority edge-colorings · Colorings from lists

1 Introduction

For a graph G, an edge-coloring $c : E(G) \to [k]$ is a *majority k-edge-coloring* if, for every vertex v of G and every color $\alpha \in [k]$, the cardinality of edges incident to v colored with α is not greater than the cardinality of edges incident to v colored with all other colors. That is, if the degree of v is finite, then at most half of the incident edges has the same color. Of course, graphs with pendant edges do not admit such a coloring. As usual, the least number of colors in a majority edge-coloring is of interest. The concept of majority edge-colorings was recently introduced in [3], and was motivated by *majority vertex-colorings*, i.e. colorings of vertices of a graph G such that each vertex v has at most as many neighbors with the color of v as with other colors. They were studied already in 1966 by Lovász in [10] for finite graphs, then extended to infinite graphs (e.g. in [2, 5, 12]), and more recently to digraphs in

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[9]. This problem for infinite graphs, including the well-known Unfriendly Partition Conjecture, is presented in more detail in Sect. 4.

In [3], the following results were proved for finite graphs.

Theorem 1 ([3]) *Every finite graph of minimum degree at least* 2 *admits a majority* 4*-edge-coloring.*

Theorem 2 ([3]) *A finite connected graph G admits a majority* 2*-edge-coloring if and only if all vertices of G have even degrees and the size of G is even.*

In this paper, we extend these results to infinite graphs whose orders are arbitrary cardinal numbers, considering also the list version of edge-colorings. Section 2 contains some auxiliary results for Sect. 3, where we prove the list version of Theorem 2 for finite and infinite graphs. Next in Sect. 4, we confirm the well-known Unfriendly Partition Conjecture for line graphs, which we derive from the results of the previous section. Actually, we prove there a stronger result that line graphs of any infinite cardinality admit majority vertex-colorings from lists of size 2. In Sect. 5, we show that Theorem 1 holds for infinite graphs, also in the list setting, and we formulate a conjecture.

We use standard terminology and notation of graph theory (cf. [6]). A *double ray* in an infinite graph is a two-sided infinite path, i.e. an infinite connected 2-regular graph. A *ray* is a one-sided infinite path, and a unique vertex of degree one in a ray is its *startvertex*.

Following Schmidt [11], we define the *rank* of a graph without rays as follows. We assign rank 0 to all finite graphs. Next, we assign the smallest ordinal ρ as the rank of any infinite graph G that does not already have rank less than ρ and contains a finite set S of vertices such that each component of $G - S$ has some rank less than ρ. We shall make use of the following facts (cf. [5, 7, 11]). For every rayless graph G with rank $\rho(G) > 0$, there exists a unique minimal set S that works, called the *core* of G. If S is the core of G, then the number of components of $G - S$ is infinite, and each vertex of S has infinite degree in G. Moreover, if $C_1, \ldots, C_n$ is a finite set of components of $G - S$, then the subgraph of G induced by $S \cup \bigcup_{i=1}^{n} V(C_i)$ has rank smaller than the rank of G.

For convenience, we introduce some more notation and terminology. If W is an oriented walk of a graph and e is an edge in W, then we denote the subsequent edge by e^+.

Assume that an edge-coloring of a graph is given. We say that a vertex v is *majority colored* if there is no color α such that v is incident with more edges of color α than with the remaining edges. Note that if the degree of v is infinite, then v is majority colored if and only if for every color α, the cardinality of the set of edges incident with v that are not colored with α equals the degree of v. We say that a vertex v is *almost majority colored* if there exists a color α, called the *o*verwhelming color, such that of the edges incident with v, strictly more have color α than do not, but at most two more have color α than do not. Notice that if v is an almost majority colored vertex, then the degree of v is finite, and the number of its incident edges with overwhelming color exceeds the number of remaining incident edges by 1 or 2, depending on the parity of the degree of v. If all vertices of a graph G are majority colored, then the graph G is majority colored.

