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Hierarchical product graphs and their prime factorization*

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Dedicated to Dragan Marušič on the occasion of his 70^{th} ***birthday.***

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Abstract

The hierarchical and the generalized hierarchical product of graphs, together with their multiary and rooted versions, are variants of the Cartesian product. They are not commutative, and only the rooted versions are associative. We prove that finite connected graphs have unique prime factorizations with respect to the rooted hierarchical product, the multiary hierarchical product, and the rooted generalized hierarchical product. For the generalized hierarchical product, we disprove a claim about unique prime factorization by Anderson, Guo, Tenney, and Wash from 2017.

We also describe the interrelation between the automorphism groups of connected graphs with the groups of their prime factors in the cases of unique prime factorization, and in the case of the standard prime factorization with respect to the hierarchical product. For finite trees, we show that their prime factors can be computed in subquadratic time.

Keywords: Hierarchical products of graphs, prime factorizations, trees, algorithms.

Math. Subj. Class.: 05C05, 05C25, 05C75, 05C76, 05C85

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1 Introduction

In 1978, Godsil and McKay [6] defined a product of graphs, of which a special case was reintroduced in 2009 by Barrière, Comellas, Dalfó, and Fiol [3] as the *hierarchical product*. In both papers, spectral properties of the product were the main subject of investigation. The generalized hierarchical product was also introduced in [3], and further studied in 2009 by Barrière, Dalfó, Fiol, and Mitjana [5].

The topic of this paper is prime factorization. We prove that all finite connected graphs have unique prime factorizations with respect to the rooted hierarchical product, the n-ary hierarchical product, the rooted generalized hierarchical product, and a standard prime factorization with respect to the hierarchical product. For the generalized hierarchical product considered by Anderson, Guo, Tenney, and Wash in [2], we provide an example of a finite connected graph with non-unique prime factorization.

We also describe the interdependence of the automorphism groups of connected graphs with the groups of their prime factors, where the factorization is taken via any of the above products with the unique, or the standard prime factorization property.

For finite trees, we prove that their prime factorizations via the hierarchical and the rooted hierarchical products can be computed in subquadratic time.

2 Hierarchical products

Given a graph G, we use the notation $V(G)$ for its set of vertices, and $E(G)$ for its set of edges. $E(G)$ is a set of unordered pairs ab of distinct vertices of G. If $ab \in E(G)$, then we call a, b *adjacent*, in symbols $a \sim b$ or $a \sim_G b$. For a graph G with a distinguished vertex u, called the *root* of G, we use the notation $G[u]$.

We define the *hierarchical product* $G \sqcap H[v]$ of an unrooted graph G by a rooted graph $H[v]$, as an unrooted graph with vertex set $V(G) \times V(H)$, whose edges are defined by

$$(g,h)(g',h') \in E(G \sqcap H[v]) \text{ if } \begin{cases} gg' \in E(G) \text{ and } h = h' = v, \text{ or} \\ hh' \in E(H) \text{ and } g = g'. \end{cases}$$

In other words, $G \sqcap H[v]$ is formed from G by attaching to each vertex of G a copy of H via its root. Figure 1 depicts $G \sqcap H[1]$, where both G and H are copies of a K_2 with vertex set $\{0, 1\}$.

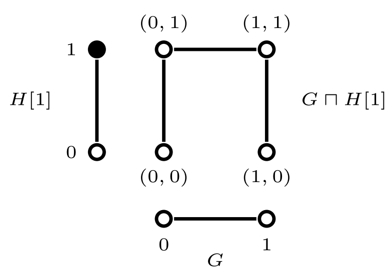

Figure 1: A hierarchical product.

This coincides with the definition of the hierarchical product of two factors by Barrière, Comellas, Dalfó, and Fiol [3].

Hierarchical multiplication is not commutative because $H[v] \sqcap G$ is not defined. Similarly, it is not associative, because $G_2[v_2] \sqcap G_3[v_3]$ is not defined as a hierarchical product. Therefore $G_1 \sqcap (G_2[v_2] \sqcap G_3[v_3])$ is also not defined, and thus cannot be equal to $(G_1 \sqcap G_2[v_2]) \sqcap G_3[v_3]$.

We call the unrooted graphs G, H the *factors* of $G \sqcap H[v]$, and note that, for given vertices $v, v' \in V(H)$, the graphs $G \sqcap H[v]$ and $G \sqcap H[v']$ need not be isomorphic. If there is no automorphism of H that maps v into v', then we say that the products $G \sqcap H[v]$ and $G \sqcap H[v']$ are different.

A graph on at least two vertices is *prime with respect to the hierarchical product* if it cannot be represented as the hierarchical product of two graphs different from K_1. We shall also synonymously use the term *indecomposable* instead of *prime*.

Clearly, each finite graph can be represented as a product of graphs that are prime with respect to the hierarchical product. We are interested in unique prime factorizations. Because the prime factorization need not be unique for disconnected graphs, we can restrict attention to connected graphs. The non-uniqueness follows from an example of Anderson, Guo, Tenney, and Wash [2]. Set $V(K_1) = \{0\}$ and $V(K_2) = \{1, 2\}$. Then, as depicted in Figure 2,

$$K_2 \sqcap (K_1 + K_2)[0] \cong (K_1 + K_1 + K_1) \sqcap K_2[1],$$

because both sides are isomorphic to $K_2 + K_2 + K_2$. As all factors have prime order, the graphs are prime.

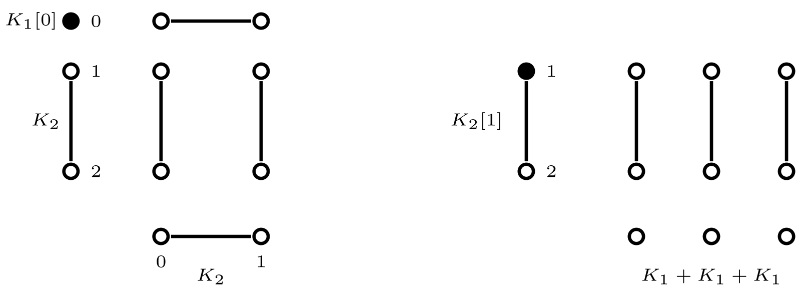

Figure 2: Different prime factorizations of $K_2 + K_2 + K_2$.

2.1 Prime factorizations for hierarchical products

Given a product $X = G \sqcap H[v]$, the subgraph of X induced by the vertices $\{(g, v) \,|\, g \in V(G)\}$ is isomorphic to G. We denote it by $G \times v$. Similarly, for each $g \in V(G)$, the set $\{(g, h) \,|\, h \in V(H)\}$ induces a subgraph isomorphic to H, which we denote by $g \times H$. We also set

$$V(G) \times H = \bigcup_{g \in V(G)} g \times H.$$

Clearly, for all $g \in V(G)$, we have $(G \times v) \cap (g \times H) = \{(g, v)\}$.

With this notation,

$$G \sqcap H[v] = (G \times v) \cup (V(G) \times H),$$

and $X - E(G \times v) = V(G) \times H$ consists of $|G|$ copies of H. In other words, the graphs $g \times H$ are uniquely determined by $G \times v$, and thus also $H[v]$.

