

On asymmetric colourings of graphs with bounded degrees and infinite motion

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Abstract

A vertex colouring of a graph is called asymmetric if the only automorphism which preserves it is the identity. Tucker conjectured that if every automorphism of a connected, locally finite graph moves infinitely many vertices, then there is an asymmetric colouring with 2 colours. This conjecture was recently confirmed by Babai, using the heavy machinery of the classification of finite simple groups. We make progress towards a purely combinatorial proof of this conjecture in the special case of graphs with bounded maximal degree. More precisely, using only elementary combinatorial methods, we prove that if every automorphism of a connected graph with maximal degree Δ moves infinitely many vertices, then there is an asymmetric colouring using $\mathcal{O}(\sqrt{\Delta} \log \Delta)$ colours. This is the first improvement which does not depend on the classification of finite simple groups over the trivial bound of $\mathcal{O}(\Delta)$.

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1 Introduction

Call a (not necessarily proper) vertex colouring of a graph *asymmetric* if the only automorphism which preserves it is the identity. This notion was first introduced by Babai [2] in 1977, who proved that 2 colours suffice for an asymmetric colouring of every regular tree. The concept was later reintroduced by Albertson and Collins [1] under the name *distinguishing* colouring, and since has received a considerable amount of attention.

In this paper we investigate connections between the maximal degree Δ of a graph and the number of colours needed for an asymmetric colouring. It is not hard to see that every connected finite graph has an asymmetric colouring with at most $\Delta + 1$ colours, and it was shown independently by Collins and Trenk [7], and Klavžar, Wong, and Zhou [11] that the only graphs which attain this bound are complete graphs, complete bipartite graphs, and the cycle on 5 vertices. In 2007, Imrich, Klavžar and Trofimov [9] extended this result to infinite graphs, showing that every connected infinite graph with maximal degree Δ has an asymmetric colouring with at most Δ colours. In [15], the authors of the present paper improved this bound to $\Delta - 1$ for infinite graphs with maximal degree at least 3 and this bound is sharp for every $\Delta \geq 3$.

It is worth noting that all known examples achieving this bound have an automorphism moving only few vertices. This motivates the concept of *motion*. We say that a graph has motion m , if the minimal number of vertices moved by a non-trivial automorphism is m . Motion turns out to be a powerful tool for bounding the number of colours needed in an asymmetric colouring. It is often used in the form of the so-called Motion Lemma, which states that if G is finite graph with motion m such that $2^{\frac{m}{2}} \geq |\text{Aut } G|$, then G has an asymmetric colouring with 2 colours. Perhaps the first to explicitly state this result were Russell and Sundaram [16], but it is worth pointing out that Cameron, Neumann, and Saxl [5] implicitly use a generalization to permutation groups to show that primitive permutation groups admit an asymmetric colouring with 2 colours, unless they are symmetric groups, alternating groups, or one of finitely many exceptions. The following conjecture due to Tucker [17] can be seen as a generalization of the Motion Lemma to infinite graphs.

Conjecture 1.1 (Tucker [17]). *Let G be a connected, locally finite graph with infinite motion. Then there is an asymmetric colouring of G with 2 colours.*

After the first preprint of the present paper was published, this conjecture was confirmed by Babai [4]. However, it is worth pointing out that the proof depends on the classification of finite simple groups; Babai poses the question whether the conjecture can also be proved by purely combinatorial methods.

Despite numerous partial results (see for example [10, 12, 13, 14, 18]) a combinatorial proof of Conjecture 1.1 remains elusive. In particular, no such proof is known if we restrict ourselves to graphs with finite maximum degree. Recently, Hüning et al. [8] provided such a proof for graphs with maximum degree at most 3, in [15] this was extended to graphs with maximum degree at most 5, and by using similar arguments one can show that there is an asymmetric colouring for connected graphs with infinite motion with at most $\frac{\Delta}{3} + 1$ colours. We note that, although this is a substantial improvement over the trivial bound of $\Delta + 1$ it is still linear in Δ .