2 Auxiliary Results

We begin with three lemmas which cover all three cases related to list majority edge-colorings of finite graphs from lists of size 2.

Lemma 3 *Let G be a connected finite graph of even size with all vertices of even degree, and $\mathcal{L}$ be a set of lists for edges of G, each of size 2. Then there exists a majority edge-coloring of G from the set $\mathcal{L}$ of lists.*

Proof From the assumptions, it follows that G has an Euler tour W. We fix an orientation of W. First, assume that all elements of $\mathcal{L}$ are the same. We color the edges in W alternately. Clearly, this yields a majority coloring.

If not every list in $\mathcal{L}$ is the same, then we choose two consecutive edges e and e^+ in W that have different lists. We color e^+ with a color that does not belong to $L(e)$. Next, we color each consecutive edge in W starting from the edge following e^+ in such a way that, for every edge f, the next edge f^+ has a different color than f has. The choice of e guarantees that each two consecutive edges in W have different colors. Therefore, the obtained coloring is a majority coloring. □

Lemma 4 *Let G be a connected finite graph of odd size with all vertices of even degree, and $\mathcal{L}$ be a set of lists for edges of G, each of size 2. Let b be an arbitrary vertex of G. Then there exist two edge-colorings of G from the set $\mathcal{L}$ of lists such that each vertex different from b is majority colored, and either b is majority colored in at least one of them, or b is almost majority colored in both colorings with two different overwhelming colors.*

Proof Again, G has an Euler tour W. We color the edges in W starting from an edge e_1 incident to b in such a way that if any edge e has a color α, then e^+ has a different color, except possibly the last edge of W and the edge e_1, which are incident to b. Now, each vertex, possibly except b, is majority colored, and b may have exactly two incident edges of one color more than the edges of all other colors.

For e_1, we can choose one of two colors in $L(e_1)$, hence we get two different edge-colorings of G as needed. □

Lemma 5 *Let G be a connected finite graph with some vertices of odd degrees, and let $\mathcal{L}$ be a set of lists for edges of G, each of size 2. Then there exists an edge-coloring from the set $\mathcal{L}$ of lists such that each vertex is majority colored, except possibly some vertices of odd degrees which are almost majority colored.*

Proof Let X be the set of all vertices of odd degrees in G. By the hand-shaking lemma, the number of vertices in X is even. Hence, we can partition X into pairs. For each such pair $\{x, y\}$, we join x and y by an extra path (with internal vertices outside of G) of length at most three to obtain a graph G' of even size and with all vertices of even degrees. We choose arbitrary lists of size 2 for the extra edges. Next, we color G' with a majority edge-coloring from the extended list set, which exists by Lemma 3. Removal of the additional paths yields a desired edge-coloring of the graph G. Indeed, the only vertices of G with deleted incident edges belong to X, and each such vertex is majority colored or almost majority colored in G. □

In the proof of our main result, we make use of the following fact. Its proof uses a standard argument in infinary combinatorics, but we include it for completeness.

Lemma 6 *Let G be a graph, and let $\mathcal{H}$ be any family of graphs. Then G contains a maximal subgraph F decomposable into elements of $\mathcal{H}$, in the sense that the remaining subgraph $G - E(F)$ contains no subgraph isomorphic to a graph of $\mathcal{H}$.*

Proof The idea of the proof is to show that there is a maximal decomposition which gives us a subgraph with the desired property rather than straightforwardly proving that such a subgraph exists. First, we define a family $\mathcal{F}$ of subsets of the set $2^{E(G)}$ of all subsets of $E(G)$ which are decompositions of some subgraphs of G into unions of some pairwise edge-disjoint elements of $\mathcal{H}$. That is, for each $X \in \mathcal{F}$ every $A \in X$ induces a subgraph isomorphic to an element of $\mathcal{H}$, and any $A, B \in X$ are disjoint. We show that every non-empty chain in $(\mathcal{F}, \subseteq)$ has an upper bound. Let $\mathcal{C}$ be a non-empty chain in $\mathcal{F}$, and let $M = \bigcup \mathcal{C}$. For every element $C \in \mathcal{C}$, we have $C \subseteq M$, and M is a decomposition of a certain subgraph of G. Hence M majorizes C. Therefore, by Zorn's Lemma there exists a maximal element in $\mathcal{F}$. The obtained maximal decomposition gives a subgraph from the statement of the theorem. □