Note that the subgraphs $g \times H$ and $G \times v$ are *convex* in $G \sqcap H[v]$. That is, each shortest path P in $G \sqcap H[v]$ between two vertices of $g \times H$ (respectively, between two vertices of $G \times v$) is already in $g \times H$ (respectively, $G \times v$).

Theorem 2.1. *To any finite connected graph $X \neq K_1$, there exists a unique graph G that is prime with respect to the hierarchical product, and a unique rooted graph $H[v]$, possibly trivial, such that*

$$X = G \sqcap H[v].$$

Furthermore, $G \times v$ is invariant under all automorphisms of X.

Proof. Let $G \sqcap H[v] \cong G^* \sqcap H^*[v^*]$ be two factorizations of a connected graph, where G and G^* are prime. Set $X = G \sqcap H[v]$, $X^* = G^* \sqcap H^*[v^*]$, and let φ be an isomorphism from X^* to X. It suffices to show that $G \cong G^*$. We consider three cases.

Case 1. $(G \times v) \cap \varphi(G^* \times v^*) = \varnothing$. Then, there exists a $g \in V(G)$ such that $\varphi(G^* \times v^*) \subseteq g \times H - (g, v)$. Hence, $\varphi^{-1}(g, v) = (g^*, h^*)$, where $g^* \in V(G^*)$, $h^* \in V(H^*)$, $h^* \neq v^*$, and $\varphi(g^* \times H^*) \supset G - (g \times H[v])$. This implies that $|H^*| > |H|$. The reversal of the roles of X and X^* yields $|H| > |H^*|$. Hence, $|H^*| > |H^*|$, which is not possible.

Case 2. The sets $(G \times v) \cap \varphi(G^* \times v^*)$, $\varphi(G \times v) - (G^* \times v^*)$, and $(G^* \times v^*) - \varphi(G \times v)$ are nonempty. Let $(x, v) \in V(G \times v) \cap \varphi(G^* \times v^*)$ and $(g, h) \in \varphi(G^* \times v^*) - V(G \times v)$. Then, $h \neq v$, and there is a path in $\varphi(G^* \times v^*)$ from (x, v) to (g, h). This path must contain the vertex (g, v), which is thus in $\varphi(G^* \times v^*)$.

Consider $\varphi^{-1}(g, v)$. As it is in $G^* \times v^*$, its second coordinate in $G^* \sqcap H^*[v^*]$ must be v^*. Let g^* be its first coordinate. Then, $\varphi(g^* \times H^*[v^*])$ cannot contain (g, v), because it is in $\varphi(G^* \times v^*)$. Hence, $\varphi(g^* \times H^*[v^*])$ is contained in $g \times H[v] - (g, v)$. This implies that $|H^*| < |H|$.

Similarly, we obtain that $|H| < |H^*|$ by interchanging the roles of X and X^*, a contradiction.

Case 3. One of the graphs $\varphi(G^* \times v^*)$ and $G \times v$ is contained in the other. If they are not equal, then one must be properly contained in the other. Without loss of generality, suppose that $\varphi(G^* \times v^*)$ properly contains $G \times v$. Then, it is easily seen that

$$\varphi(G^* \times v^*) = G \sqcap ((g \times H) \cap \varphi(G^* \times v^*))[v].$$

But then G^* is not prime. Hence, $\varphi(G^* \times v^*) = G \times v$.

To show the invariance of $G \times v$ under $\mathrm{Aut}(X)$, let $G^* \sqcap H^*[v^*]$ be another representation of $X = G \sqcap H[v]$, where G^* is prime. By the above, $\varphi(G^* \times v^*) = G \times v$ for any $\varphi \in \mathrm{Aut}(X)$. □

Corollary 2.2 (Standard prime factorization with respect to the hierarchical product)**.** *Each finite connected graph $G \neq K_1$ has a unique standard prime factorization as a hierarchical product*

$$G = G_1 \sqcap (G_2 \sqcap (G_3 \sqcap (\cdots \sqcap G_k[v_k])[v_{k-1}])\cdots[v_3])[v_2],$$

of uniquely determined prime factors $G_1, \dots, G_k$ and uniquely determined roots $v_2, \dots, v_k$, where each root v_i, for $2 \leq i \leq k$, is a vertex of the product of the last $k - i + 1$ factors, that is

$$v_i \in V\big(G_i \sqcap (G_{i+1} \sqcap (\cdots \sqcap G_k[v_k])\cdots[v_{i+2}])[v_{i+1}]\big).$$

Proof. Let $G = G_1 \sqcap H_2[v_2]$, where G_1 is prime. If H_2 is not prime, then $H_2 = G_2 \sqcap H_3[v_3]$, where G_2 is prime by Theorem 2.1. Then,

$$G = G_1 \sqcap (G_2 \sqcap H_3[v_3])[v_2]$$

and the proof is easily completed by induction. □

There may be prime factorizations that are different from the standard prime factorization. For example, let $V(K_2) = \{0, 1\}$, and 1 be a leaf of $P_4[1]$. Both K_2 and $P_4[1]$ are indecomposable by the hierarchical product, and $(K_2 \sqcap K_2[1]) \sqcap P_4[1]$ is a prime factorization of the tree with standard prime factorization $K_2 \sqcap (K_2 \sqcap K_2[1])[(0,0)]$.

2.2 Automorphisms of hierarchical products

Let $X = G \sqcap H[v]$. Suppose we are given an automorphism α of G and, for each $g \in V(G)$, an automorphism β_g of $H[v]$.

Then, the mapping $\varphi: V(X) \to V(X)$ defined by

$$\varphi(g,h) = \big(\alpha(g), \beta_g(h)\big)$$

is an automorphism of X. These automorphisms form a group, which is called the *semidirect product* $\mathrm{Aut}(G) \ltimes \mathrm{Aut}(H[v])$ of $\mathrm{Aut}(G)$ by $\mathrm{Aut}(H[v])$.

For connected graphs, we immediately obtain the following theorem.

Theorem 2.3. *Let X be a connected graph that is representable in the form $G \sqcap H[v]$, where G is prime with respect to the hierarchical product. Then,* $\mathrm{Aut}(X) = \mathrm{Aut}(G) \ltimes \mathrm{Aut}(H[v])$.