In this short note we show by elementary combinatorial methods that a graph with infinite motion and maximum degree Δ admits an asymmetric colouring with $(\sqrt{\Delta} + 1)(2 \log_2 \Delta + 3) = \mathcal{O}(\sqrt{\Delta} \log \Delta)$ colours. To our best knowledge, this is the

first combinatorial proof of a bound which is sublinear in Δ thus constituting a significant step towards an elementary proof of Conjecture 1.1 for graphs with finite maximum degree.

An anonymous referee suggested that the conclusion of our main result remains true if instead of infinite motion we assume large enough finite motion. While our techniques do not seem to be strong enough to prove this stronger statement, this leads to an interesting question for further research.

Question 1.2. Are there functions $f: \mathbb{N} \rightarrow \mathbb{N}$ with $f(n) = o(n)$, and $g: \mathbb{N} \rightarrow \mathbb{N}$ such that every graph with maximum degree Δ and motion at least $g(\Delta)$ has an asymmetric colouring with $f(\Delta)$ colours? If yes, what is the relation between f and g ? What is the smallest possible function f ?

2 Definitions and auxiliary results

Throughout this paper G will denote an infinite, connected graph with vertex set V , edge set E , and finite maximal degree Δ . Let d denote the usual geodesic distance between vertices of G . For a vertex $v_0 \in V$ denote the *ball* with centre v_0 and radius k by $B(v_0, k) = \{v \in V \mid d(v_0, v) \leq k\}$, and the *sphere* with centre v_0 and radius k by $S(v_0, k) = \{v \in V \mid d(v_0, v) = k\}$. As usual, we write $N(v)$ for the set of neighbours of a vertex v .

For a set Ω , denote by $\text{Sym}(\Omega)$ the group of all bijective functions from Ω to itself. Further denote by $\text{Sym}_n := \text{Sym}(\{1, \dots, n\})$.

Let Γ be a group acting on a set Ω . For $S \subseteq \Omega$, denote the *setwise stabilizer* of S by $\Gamma_S = \{\gamma \in \Gamma \mid \forall s \in S: \gamma s \in S\}$ and denote the *pointwise stabilizer* of S by $\Gamma_{(S)} = \{\gamma \in \Gamma \mid \forall s \in S: \gamma s = s\}$. Observe that $\Gamma_{(\Omega)}$ is a normal subgroup of Γ , but the same isn't necessarily true for $\Gamma_{(S)}$ for $S \subsetneq \Omega$. In particular, it makes sense to speak about the quotient $\Gamma/\Gamma_{(\Omega)}$, which is isomorphic to the group of all different permutations induced by Γ on Ω . The *stabilizer* of a colouring c of Ω is defined by $\Gamma_c = \{\gamma \in \Gamma \mid \forall v \in \Omega: c(v) = c(\gamma v)\}$. A colouring c of Ω is called *asymmetric* if $\Gamma_c = \Gamma_{(\Omega)}$. Note that in the special case of $\text{Aut } G$ acting on the vertices of G , a vertex colouring is asymmetric if and only if its stabilizer only contains the identity automorphism.

The proof of our main result relies on the following result proved in [3]. We point out that unlike the proof of the optimal bound $\frac{3n}{2}$ in [6], the proof of this result does not depend on the classification of finite simple groups.

Lemma 2.1. *The length of a chain of subgroups in Sym_n is at most $2n$.*

The following corollary to Lemma 2.1 is one of our main tools.