3 Lists of Size 2

Let us formulate the main result of this section.

Theorem 7 *A graph G of arbitrary order admits a majority edge-coloring from any set $\mathcal{L}$ of lists, each of size 2, if and only if no vertex of G has odd degree and G has no finite component of odd size.*

The "only if" part is obvious. For the "if" part, we prove the following slightly stronger result.

Theorem 8 *Let G be a graph, and $\mathcal{L}$ be a set of lists for the edges of G, each of size 2. Then either there exists an edge-coloring of G from the set $\mathcal{L}$ of lists in which every vertex of G which is not of odd degree is majority colored and each vertex of odd degree in G is either majority colored or almost majority colored, or G has a finite component of odd size with all vertices of even degree.*

Proof We may assume that G is connected; otherwise we can argue component-wise. Let us call an edge-coloring from the lists $\mathcal{L}$ *good* if it satisfies the claim, i.e. only vertices of odd degrees need not be majority colored, but then they have to be almost majority colored.

The case where G is finite follows from Lemma 3 and Lemma 5. Assume that G is infinite.

If G contains a double ray, then let F be a maximal subgraph of G decomposable into double rays, which exists by Lemma 6. Fix a decomposition $\mathcal{P}$ of F into double rays. We color each double ray $P \in \mathcal{P}$ from the lists in $\mathcal{L}$ in so that adjacent edges get distinct colors. Hence, every vertex of F is majority colored in F since no vertex has odd degree in F.

Now we consider every component H of $G - E(F)$. Clearly, each vertex $v \in V(F)$ belongs to exactly one component of $G - E(F)$ (which can be a trivial graph K_1 but then we have nothing to do because all edges incident to v in G are already colored). To color the edges of H, we distinguish three cases.

Case 1. Component H is a finite graph.
Let X be the set of vertices of odd degrees in H. If the set X is non-empty, then we apply Lemma 5 for a good coloring of H. If X is empty and the size of H is even, then H admits a majority coloring by Lemma 3.

Suppose now that X is empty but the size of H is odd. For each such component H, we select a vertex $b = b(H) \in V(H) \cap V(P)$, for some $P \in \mathcal{P}$. For $P \in \mathcal{P}$, let $B(P)$ denote the set of all vertices $b(H)$, for some component H, which belong to P. In view of Lemma 4, there exists a coloring from $\mathcal{L}$ of each such component H such that every vertex of H is majority colored, except possibly the vertex $b(H)$ which is almost majority colored and overwhelmed by a certain color. We now show how to make these vertices majority colored.

To make all vertices of a given double ray P majority or almost majority colored in G, we perform the following procedure of shifting the edge-coloring of P. We consecutively enumerate the vertices of P by integers, i.e. $V(P) = \{\ldots, v_{-1}, v_0, v_1, \ldots\}$. For $i = 0, 1, 2, \ldots$, we verify whether $v_i \in B(P)$ and v_i is not majority colored. Suppose that this is the case and v_i is overwhelmed by a certain color in H that is also a color of an edge e of P incident to v_i. Without loss of generality, we can assume that $e = v_i v_{i+1}$. Indeed, if $e = v_{i-1}v_i$, then, by Lemma 4, we can recolor the edges of H such that v_i is not overwhelmed by the color of the edge $v_{i-1}v_i$. Next, we put another admissible color from $L(e)$ on $e = v_i v_{i+1}$, and we consecutively recolor the edges $v_j v_{j+1}$ of P, for $j = i + 1, i + 2, \ldots$, so that the color of $v_j v_{j+1}$ is distinct from the color of the previous edge $v_{j-1}v_j$. Next, we examine in a similar way the vertices v_{-i} for $i = 1, 2, \ldots$. If $v_{-i} \in B(P)$, then we may analogously assume that v_i is overwhelmed by a color of the edge $v_{-i-1}v_{-i}$. Then we consecutively recolor the edges $v_{-j-1}v_{-j}$, for $j = i + 1, i + 2, \ldots$, so that the color of $v_{-j-1}v_{-j}$ is distinct from that of the edge $v_{-j}v_{-j+1}$.