Proof. By Theorem 2.1, each automorphism φ of X stabilizes $G \times v$ and, thus, induces an automorphism α of G. Furthermore, it maps $g \times H$ into $\alpha(g) \times H$, where $\varphi(g,v) = (\alpha(g), v)$. Let $\varphi(g,h) = \big(\alpha(g), \beta_g(h)\big)$. Then, β_g clearly is an automorphism of $H[v]$. □

Corollary 2.4. *Let*

$$G = G_1 \sqcap (\, G_2 \sqcap \, (G_3 \sqcap (\, \cdots \, \sqcap \, G_k[v_k] \,)[v_{k-1}])\cdots[v_3])[v_2]$$

be the standard prime factorization of a connected finite graph G. Then,

$$\mathrm{Aut}(G) = \mathrm{Aut}(G_1) \ltimes (\mathrm{Aut}(G_2) \ltimes (\mathrm{Aut}(G_3) \ltimes (\cdots \\ \cdots \ltimes \mathrm{Aut}\,(G_k[v_k])[v_{k-1}])\cdots[v_3])[v_2]).$$

2.3 Unique prime factorization for rooted hierarchical products

Given two rooted graphs $G[u], H[v]$, we define the *rooted hierarchical product*

$$(G[u] \sqcap H[v])[(u,v)]$$

as the graph $G \sqcap H[v]$ with root (u,v). For the first factor, we also allow that it has a root set U_G consisting of several vertices. In that case, $G[U_G] \sqcap H[v]$ is $G \sqcap H[v]$ with root set $U_G \times v$.

It is easily seen that the rooted hierarchical product is associative.

We call a graph $X[U_X]$ on at least two vertices *prime with respect to the rooted hierarchical product* if it cannot be represented as a rooted hierarchical product $G[U_G] \sqcap H[v]$ of non-trivial graphs $G[U]$ and $H[v]$.

Theorem 2.5. *To any finite connected graph $X \neq K_1$ with given, nonempty root set U, there exists a unique rooted graph $G[U_G]$ that is prime with respect to the rooted hierarchical product and a unique rooted graph $H[v]$ such that*

$$X[U] = G[U_G] \sqcap H[v].$$

Furthermore, $G \times v$ is invariant under all automorphisms of X.

Proof. The proof follows the lines of the proof of Theorem 2.1, in Case 3 one has to take the intersection of all $G \times v$ that contain U. □

As a consequence of Corollary 2.2, we obtain the unique prime factorization property of finite connected graphs via the rooted hierarchical product.

Corollary 2.6 (Unique prime factorization with respect to the rooted hierarchical product)**.** *Each finite rooted connected graph G has a presentation as a hierarchical product*

$$G = G_1[U_1] \sqcap G_2[u_2] \sqcap \cdots \sqcap G_k[u_k]$$

of uniquely determined rooted prime graphs $G_1[U_1], \dots, G_k[u_k]$.

2.4 The hierarchical product of several factors, as defined by Barrière, Comellas, Dalfó and Fiol

Let G_1 be an unrooted graph, and $G_2[v_2], \dots, G_k[v_k]$ be $k-1$ rooted graphs. Then the hierarchical product

$$G_1 \sqcap G_2 \sqcap \cdots \sqcap G_k$$

of $G_1, G_2[v_2], \dots, G_k[v_k]$, as originally defined by Barrière, Comellas, Dalfó, and Fiol [3], is, in our notation, the unrooted graph

$$G_1 \sqcap (G_2[v_2] \sqcap \ldots \sqcap G_k[v_k]),$$

that is, the hierarchical product of G_1 by the rooted hierarchical product $G_2[v_2] \sqcap \cdots \sqcap G_k[v_k]$.

It is a k-ary operation that coincides with the hierarchical product for $k = 2$. Because k is not fixed, we call it the *multiary hierarchical product.* It is not commutative. Concerning associativity, Barrière, Dalfó, Fiol, and Mitjana [4, Proposition 2.1] showed that

$$G_1 \sqcap G_2[u_2] \sqcap G_3[u_3] = G_1 \sqcap (G_2 \sqcap G_3[u_3])[(u_2, u_3)] = (G_1 \sqcap G_2[u_2]) \sqcap G_3[u_3].$$

In our notation this is

$$G_1 \sqcap (G_2[u_2] \sqcap G_3[u_3]) = (G_1 \sqcap G_2[u_2]) \sqcap G_3[u_3],$$

which looks like the associative law. This is deceptive, because $G_2[u_2] \sqcap G_3[u_3]$ is a rooted generalized product, and not a multiary generalized hierarchical product. Nonetheless, for any given ℓ between 1 and $k-1$, we have

$$G_1 \sqcap \cdots \sqcap G_k = (G_1 \sqcap \cdots \sqcap G_\ell) \sqcap G_{\ell+1} \sqcap \cdots G_k.$$

A graph on at least two vertices is prime with respect to this product, if it cannot be presented as a non-trivial multiary hierarchical product. This is equivalent to being prime with respect to the hierarchical product.

Theorem 2.7 (Unique prime factorization of graphs with respect to the multiary hierarchical product)**.** *Prime factorization with respect to the multiary hierarchical product, as defined by Barrière, Comellas, Dalfó, and Fiol* [3]*, is unique.*

Proof. Let G be a given graph. If it is rooted, then it is prime with respect to the associative hierarchical product. Otherwise, by Theorem 2.1, there is a unique prime graph G_1 together with a unique rooted graph $H[v]$, such that $G = G_1 \sqcap H[v]$. The observation that the prime factorization of $H[v]$ is unique by Corollary 2.6 completes the proof. □

Prime factorizations with respect to the hierarchical product, and the multiary hierarchical product, need not be the same. For example, let K_2 be the edge 01, and $P_4 = 1234$. Then

$$P_8 = K_2 \sqcap P_4[1]$$

is the prime factorization of P_8 by the multiary hierarchical product, because $P_4[1]$ is indecomposable by the rooted hierarchical product. However, the prime factorization of P_8 via the hierarchical product is

$$P_8 = K_2 \sqcap (K_2 \sqcap K_2[0])[0,1].$$

Let $G_1 \sqcap \cdots \sqcap G_k$ be the prime factorization of a graph G via the multiary hierarchical product. Then, we infer from Corollary 2.4 that

$$\mathrm{Aut}(G) = \mathrm{Aut}(G_1) \ltimes (\mathrm{Aut}(G_2[v_2]) \ltimes \cdots \ltimes \mathrm{Aut}(G_k[v_k])\cdots). \tag{2.1}$$

As an application, consider the group of P_r^m, that is, the multiary hierarchical product of m copies of the path $P_r = 1, 2, \ldots, r$, for $r \geq 2$, where the first factor is unrooted and the other factors are rooted at an endpoint and multiplied via the rooted hierarchical product.

Barrière, Comellas, Dalfó, and Fiol [4, Proposition 4.2] proved at length that for each $m \geq 1$ and each $r > 1$, the automorphism group of P_r^m is the symmetric group S_2. However, it also is a direct consequence of Equation (2.1). Just observe that $|\mathrm{Aut}(P_r)| = 2$, and $|\mathrm{Aut}(P_r[v])| = 1$ unless the root v is the center of P_r.

3 Trees

Finite trees are connected and, thus, have unique prime factorizations with respect to the hierarchical and the rooted hierarchical products. We show that the factorizations can be found in subquadratic time.

Each finite tree T has at least two leaves and a center consisting of a single vertex or an edge that can be constructed by recursively removing leaves from T. To be more precise, let T be a tree and $\{\ell_1, \ell_2, \ldots, \ell_k\}$ its set of leaves. We set $T^{(1)} = T - \{\ell_1, \ell_2, \ldots, \ell_k\}$ and, for $k > 1$, recursively define $T^{(k)}$ by

$$T^{(k)} = (T^{(k-1)})^{(1)}.$$

It is well known that the $T^{(k)}$ are connected and that the center of T is T^{k_0}, where k_0 is the smallest k for which $T^{(k)}$ has one or two vertices. We set $E_k = E(T^{(k-1)}) - E(T^{(k)})$.