Lemma 2.2. *Let Γ be a group acting on two non-empty finite sets Ω and Ω' . If $\Gamma_{(\Omega')} \leq \Gamma_{(\Omega)}$, then there is $S \subseteq \Omega$ such that $|S| \leq 2|\Omega'|$, and $\Gamma_{(S)} = \Gamma_{(\Omega)}$.*

Proof. If Γ acts trivially on Ω we can choose $S = \emptyset$. Otherwise, pick $s_1 \in \Omega$ with $\Gamma_{(s_1)} \subsetneq \Gamma$ and set $S_1 = \{s_1\}$. If $\Gamma_{(S_1)} \neq \Gamma_{(\Omega)}$, we can inductively pick $s_{i+1} \in \Omega$ which is not stabilized by $\Gamma_{(S_i)}$ and set $S_{i+1} = S_i \cup \{s_{i+1}\}$. This process terminates when $\Gamma_{(S_k)} = \Gamma_{(\Omega)}$.

Clearly, $\Gamma_{(\Omega')} \leq \Gamma_{(\Omega)} \leq \Gamma_{(S_i)} \leq \Gamma$, and since $\Gamma_{(\Omega')} \trianglelefteq \Gamma$ we also have $\Gamma_{(\Omega')} \trianglelefteq \Gamma_{(S_i)}$. In particular, we have a chain of subgroups

$$\text{Sym}(\Omega') \geq \Gamma/\Gamma_{(\Omega')} \geq \Gamma_{(S_1)}/\Gamma_{(\Omega')} \geq \dots \geq \Gamma_{(S_k)}/\Gamma_{(\Omega')},$$

and thus $k \leq 2|\Omega'|$ by Lemma 2.1. □

3 Proof of the main result

Theorem 3.1. *Every connected graph with infinite motion and finite maximal degree Δ has an asymmetric colouring with at most $(\sqrt{\Delta} + 1)(2 \log_2 \Delta + 3)$ colours.*

Proof. We will inductively define colourings c_k using colours in the set $\{0, 1, 2, 3, \dots\} \cup \{\bar{1}, \bar{2}, \bar{3}, \dots\} \cup \{\infty\}$. This set is infinite, but we will later determine bounds on the number of colours that we actually use.

Choose an arbitrary root $v_0 \in V$. The colouring c_k will satisfy the following properties for every $k \geq 0$.

- (a) $c_k(v) = 0$ if and only if $v = v_0$,
- (b) $c_k(v) = \infty$ if and only if $d(v, v_0) > k$,
- (c) $c_k|_{B(v_0, k-1)} = c_{k-1}|_{B(v_0, k-1)}$ if $k > 0$,
- (d) Let Γ_k be the stabilizer of c_k in $\Gamma = \text{Aut } G$, then
 - $\Gamma_k \leq \Gamma_{(B(v_0, k-1))}$ if $k > 0$,
 - all orbits under Γ_k in $B(v_0, k)$ have size at most $\lceil \sqrt{\Delta} \rceil$.

Note that (a) implies that every automorphism which preserves c_k must fix $B(v_0, k)$ setwise, and thus it makes sense to speak of orbits of Γ_k in $B(v_0, k)$.

3.1 An asymmetric limit colouring

Before we turn to the construction of the colourings c_k , let us show that they yield an asymmetric colouring of G . By (c) we can define a colouring $c_\infty = \lim_{k \rightarrow \infty} c_k$ by $c_\infty|_{B(v_0, k)} = c_k|_{B(v_0, k)}$ for every k . By (a), the only vertex with $c_\infty(v) = 0$ is v_0 , hence v_0 is fixed by every automorphism preserving c_∞ . Consequently, $B(v_0, k)$ is fixed setwise by each colour preserving automorphism. Since $c_\infty|_{B(v_0, k)} = c_{k+1}|_{B(v_0, k)}$, property (d) implies that every automorphism in the stabilizer of c_∞ fixes $B(v_0, k)$ pointwise. This is true for every k whence c_∞ is asymmetric.

3.2 Inductive construction

In this section, we inductively construct colourings c_k satisfying (a) to (d). For $k = 0$ let $c_0(v_0) = 0$ and $c_0(v) = \infty$ for $v \neq v_0$. Properties (a) to (d) are trivially satisfied for this colouring.