Thus, for any double ray $P \in \mathcal{P}$, we obtain a good edge-coloring of the subgraph induced by the edges of P and all finite components sharing a vertex with P.

Case 2. Component H is an infinite rayless graph.
It suffices to prove that every rayless graph, finite or infinite, admits a good edge-coloring. To this end, we proceed by transfinite induction on the rank of H. If the rank of H is 0, then H is a finite graph, which we settled in Case 1. Assume then that the rank of H is greater than 0. Let S be the core of H. Then each vertex of S has infinite degree in H. Let $\mathcal{C}$ be the set of all components of $H - S$. For $C \in \mathcal{C}$, denote $C' = H[S \cup V(C)] - E(H[S])$, the subgraph consisting of edges of the component C and edges between C and S.

Denote by S_1 the set of all vertices $s \in S$ that are joined to $d_H(s)$ components from $\mathcal{C}$ by exactly one edge. For each $s \in S_1$, select a set $\mathcal{C}(s)$ of cardinality $d_H(s)$ of such components. As the set S_1 is finite, this can be done in such a way that the sets $\mathcal{C}(s)$, $s \in S_1$, are pairwise disjoint. Next, partition each set $\mathcal{C}(s)$ into pairs (C_1, C_2). For each such pair, take a good edge-coloring of the subgraph $C_1' \cup C_2'$, in which s has degree 2. Such a good coloring exists by the induction hypothesis. It is easy to see that each vertex of S_1 is already majority colored no matter how the remaining edges incident to it will be colored.

Now, consider the components of $\mathcal{C} \setminus \bigcup_{s \in S_1} \mathcal{C}(s)$. For each such component C take a good coloring of C', which exists by the induction hypothesis, and color $H[S]$ arbitrarily. Hence, all edges in H are colored.

Observe that each vertex v of any component $C \in \mathcal{C}$ is majority colored, unless v is of odd degree in H but then v is almost majority colored. This is because $N_H(v) \subset V(C')$. Also, every vertex $s \in S \setminus S_1$ is majority colored, even if its degree is odd for infinitely many components $C \in \mathcal{C}$.

Case 3. Component H is an infinite graph with a ray.

Let F' be a maximal subgraph of H decomposable into rays that exists by Lemma 6. We select a decomposition $\mathcal{R}$ of F' into rays. Observe that the decomposition $\mathcal{R}$ may contain any number of rays, but any two rays share infinitely many vertices since F' does not contain a double ray. Each component of $H - E(F')$ is a rayless graph, finite or infinite. Let B denote the set of those finite components that are of odd size and with all vertices of even degrees. For each $K \in B$, we assign exactly one ray R and a vertex $b(K) \in V(K) \cap V(R)$. Let $B(R)$ be the set of all vertices $b(K) \in V(R)$ of components $K \in B$ to which R is assigned.

First, we color all components K of $H - E(F')$. If $K \in B$ and $b = b(K)$, then we color the edges of K such that only b can be a vertex of K overwhelmed by some color (by Lemma 4). Otherwise, if $K \notin B$, then we choose a good coloring of K as in Case 1 or 2.