For rooted trees $S[s]$, we slightly change the definition and define $S^{(1)}[s]$ as the tree with root s obtained from S by removing all leaves that are different from s. For $k > 1$, we recursively define $S^{(k)}[s]$ as we have done before. Furthermore, the smallest k for which $S^{(k)} = \{s\}$ is the *height* of $S[s]$, denoted by $h(S[s])$.

Theorem 3.1. *Let T be a finite tree of order $n > 1$. Then, the prime factorization of T with respect to the hierarchical product can be computed in $O(|T|^{3/2})$ time.*

Proof. Let T be a finite tree of order n. If T is prime, then there is nothing to show. Hence, we can assume by Theorem 2.1 that T has a unique factorization of the form

$$T = R \sqcap S[s],$$

where $R \neq K_1$ is prime and $S[s]$ has at least two vertices. Clearly no leaf of T is in $R \times s$. Hence, $T^{(1)}$ contains $R \times s$, and $T^{(1)} = R \sqcap S^{(1)}[s]$. By induction, it is easily seen that $T^{(k)} = R \sqcap S^{(k)}$ if $1 \leq k \leq h$, where h denotes the height of $S[s]$. Because s is the only vertex of $S^{(h)}[s]$, we infer that

$$R \times s = T^{(h)}.$$

In order to compute R, we thus have to determine h. Clearly, h can be smaller than k_0, for example, when the center T^{k_0} of T consists of a single vertex x, because $R \times s = T^{(h)}$ has at least two vertices. Hence, $T^{(h)} \supseteq T^{(k_0-1)} \supset T^{(k_0)}$ and thus $x \in V(R \times s)$. Therefore, the set of edges E_{k_0} that is removed from T at step k_0 is in $R \times s$. If these are all edges of $R \times s$ that are incident with x, then the component of $T - E_{k_0}$ that contains $x = (x', s)$ is $x' \times S$ with root s. We can thus determine $h = h(S[s])$ and check whether all components of $T - E(T^{(h)})$ are isomorphic. If so, then $R \times s = T^{(k_0-1)}$.

$R \times s$ contains edges of all E_k for $h < k \leq k_0$, and if $h < k_0 - 1$ we have to repeat this process. As such, we have to check whether all components of $T - (E_k \cup \cdots \cup E_{k_0})$ are isomorphic for $h < k \leq k_0$, but because we do not know h, we might have to consider all $k \in \{1, \ldots, k_0\}$.

For illustration, consider Figure 3. It shows a tree $T = S \sqcap R$, where S is a tree of order 5 and $R = K_2[1]$. To reach the center e of T, the algorithm first removes the full lines, and then the dashed lines. Clearly $k_0 = 2$. The components of $T_e = T^{k_0}$ are not isomorphic, hence, $h < k_0$ (and 1 in this case).

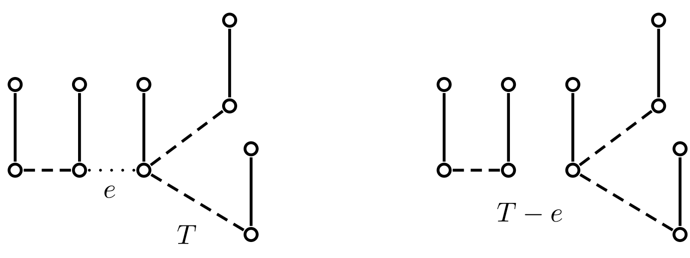

Figure 3: T and $T - e$.

To estimate the number of iterations, let $E_{k_0} \cup E_{k_0-1} \cup \cdots \cup E_k \subseteq R \times s$. Then, $E_{k_0} \cup E_{k_0-1} \cup \cdots \cup E_k$ must contain at least $1 + 2 + \cdots + (k_0 - k + 1) = (k_0 - k + 1)(k_0 - k + 2)/2$ edges, and $|T|$ is at least twice that number if $|S|$ is non-trivial. Hence, $|T| - 1 \geq (k_0 - k + 1) = (k_0 - k + 1)(k_0 - k + 2)$ and $\sqrt{|T|} > k_0 - k + 1$, which limits the number of iterations to $\sqrt{|T|}$.

At each iteration, we have to determine the height of the component of x in $T - E(T^{(k)})$, say h_k, and check whether the components of $T - E(T^{(h_k)})$ are isomorphic. Because the time complexity of the isomorphism checking of trees is linear in their order, as shown by

Aho, Hopcroft, and Ullman [1], the time complexity of each iteration is proportional to the number of components times their size, which is $|T|$.

Beginning with $T^{(1)}$ and the components of $T - E(T^{(1)})$, we can successively compute all $T^{(k)}$ and the components of $T - E(T^{(k)})$ by overwriting them for all k between 1 and k_0 in total time that is linear in $|T|$.

Hence, we need $O(|T|)$ time at each of the at most $\sqrt{|T|}$ iterations to compute R, and there is a constant c such that the time needed for computing R is at most $c|T|^{3/2}$.

This also determines S, which we represent as $S = R_2 \sqcap S_2[s_2]$, where R_2 is prime. Because the size of S is at most half the size of T, we need at most $\frac{1}{2}c|T|^{3/2}$ time for this step. Continuing the process, we see that we need at most $(1 + \frac{1}{2} + \frac{1}{4} + \cdots)\, c|T|^{3/2} < 2c|T|^{3/2}$ time to represent T in the form

$$T = R \sqcap (R_2 \sqcap (R_3 \sqcap \cdots \sqcap R_r[s_r])\cdots)[s_3])[s_2].$$

□

Corollary 3.2. *Let $T[U_T]$ be a finite rooted tree of order n. Then, the prime factorization of $T[U_T]$ with respect to the rooted hierarchical product can be computed in $O(|T|^{3/2})$ time.*

Proof. Just replace the center of $T[U_T]$ with U_T and use the same arguments as in the proof of Theorem 3.1. □

4 Generalized hierarchical products

Given two graphs G and H and a subset V of the vertex set of H, the *generalized hierarchical product* $G \sqcap H[V]$ is defined in Barrière, Comellas, Dalfó, and Fiol [3] on $V(G) \times V(H)$ by, in our notation,

$$G \sqcap H[V] = (G \times V) \cup (V(G) \times H).$$

For $|V| = 1$, this coincides with the hierarchical product; for $V = V(H)$, one obtains the Cartesian product. For $|V| > 1$, we speak of a *proper generalized hierarchical product.* Clearly, no tree is a proper generalized hierarchical product.

Similarly to the case of the hierarchical product $G \sqcap H[v]$, we observe that $G \sqcap H[V]$ and G are unrooted, but H is rooted (with root set V). Furthermore, given two rooted graphs $G[U], H[V]$, we define the *rooted generalized hierarchical product*

$$G[U] \sqcap H[V] \tag{4.1}$$

as the product $G \sqcap H[V]$ with root set $U \times V$. This product is associative.