For the inductive definition of c_{k+1} , assume that we already have defined c_k with the desired properties. By (b) and (c), the only vertices where c_{k+1} and c_k can differ are those in $S(v_0, k+1)$, hence it suffices to describe the colouring c_{k+1} on $S(v_0, k+1)$.

Recall that by (a), the stabilizer Γ_k of c_k must fix $S(v_0, k)$ and $S(v_0, k+1)$ setwise. Let $A_1, \dots, A_n$ be the orbits of Γ_k on $S(v_0, k)$. Define equivalence relations R_i for $0 \leq i \leq n$ on $S(v_0, k+1)$ by

$$xR_iy \iff \forall j \leq i: N(x) \cap A_j = N(y) \cap A_j,$$

see Figure 1 for an illustration.

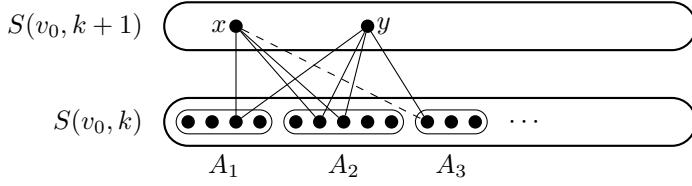


Figure 1: An example illustrating the equivalence relations R_i . The vertices x and y have the same neighbours in A_1 and A_2 , so xR_1y and xR_2y , but y has a neighbour in A_3 which is not adjacent to x , so xR_iy for every $i \geq 3$.

Note that the condition is void for $i = 0$, hence all vertices are equivalent under R_0 . Further note that R_i has at most one equivalence class of size larger than Δ , namely the class of vertices which have no neighbours in A_j for any $j \leq i$. All other equivalence classes are contained in the neighbourhood of some vertex and thus have size at most Δ . This argument also shows that all equivalence classes with respect to R_n have size at most Δ since every vertex in $S(v_0, k+1)$ has a neighbour in $S(v_0, k) = \bigcup_{j \leq n} A_j$.

Denote by $\mathcal{B}_i$ the set of equivalence classes with respect to the relation R_i . Observe that $\mathcal{B}_{i+1}$ is a refinement of $\mathcal{B}_i$, that is, for every $B \in \mathcal{B}_{i+1}$ there is a unique $B' \in \mathcal{B}_i$ with $B \subseteq B'$.

Let xR_iy and $\phi \in \Gamma_k$. Then $\phi xR_i\phi y$ because ϕ maps neighbours of x and y to neighbours of ϕx and ϕy while fixing every A_j setwise. This shows that ϕ permutes the equivalence classes in $\mathcal{B}_i$, and hence Γ_k in its natural action on subsets of $S(v_0, k+1)$ preserves the partition $\mathcal{B}_i$, thereby inducing a natural action of Γ_k on $\mathcal{B}_i$. Let $\Gamma_{k,i} := (\Gamma_k)_{(\mathcal{B}_i)}$, in other words, $\Gamma_{k,i}$ is the pointwise stabilizer of $\mathcal{B}_i$ in Γ_k with respect to this action.

Before we turn to the construction of c_{k+1} , we need one last definition. For every colouring $c: S(v_0, k+1) \rightarrow \mathbb{N}$ we define the *induced colouring* $c[\mathcal{B}_i]: \mathcal{B}_i \rightarrow \mathbb{N}$ by

$$c[\mathcal{B}_i](B) = \min_{v \in B} c(v).$$

We note that if $c: S(v_0, k+1) \rightarrow \mathbb{N}$ is a colouring, then $(\Gamma_k)_c \leq \bigcap_{1 \leq i \leq n} (\Gamma_k)_{c[\mathcal{B}_i]}$ due to the definition of induced colourings and the fact that Γ_k acts on $\mathcal{B}_i$.

We now inductively define colourings $c_{k,i}$ of $S(v_0, k+1)$ for $0 \leq i \leq n$ satisfying the following properties.