Denote by D the set of all startvertices of rays in $\mathcal{R}$. Clearly, a vertex v may have odd degree in F' only if $v \in D$. For each $v \in D$, let $r(v)$ be the cardinality of the set $\mathcal{R}(v)$ of rays in $\mathcal{R}$ starting at v.

Let $v_0 \in D$. If $r(v_0)$ is finite, then we color initial edges of consecutive rays in $\mathcal{R}(v_0)$ such that v_0 is either majority colored or it has odd degree and is almost majority colored in the subgraph induced by the rays of $\mathcal{R}(v_0)$ and the component K of $H - E(F')$ to which v_0 belongs.

Let $r(v_0)$ be infinite. We partition the set $\mathcal{R}(v_0)$ into two subsets $\mathcal{R}^1(v_0)$ and $\mathcal{R}^2(v_0)$ of the same cardinality. That is, to each ray in $\mathcal{R}^1(v_0)$ there corresponds a unique ray in $\mathcal{R}^2(v_0)$. We color the initial edges of rays in $\mathcal{R}(v_0)$ in such a way that the corresponding rays do not get the same color. Thus, the vertex v_0 is majority colored, and each ray in $\mathcal{R}(v_0)$ has its initial edge already colored.

Let $R = v_0v_1v_2\ldots$ be a ray in $\mathcal{R}(v_0)$. We successively color edges of R so that, for each $i \geq 1$, the edges $v_{i-1}v_i$ and v_iv_{i+1} get distinct colors, unless $v_i = b(K)$ for some $K \in B$ and v_i is overwhelmed by the color α. Again, by Lemma 4, we may recolor K if necessary so that α is different to the color of $v_{i-1}v_i$. Then we color the edge v_iv_{i+1} with any color different from α, perhaps with the color of $v_{i-1}v_i$.

Hence, we produced a good coloring of the subgraph R_H consisting of the ray R and all components K of $H - E(F')$ that meet R at the vertex $b(K)$. It is easy to see that in this way we obtain a good coloring of the union $\mathcal{R}_H(v_0)$ of subgraphs $\{R_H : R \in \mathcal{R}(v_0)\}$. Indeed, each vertex $v \neq v_0$ that belongs to at least one ray in $\mathcal{R}(v_0)$ has either even or infinite degree in $\bigcup \mathcal{R}(v_0)$, and all other vertices of $\mathcal{R}_H(v_0)$ are well colored.

For every $v \in D$, we color the subgraphs $\mathcal{R}_H(v)$ in the same way. Now, we want to show that this gives a good edge-coloring of the component H. Consider the set Z of all good edge-colorings of subgraphs of H that are unions of $\mathcal{R}_H(v)$ for some $v \in D$. For $c, c' \in Z$, we write $c \leq c'$ if the domain of c is contained in the domain of c' and the restriction of c' to the domain of c coincides with c. Let C be a non-empty chain in $(Z, \leq)$, and let $m = \bigcup C$. For every element $c \in C$, we have $c \leq m$, and $m \in Z$. Therefore, by Zorn's Lemma, there exists a maximal element M in Z. We claim that M is a good coloring of H. Suppose, contrary to the claim, that there is a subgraph $\mathcal{R}_H(v)$ of H, for some $v \in D$, that does not belong to the domain of M. However, it is easy to see that M and any good coloring of $\mathcal{R}_H(v)$ gives a good coloring because common vertices are well colored, contrary to the assumption. Indeed, the only problem might occur when a certain vertex w has odd degrees in both $\mathcal{R}_H(v)$ and the domain of M, with disjoint sets of incident edges. Then w has to be the startvertex of a ray in $\mathcal{R}(v)$. However, then w could not have odd degree in the union of rays in the domain of M.

We have just shown that, for every double ray $P \in \mathcal{P}$, there exists a good edge-coloring of P and all components of $G - E(F)$ sharing a vertex with P. Analogously as above, application of Zorn's Lemma gives a good coloring of the whole graph G. □

We conclude with an obvious consequence of Theorem 7 concerning usual majority colorings without lists.