The multiary product

$$G_1 \sqcap G_2[U_2] \sqcap \cdots \sqcap G_k[U_k]$$

is understood in Barrière, Dalfó, Fiol, and Mitjana [5] and Anderson, Guo, Tenney, and Wash [2] to mean the unrooted graph

$$G_1 \sqcap (G_2[U_2] \sqcap \cdots \sqcap G_k[U_k]). \tag{4.2}$$

We refer to it as the *multiary generalized hierarchical product*. For $k = 2$, it coincides with the generalized hierarchical product. As the multiary hierarchical product, which is a special case, it is neither commutative, nor associative. A graph on at least two vertices that is not presentable as product of two non-trivial graphs via any of the three products just defined, is called *prime* with respect to this product.

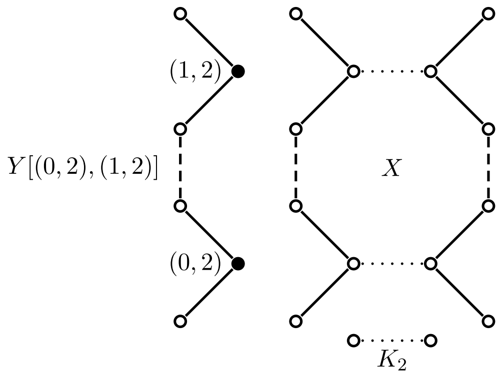

Figure 4: X represented as $K_2 \sqcap Y[(0,2),(1,2)]$.

4.1 Non-unique prime factorization of graphs with respect to the generalized, and the multiary generalized hierarchical product

In [2], the unique prime factorization property of connected graphs with respect to the multiary generalized hierarchical product, see (4.2), was claimed. It would extend the well-known result of Sabidussi [8] that prime factorization of finite connected graphs by the Cartesian product is unique. The following example shows that this claim does not hold.

Example 4.1. Let K_2 be the path 01, $P_3 = 123$, $Y = K_2 \sqcap P_3[1]$, and $Z = K_2 \sqcap P_3[2]$. Then, Y is a P_6 and Z consists of two P_3 whose center vertices are joined by an edge. In Y, the vertices $(0,2),(1,2)$ are the neighbors of the endpoints of Y and, in Z, the vertices $(0,1),(1,1)$ are leaves.

Clearly $Y \ncong Z$, and both $Y[(0,2),(1,2)]$ and $Z[(0,1),(1,1)]$ are indecomposable by the rooted generalized hierarchical product. For a proof, we observe that both graphs have order 6, which implies that all factorizations have two factors, one with two, the other with three vertices. Moreover, both X and Y are trees with root sets of order 2. If the second factor had a root set of order 2, then the product would not be a tree. Hence, both second factors have only one root, and the first factor have two.

If the first factor had order three, the second would be a K_2, and the product would have three leaves. But, Y has two and Z four. We infer that the first factor is a K_2 with the root set $V(K_2)$. Then, the roots in the product are adjacent, which is neither the case in Y, nor Z. Therefore,

$$K_2 \sqcap Y[(0,2),(1,2)] \cong K_2 \sqcap Z[(0,1),(1,1)]$$

are two different prime factorizations of the unrooted connected graph $X = K_2 \sqcap Y[(0,2),(1,2)]$ by the multiary generalized hierarchical product, compare with Figures 4 and 5.

The dashed lines in the figures indicate the layers of the factor K_2 in $Y = K_2 \sqcap P_3[1]$ and $K_2 \sqcap Z[(0,1),(1,1)]$, whereas the dotted lines indicate the layers of the factors K_2 in $Z = K_2 \sqcap P_3[2]$ and $K_2 \sqcap Y[(0,2),(1,2)]$.

Recalling that $Y = K_2 \sqcap P_3[1]$, and $Z = K_2 \sqcap P_3[2]$, we also have presentations of X by three factors, namely

$$K_2 \sqcap (K_2 \sqcap P_3[1])[(0,2),(1,2)] \text{ and } K_2 \sqcap (K_2 \sqcap P_3[2])[(0,1),(1,1)].$$

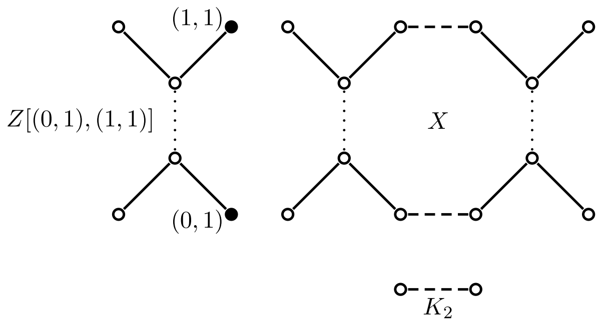

Figure 5: X represented as $K_2 \sqcap Z[(0,1),(1,1)]$.

The factors are prime and isomorphic. However, the root sets are different and inequivalent under isomorphisms of the factors. This means that X also has at least two different prime factorizations by the generalized hierarchical product of graphs.

If we replace the product from Equation (4.2) by the rooted generalized hierarchical product, then the prime factorization is unique, as we will show in Theorem 4.2.

For our graph X, this means that each $X[U]$, where $U \subseteq V(X)$, is prime or uniquely representable as a rooted generalized hierarchical product of prime rooted graphs. For example, the rooted generalized products

$$K_2[0] \sqcap Y[(0,2),(1,2)] \text{ and } K_2[0] \sqcap Z[(0,1),(1,1)]$$

of prime graphs are both isomorphic to X, but the root sets are inequivalent under automorphisms of X. It is easy to check that they have no other factorizations with respect to the rooted generalized hierarchical product.

4.2 Unique prime factorization for the rooted generalized hierarchical product

We shall prove the following theorem, which shows that the prime factors with respect to the generalized hierarchical product obey an order hierarchy that is similar to the one of the prime factors with respect to the hierarchical product. In the proof, we shall invoke the fact that connected graphs have unique prime factorizations by both the hierarchical product and the Cartesian product.

Theorem 4.2. *Each rooted, connected finite graph $G[U]$ is uniquely representable as a rooted generalized hierarchical product $G_1[U_1] \sqcap G_2[U_2] \sqcap \cdots \sqcap G_k[U_k]$ of rooted prime graphs, where two factors $G_i[U_i]$, $G_{i+1}[U_{i+1}]$, for $1 \le i < k$, commute if they are isomorphic as rooted graphs, or if $U_i = V(G_i)$ and $U_{i+1} = V(G_{i+1})$.*

Let us first have a look at the erroneous proof given in Anderson, Guo, Tenney, and Wash [2] for the uniqueness of a prime factorization in the form of Equation (4.2). It is modelled after a proof in Hammack, Imrich, and Klavžar [7], which uses properties of convex subgraphs H of a graph G, where H is convex in G if each shortest path in G between two vertices a, b of H is already in H. It is important that convexity is defined independently of any factorization of G, it only depends on the metric of G. Nonetheless, in Anderson, Guo, Tenney, and Wash [2], convexity is replaced by *hierarchical convexity*, defined as follows: let a, b be two vertices of $G = G_1 \sqcap (G_2[U_2] \sqcap \cdots \sqcap G_k[U_k])$, and P be a shortest (a,b)-path in the Cartesian product $G^{\square} = G_1 \square G_2 \square \cdots \square G_k$. Then, a subgraph H of G is hierarchically convex if, to each pair of vertices $a,b \in V(H)$ and each shortest (a,b)-path P in $G^{\square}$, all edges in $P \cap G$ are already in H.