Corollary 9 *A graph G of arbitrary order admits a majority* 2*-edge-coloring if and only if no vertex of G has an odd degree and no component of G is a finite graph of odd size.*

4 Unfriendly Partition Conjecture Holds for Line Graphs

Recall that a majority k-vertex-coloring of a graph G is a mapping $c : V(G) \to [k]$ such that, for every vertex $v \in V(G)$, the cardinality of neighbors of v with the color $c(v)$ is not greater than the cardinality of neighbors in other colors. Equivalently, a graph G admits a majority k-vertex-coloring if there is a partition $V_1, \dots, V_k$ of $V(G)$ such that, for each vertex $v \in V_i$, one has $|N(v) \cap V_i| \leq |N(v) \cap (V \setminus V_i)|$, for $i = 1, \dots, k$. The following result was proved by Lovász [10] for usual coloring but can be extended to the list setting with essentially the same proof, which we include for completeness.

Theorem 10 ([10]) *Every finite graph admits a majority vertex-coloring from any lists of size* 2.

Proof Let G be a finite graph with a set $\mathcal{L}$ of lists of size 2 assigned to every vertex of G. Let c be a vertex-coloring of G from $\mathcal{L}$ with the least number of edges with both endvertices of the same color. Then c is a majority vertex-coloring. Indeed, if there existed a vertex v with more than half of its neighbors with the color of v, then recoloring of v would decrease the number of edges with the same color of endvertices. □

Cowan and Emerson, in an unpublished work, conjectured that every infinite graph has a majority 2-vertex coloring. It was disproved by Shelah and Milner by showing the following.

Theorem 11 ([12]) *There exist uncountable graphs without majority* 2*-vertex-coloring. However, every infinite graph has a majority* 3*-vertex-coloring.*

The question whether countably infinite graphs have a majority 2-vertex-coloring remains open, and is known as the Unfriendly Partition Conjecture, which reads as follows.

Conjecture 12 Every countably infinite graph admits a majority 2-vertex-coloring.

The Unfriendly Partition Conjecture has been confirmed for graphs:

- with finitely many vertices of infinite degree (Aharoni, Milner, and Prikry [1]),
- with finitely many vertices of finite degree (Aharoni, Milner, and Prikry [1]),
- without rays (Bruhn, Diestel, Georgakopoulos, and Sprüssel [5]),
- without a subdivision of an infinite clique (Berger [2]).

Let us add that in 2023 Haslegrave proved the following.

Theorem 13 ([8]) *Every countable graph admits a majority vertex-coloring from any lists of size* 3.

Our Theorem 8 easily implies the following result.

Theorem 14 *Every line graph of arbitrary infinite order admits a majority vertex-coloring from any lists of size* 2.

Proof Let G be a line graph of a graph H. Let $\mathcal{L}$ be any set of lists for vertices of G, each of size 2. Simultaneously, $\mathcal{L}$ is the set of lists for edges of H.

Obviously, it is enough to prove that every component of G satisfies the claim. That is, it suffices to prove the theorem for connected graphs G. For finite graphs, the claim follows from Theorem 10.

Then suppose that $G = L(H)$ is an infinite connected graph. Hence, H is also an infinite connected graph. By Theorem 8, there exists an edge-coloring c of H from the set $\mathcal{L}$ of lists such that all vertices are majority colored, except possibly some vertices of odd degrees which are almost majority colored. Naturally, we color each vertex e of G with the color $c(e)$ of the corresponding edge $e = uv$ of H. It is easy to see that the cardinality of neighbors of e in G colored with $c(e)$ cannot be greater than that of other colors, even if one or two endvertices u, v of e have odd degree in H and $c(e)$ is an overwhelming color. Thus, we obtained a majority 2-vertex-coloring of the graph G. □

Consequently, line graphs form another class of graphs satisfying the Unfriendly Partition Conjecture.