Clearly, the hierarchical convexity of H in G depends not only on H and G, but also on the given factorization of G. Thus, H may be hierarchically convex with respect to one factorization, but not with respect to another one. For example, the path $(0,0,1)(0,0,2)(1,0,2)$ in $K_2 \sqcap Y[(0,2),(1,2)]$ is not convex, because $(0,0,1)\,(1,0,1)(1,0,2)$ is a path in $K_2 \square Y$, but the corresponding path in the representation $K_2 \sqcap Z[(0,1),(1,1)]$ of X is convex.

The proof in Hammack, Imrich, and Klavžar [7] strongly depends on the fact that convexity is defined independently of factorizations.

Before we prove Theorem 4.2, let us recall that the vertex set of the Cartesian product $G = G_1 \square G_2 \square \cdots \square G_k$ is

$$V(G) = V(G_1) \times V(G_2) \times \cdots \times V(G_k),$$

and that it consists of k-tuples $(x_1, x_2, \ldots, x_k)$, where $x_i \in V(G_i)$ for $1 \leq i \leq k$. One says x_i is the *i-th coordinate* of x or the *projection* $p_i(x)$ of x into G_i. Furthermore, let $a \in V(G)$. Then the *G_i-layer* through a is the induced subgraph

$$G_i^a = \langle \{x \in V(G) \,|\, p_j(x) = a_j \text{ for } j \neq i\} \rangle.$$

These definitions extend verbatim to all types of hierarchical and generalized hierarchical products. For example, let

$$G_1[U_1] \sqcap G_2[U_2] = (G_1 \times U_2) \cup (V(G_1) \times G_2),$$

and $a \in V(G_1) \times V(G_2)$. Then, $G_1^a = G_1 \times a_2$ is isomorphic to G_1 if $a_2 \in U_2$, and edgeless otherwise. However, and less restrictive, for all $a_1 \in V(G_1)$, $G_2^a = a_1 \times G_2$ is isomorphic to G_2. We call edgeless layers *empty*, and the others *full*.

For the product

$$G[U] = G_1[U_1] \sqcap G_2[U_2] \sqcap \cdots \sqcap G_k[U_k], \tag{4.3}$$

we observe that

$$G_i^a \cong G_i$$

for all i, for $1 \leq i \leq k$, if $a \in U$. In other words, each G_i-layer through an a in U is full. However, it is possible that there are vertices $v \in V(G) - U$ for which all layers G_i^v are full.

In Cartesian products, layers are *convex*, and this extends to full layers of all types of hierarchical products. Actually we can say more. Let $G = G_1[U_1] \sqcap \cdots \sqcap G_k[U_k]$, and set $G^{\square} = G_1 \square \cdots \square G_k$. Then, each full layer of G is also a layer of $G^{\square}$, and each shortest path in $G^{\square}$ that is in G is also a shortest path in G.

Furthermore, in a Cartesian product $G_1 \square \cdots \square G_k$, any two layers for the same factor G_i, say G_i^a and G_i^b, are isomorphic via the mapping

$$(x_1, \ldots, x_{i-1}, a_i, x_{i+1}, \ldots, x_k) \mapsto (x_1, \ldots, x_{i-1}, b_i, x_{i+1}, \ldots, x_k).$$

We call it the *natural mapping between layers*. In a Cartesian product, it is a uniquely defined isomorphism, in a hierarchical product, it is an isomorphism if both G_i^a and G_i^b are full, but depends on the embedding of the hierarchical product $G_1[U_1] \sqcap \cdots \sqcap G_k[U_k]$ into

$G_1 \square \cdots \square G_k$. As an example, consider $P_3[v] \sqcap P_3[v]$, where v is the center of P_3. Clearly, there are four different embeddings of $P_3[v] \sqcap P_3[v]$ into $P_3 \square P_3$.

Our proof of Theorem 4.2 uses the order of the factors. While Cartesian multiplication is commutative, two factors $G_i[U_i]$, $G_{i+1}[U_{i+1}]$ in a hierarchical product can only be interchanged under certain conditions, as the following argument shows. Recall that two vertices $(x_1, x_2, \ldots, x_k)$, $(y_1, y_2, \ldots, y_k)$ in the Cartesian product $G_1 \square \cdots \square G_k$ are adjacent if there is an i, for $1 \leq i \leq k$, such that $x_i y_i \in E(G_i)$ and $x_j = y_j$, for $1 \leq j \leq k$, for $j \neq i$. In $G_1[U_1] \sqcap \cdots \sqcap G_k[U_k]$, two vertices x, y are adjacent if there exists an i, for $1 \leq i \leq k$, such that the following two conditions are satisfied:

(i) x, y differ only in coordinate i, for which $x_i \sim y_i$.

(ii) $x_j = y_j \in U_j$ for all j with $i < j \leq k$.

This implies that two factors $G_i[U_i]$ and $G_{i+1}[U_{i+1}]$ commute if they are isomorphic as rooted graphs, or if $U_i = V(G_i)$ and $U_{i+1} = V(G_{i+1})$. In the latter case, we call the factors *Cartesian*.

Proof of Theorem 4.2. Let

$$G[U] = G_1[U_1] \sqcap \cdots \sqcap G_k[U_k]$$

be a prime factorization of the rooted connected graph $G[U]$ via the rooted generalized hierarchical product. We wish to show that it is unique up to isomorphisms of the factors $G_i[U_i]$ and the interchange of neighboring factors $G_i[U_i]$, $G_{i+1}[U_{i+1}]$ if they are isomorphic or if $U_i = V(G_i)$ and $U_{i+1} = V(G_{i+1})$.

If $|U| = 1$, then the factorization is unique by Corollary 2.6. We can thus assume that $|U| > 1$, and use induction with respect to $|G|$. Clearly, the statement is true for $|G| = 2$. Suppose it is true for all connected graphs with fewer vertices than G.

Consider two presentations $G_1[U_1] \sqcap G_*[U_*]$ and $H_1[V_1] \sqcap H_*[V_*]$ of G, where $G_1[U_1]$ and $H_1[V_1]$ are prime first factors. If $G_1[U_1] \cong H_1[V_1]$, then $G_* \cong H_*$. To see this, observe that, for any first factor $G_1[U_1]$ and any full layer G_1^x, the layer G_*^x consists of all vertices $g \in G$ whose nearest neighbor in G_1^x is x, and that this holds whether x is in U or not. Then, we can invoke the induction hypothesis, by which G_* is uniquely factorable. This solves the case when $G_1[U_1] \cong H_1[V_1]$.

For the case when $G_1[U_1] \ncong H_1[V_1]$, we show that $G_1[U_1]$ and $H_1[V_1]$ are interchangeable.

To simplify notation, we shall henceforth not explicitly indicate that the graphs G_i, H_i, G_*, H_* have nonempty root sets U_i, V_i, U_*, V_*.

If $|U_*| = 1$, then G is a rooted hierarchical product and $G_1 \cong H_1$ by Corollary 2.6. Because G_1, H_1 uniquely determine G_*, H_*, we infer that $G_* \cong H_*$, and invoke the induction hypothesis.

Hence, we can assume that $|U_*| > 1$, and $|V_*| > 1$ by symmetry.

Suppose $|H_1^x \cap G_1^x| > 1$ for some $x \in U$. We will show that this implies that neither G_1 nor H_1 are prime.

Let $x \in U$ with the coordinates x_1, x_2 pertaining to the factorization $G_1 \sqcap G^*$. We can assume that $H_1^x \neq G_1^x$ and, first, consider the case $H_1^x \subset G_1^x$, where H_1^x is the H_1-layer through x in the factorization $H_1 \sqcap H^*$.

Because G_*^x and G_1^x are layers of $G_1 \square G_*$, all shortest G-paths from vertices in G_*^x to G_1^x meet G_1^x in x. Hence, all shortest G-paths from vertices in G_*^x to H_1^x meet H_1^x in x, and thus $G_*^x \subset H_*^x$. Let $v \in G_*^x \subset H_*^x$. It has a neighbor g in some H_*^x that is not in G_*^x. It must be in G_1^v, hence, G_1^v must contain a vertex of U, but then $v \in U$ and $H_1^v \subset G_1^v$. As $v \in G_*$ was arbitrarily chosen, all vertices of G_*^x are in U.

If all H_1^v-layers for $v \in G_1^x$ are in G_1^x, then H_1 is a factor of G_1, and G_1 is not prime. If this is not the case, then there must be an H_1^v-layer that meets more than one G_1-layer. Then, H_1 is not prime, because all vertices of G_*^x are in U and H_1^v is convex.

Thus, $H_1^x \notin G_1^x$ and there must be vertices $b \in H_1^x - G_1^x$. Consider the shortest path P in G_*^b from b to $p_{G_1^x}(b)$, and the shortest path Q in G_1^x from $p_{G_1^x}(b)$ to x. Then, $P \cup Q$ is a shortest (b, x)-path in G, and P is in H_1^x, because H_1^x is convex in G. This implies that, for each $b \in H_1^x$, the intersection $H_1^b \cap G_*^b$ meets G_1^x.

Clearly, H_1^x and all G_*^b, for $b \in G$, are full, and if any two $H_1^a \cap G_*^a$ and $H_1^b \cap G_*^b$ are isomorphic, then $H_1^x \cap G_1^x$ divides G_1^x. Then, G_1 is not prime.

Hence, we can assume that there exist two vertices a, b in the same G_1-layer, where $a \in H_1^x$, $b \notin H_1^x$, and, where G_*^a and G_*^b are at distance 1. Of course, G_1^a is empty in this case, and we can suppose that b has minimal distance from G_1^x amongst all vertices with these properties.

Let P be a shortest path from a to $r = G_1^x \cap G_*^a$, and Q the path from b to $s \in G_1^x \cap G_*^b$ that corresponds to P via the natural isomorphism of the layers G_*^a and G_*^b. Let d be the neighbor of b in Q and c the corresponding vertex in P. Then, $b \in H_1^x$.

As $|H_*| \geq |G_*|$, the layer H_*^d cannot be contained in G_*^d. Hence, there must be a full G_1-layer, say G_1^f, that meets H_*^d. We can choose the notation such that $f \in H_*^b$. We choose an f of shortest distance from b and let $e = G_1^f \cap G_*^a$. By convexity, f is not closer to G_1^x than d. Let Q' be the path in H_*^b to f. Clearly, it is also in G_*^b and, if P' is the corresponding path in G_*^a, then it is in H_*^a. Hence, the edge ef connects two H_*-layers and must be in H_1^e. As $c, d \in H_1^x$, this is only possible if $s = c$ and $r = d$.

Now we consider H_*^b. Similarly to the previous argument, there is a G_1^h of minimal distance from b that contains H_*^b. We choose the notation such that $h \in H_*^b$ and let $g = G_1^h \cap G_*^a$. Note that h and f are in different H_*-layers, and thus each shortest (h, d)-path contains b and, therefore, $d(h, d) = d(h, b) + 1$. Because ac is an edge not in H_1^x, the layer H_*^a, which contains g, is H_*^c. As before, we see that $gh \in H_1^g$ and $d(g, c) = d(h, d)$. Then, the shortest distance from g to H_1^x is $d(g, c) = d(h, d) = d(h, b) + 1$, whereas the shortest distance from h to H_1^x is $d(h, b)$, which is not possible.

Therefore, it suffices to treat the case where $|H_1^x \cap G_1^x| = 1$. If H_1^x contains a vertex $w \in G_*^a \neq G_*^x$, then $d_G(x, w) = d_{G_1}(x_1, a_1) + d_{G_*}(x_2, w_2)$, and $(a_1, x_2) \in H_1^x$ by the convexity of H_1^x, but it is also in G_1^x and different from x, contrary to the assumption. Hence, $H_1^x \subseteq G_*^x$. By the same argument, $G_1^x \subseteq H_*^x$.

Note that we still have two coordinatizations of each vertex of G, one pertaining to the representation $G_1 \sqcap G_*$, and one pertaining to the representation $H_1 \sqcap H_*$. To be more precise, let $a \in G_1^x$ and $\varphi_{x,a}$ be the natural isomorphism from G_*^x to G_*^a. Because $H_1^x \subseteq G_*^x$, its image $\varphi_{x,a}(H_1^x)$ is in G_*^a, but we do not know yet whether it is H_1^a. However, if $a \in U$, then this is the case by the same argument that we used to show that $H_1^x \subset G_*^x$. Similarly, if $b \in H_1^x$, and $\psi_{x,b}$ is the natural isomorphism between H_*^x and H_*^b, then $\psi_{x,b}(G_1^x)$ is in H_*^b and is isomorphic to G_1^x. As such, it need not be equal to G_1^b, unless $b \in U$. This implies that any two vertices w, z coincide if $p_{G_1^x}(w) = p_{H_*^x}(z)$ and $p_{H_1^x}(w) = p_{G_*^x}(z)$. In other words, they have the same coordinates in the representations $G_1 \sqcap G_*$ and $H_1 \sqcap H_*$.

We now show that all vertices of $G_1^x \cup H_1^x$ are in U. We begin with H_1^x and assume $|H_*| \geq |G_*|$. Let $x \neq a$, with $a \in H_1^x$, be a vertex that is not in $G_*^x \cup H_*^x$. Because $x \in G_*^x - H_*^a$, there must be a vertex $b \in H_*^a$ that is not in G_*^a. Let $P_{a,b}$ be a shortest (a,b)-path in H_*^a, and $P_{a,d}$ be the longest subpath of $P_{a,b}$ that is in G_*^x. Let e be the neighbor of d in $P_{a,b}$ that is not in $P_{a,d}$. Clearly, $G_*^e \neq G_*^x$ and G_1^e is full. Then, it must contain a vertex u that is in U. Its projection into G_*^x is d and must also be in U. But d is also in H_*^x. As a is in H_1^x with respect to the coordinates of $H_1 \sqcap H_*$, we infer that $a \in U$.