Corollary 15 *The Unfriendly Partition Conjecture holds for line graphs.*

5 General Bound

In this section, we discuss a general upper bound for the least size of lists in a majority edge-coloring of graphs of arbitrary order. Note that this problem is related to the well-known List Coloring Conjecture, which is still open. However, for our purpose, the following result of Borodin, Kostochka and Woodall [4] suffices.

Theorem 16 ([4]) *Let G be a finite graph with maximum degree $\Delta(G)$, and let $\mathcal{L}$ be a set of lists, each of size $\lfloor \frac{3}{2}\Delta(G) \rfloor$ assigned to the edges of G. Then G has a proper edge-coloring from these lists.*

By compactness, this theorem is also true for infinite graphs. This result enables us to prove the following lower bound for the size of lists, allowing a majority edge-coloring.

Theorem 17 *Let G be a graph of arbitrary order and without pendant edges. Then G admits a majority edge-coloring from any collection of lists, each of size* 4.

Proof Given a graph G, analogously as in [3] we construct a graph G^* in the following way. We split every vertex v of degree greater than 3 in G into a set of vertices $\{v_i : i \in I(v)\}$ of degrees 2 or 3 in G^* by a suitable partition of its neighborhood $N_G(v)$. Naturally, if the degree $d_G(v)$ is infinite, then the cardinality of the index set $I(v)$ is equal to $d_G(v)$. Observe that there is a one-to-one correspondence between the edges of G and G^*. So, the list for an edge of G^* is the same as the list for its counterpart in G. Each component of the graph G^* is a countable subcubic graph. By the above result of Borodin, Kostochka and Woodall for countable graphs, there exists a proper coloring of G^* from any collection of lists of size 4. This coloring transferred to the graph G yields a majority coloring since for every $v \in V(G)$, the edges incident to any single vertex of $\{v_i : i \in I(v)\}$ have distinct colors. $\square$

This bound is tight. Namely, for each infinite cardinal κ, there exists a graph of order κ that needs four colors in any majority edge-coloring. To see this, consider the following construction. Take any graph G' of order κ and a subcubic Class 2 finite graph H. Then join them by an edge vu, where $v \in V(H)$ and $u \in V(G')$, to create a graph G. If there existed an edge $vw \in E(H)$ such that $H - vw$ was Class 1, then we could easily extend a proper 3-edge-coloring of $H - vu$ to a majority 3-edge-coloring of $H + vu$ by putting the same color on two edges vw, vw' incident to v in H. Hence, to obtain a graph G without a majority 3-edge-coloring, we take a vertex v such that for every incident edge vw the subgraph $H - vw$ is still of Class 2 (the Petersen graph is an example). Clearly, the graph H may contain more vertices $v_1, \ldots, v_k$ such that the deletion of an arbitrary edge $v_i w_i$, for each $i = 1, \ldots, k$, results in a Class 2 graph. Then we can join each vertex v_i by an edge to a vertex $u_i \in V(G')$ to obtain a graph G that does not admit a majority 3-edge-coloring. However, k has to be smaller than $|V(H)|$ since if, for each $v \in V(H)$, we delete an incident edge, then we get a graph with maximum degree two which clearly admits a majority 3-edge-coloring. Therefore, some vertices of H have to maintain their degrees also in G, otherwise G could have a majority 3-edge-coloring. This observation justifies the following conjecture showing that subcubic Class 2 subgraphs are the only obstacle for the existence of a majority 3-edge-coloring. For finite graphs, this conjecture was already formulated in [3].

Conjecture 18 Every graph G without pendant edges admits a majority 3-edge-coloring unless G contains a Class 2 graph H with $\Delta(H) = 3$ as an induced subgraph, and with some vertices of H with degrees at most 3 in G.

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