Because $V(H_1^x) \subseteq U$, all layers G_1^a are full for $a \in H_1^x$. Let w be a neighbor of x in G_1, and φ be the natural isomorphism from G_*^x to G_*^w. Then, $a \sim \varphi(a)$ for all $a \in H_1^x$, and $\varphi(H_1^x) = H_1^w$. This is only possible if some $z \in H_1^w$, which is in G_*^w, is in U. Then, the projection of z to G_1^w, which is w, is in U. Because G_1^x is connected, it is clear that all vertices of G_1^x are in U.

This means that we can consider $G_1[V(G_1)] \sqcap H_1[V(H_1)]$ as a subgraph of G, and identify it with $G_1^x \sqcap H_1^x$, but it still remains to show that $G_1 \sqcap H_1$ is a factor of G. To this end, we first prove that, to each vertex $z \in G$, there is exactly one vertex of shortest distance to z in $G_1^x \sqcap H_1^x$.

Recall that in a generalized hierarchical product $A \sqcap B$, each (w,z)-path P in $A \sqcap B$ is a shortest path in $A \sqcap B$ if it is a shortest path in $A \,\square\, B$. By the Distance Formula, see Hammack, Imrich, and Klavžar [7, Corollary 5.2], the distance $d_{A\square B}((p_A(w), p_B(w)), (p_A(z), p_B(z)))$ between two vertices w and z in $A \square B$ is $d_A(p_A(w), p_A(z)) + d_B(p_B(w), p_B(z))$. In our case, $A = G_1$ and $B = G_*$, and G_1^z is full. Hence, there is a shortest (w,z)-path in G that consists of the union of a shortest path in G_*^w from w to $w' = G_*^w \cap G_1^z$ with a shortest path in G_1^z from w' to z. Because w' is also in $G_1^x \sqcap H_1^x$, the vertex z can only be of shortest distance from $G_1^x \sqcap H_1^x$ if it is in G_*^w. By the same argument, z must also be in H_*^w. Now, we observe that $z \in H_1^z \subseteq G_*^z = G_*^w$ and $z \in G_1^z \subseteq H_*^z = H_*^w$. Hence, $z \in H_1^z \cap G_1^z$, and because $H_1^z \cap G_1^z$ consists of only one vertex, z is unique.

This means that the set of all vertices in G whose nearest neighbor in $G_1^x \sqcap H_1^x$ is z is $G_*^z \cap H_*^z$. Setting $a = H_1^z \cap G_1^x$ and $b = G_1^z \cap H_1^x$, we can also consider the pair (a,b) as the coordinates of z in $G_1^x \sqcap H_1^x$. Consider the intersections $G_*^a \cap H_*^b$ for all such pairs (a,b). Any two of them are isomorphic via the natural isomorphism between full layers, although $\psi_{x,b}(\varphi_{x,a}(w))$ may be different from $\varphi_{x,b}(\psi_{x,a}(w))$ for $w \in G_*^x \cap H_*^x$. We can choose any of these embeddings to embed

$$G \cong G_1^x[V(G_1^x)] \sqcap H_1^x[V(H_1^x)] \sqcap (G_*^x \cap H_*^x)[U \cap G_*^x \cap H_*^x]$$

into $G_1 \,\square\, H_1 \,\square\, (G_*^x \cap H_*^x)$. We wish to note that all vertices of G_1^u and H_1^u are in U if $u \in U$. It follows by the same argument that we used for G_1^x and H_1^x. Then, all vertices of $G_1^u \sqcap H_1^u$ are in U, and $\psi_{x,b}(\varphi_{x,a}(u)) = \varphi_{x,b}(\psi_{x,a}(u))$.

If G has other first factors that are prime, they commute with G_1 and H_1. We can thus collect all first factors $G_1, \dots, G_f$. We call their product Q. As this is a Cartesian product, there is no other representation of Q as a product of prime factors. As before, we can prove that Q is a factor, and there is a uniquely defined $Q_* \subseteq G$ with $G = Q[V(Q)] \sqcap Q_*[U \cap V(Q_*)]$.

By the induction hypothesis, Q_* is uniquely representable as a product of prime graphs and, hence, also G. □

4.3 Automorphisms of rooted generalized hierarchical products

Let $G = G_1[U_1] \sqcap G_2[U_2]$ be a rooted generalized hierarchical product, where $G_1[U_1]$ is the power of a prime graph $H[U_H]$, and G has no other first factor isomorphic to $H[U_H]$, or $G_1[U_1]$ is the product of all prime first Cartesian factors. Then, each automorphism α of G has the following properties:

(1) The restriction of α to G_1^u, where $u \in U_1 \times U_2$, preserves $U_1 \times u_2$.

(2) α maps each layer G_2^a into the layer $G_2^{\alpha(a)}$.

(3) The restriction α^a of α to G_2^a maps $a_1 \times U_2$ into $\alpha(a)_1 \times U_2$. It thus induces a mapping $(\alpha^a)'$ of U_2 into U_2, and if b is another vertex of G, then $(\alpha^b)' = (\alpha^a)'$.

We denote this group by $\mathrm{Aut}(G_1[U_1]) \ltimes_{U_2} \mathrm{Aut}(G_2[U_2])$.

Furthermore, let $G_1[U_1]$ be the product $H_1[V_1] \sqcap H_2[V_2] \sqcap \cdots \sqcap H_r[V_r]$, where H_j are prime and $V_j = V(H_j)$ for $1 \le j \le r$. In this case, the hierarchical product coincides with the Cartesian one. Therefore, by Hammack, Imrich, and Klavžar [7, Theorem 6.10], if $\varphi \in \mathrm{Aut}(G_1[U_1])$, then there exists a permutation π of $\{1, 2, ..., k\}$, together with isomorphisms $\varphi_i\colon H_{\pi(i)} \to H_i$, such that for each $x \in G_1$

$$\varphi(x_1, x_2, ..., x_r) = (\varphi_1(x_{\pi(1)}), \varphi_2(x_{\pi(2)}), ..., \varphi_r(x_{\pi(r)})).$$

With this definition, it is easily seen that the following theorem holds.

Theorem 4.3. *Let G be a finite connected graph with factorization $G[U] = G_1[U_1] \sqcap \cdots \sqcap G_k[U_k]$, where each $G_i[U_i]$ is either the highest power of a prime factor, or the product of a maximal number of Cartesian factors. Then*

$$\mathrm{Aut}(G) = \mathrm{Aut}(G_1[U_1]) \ltimes_{U_2} \mathrm{Aut}(G_2[U_2]) \ltimes_{U_3} \cdots \ltimes_{U_k} \mathrm{Aut}(G_k[U_k]).$$

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