- (i) $c_{k,i}[\mathcal{B}_j] = c_{k,i-1}[\mathcal{B}_j]$ for $i \geq 1$ and $j < i$.
- (ii) Every $B \in \mathcal{B}_i$ is monochromatic under $c_{k,i}$.
- (iii) $\tilde{\Gamma}_{k,i} := \bigcap_{j \leq i} (\Gamma_k)_{c_{k,i}[\mathcal{B}_j]} \leq \Gamma_{k,i}$.

Define $c_{k,0} \equiv 1$. The statement (i) is vacuously true for $i = 0$, and properties (ii) and (iii) are readily verified.

For the inductive definition of $c_{k,i+1}$ let us assume that we already defined $c_{k,i}$ with the desired properties. Recall that $\Gamma_{k,i} = (\Gamma_k)_{(\mathcal{B}_i)}$. Note that if two vertices in $S(v_0, k+1)$ are contained in the same orbit with respect to $\Gamma_{k,i}$, then they are equivalent with respect to R_i . Therefore two such vertices can only be inequivalent with respect to R_{i+1} , if they have different neighbours in A_{i+1} . This shows that the pointwise stabilizer of A_{i+1} in

$\Gamma_{k,i}$ fixes $\mathcal{B}_{i+1}$ pointwise, and hence $(\Gamma_{k,i})_{(A_{i+1})} \leq \Gamma_{k,i+1}$. Since $\tilde{\Gamma}_{k,i} \leq \Gamma_{k,i}$ we have $(\tilde{\Gamma}_{k,i})_{(A_{i+1})} \leq \tilde{\Gamma}_{k,i} \cap \Gamma_{k,i+1} \leq (\tilde{\Gamma}_{k,i})_{(\mathcal{B}_{i+1})}$. Therefore, by Lemma 2.2 there is a subset $\mathcal{S} \subseteq \mathcal{B}_{i+1}$ such that

$$|\mathcal{S}| \leq 2|A_{i+1}|$$

and

$$(\tilde{\Gamma}_{k,i})_{(\mathcal{S})} = (\tilde{\Gamma}_{k,i})_{(\mathcal{B}_{i+1})}.$$

Note that this implies that $|\mathcal{S}| \leq 2\lceil\sqrt{\Delta}\rceil$ because by induction hypothesis $|A_{i+1}| \leq \lceil\sqrt{\Delta}\rceil$.

Let $\mathcal{S} = \{S_1, \dots, S_\ell\}$ be a subset of $\mathcal{B}_{i+1}$ of minimal cardinality satisfying $(\tilde{\Gamma}_{k,i})_{(\mathcal{S})} = (\tilde{\Gamma}_{k,i})_{(\mathcal{B}_{i+1})}$. For $j \in \{1, \dots, \ell\}$ and each $v \in S_j$ let $c_{k,i+1}(v) = c_{k,i}(v) + j$. For all other vertices set $c_{k,i+1}(v) = c_{k,i}(v)$. Property (ii) for $c_{k,i+1}$ trivially follows from (ii) for $c_{k,i}$. It remains to show that $c_{k,i+1}$ satisfies (i) and (iii).

For (i), let $j < i$ and recall that every $B_i \in \mathcal{B}_i$ can be written as a disjoint union of members of $\mathcal{B}_{i+1}$. Since $c_{k,i}$ satisfies (ii), for every $B_j \in \mathcal{B}_j$ there is some $B_i \in \mathcal{B}_i$ with $B_i \subseteq B_j$ such that all vertices in B_i are coloured with colour $c_{k,i}B_j$. It thus suffices to show that for every $B_i \in \mathcal{B}_i$ there is some $B \in \mathcal{B}_{i+1}$ such that $B_i \supseteq B \notin \mathcal{S}$.

Assume for a contradiction that there is some $B_i \in \mathcal{B}_i$ for which every $B \in \mathcal{B}_{i+1}$ with $B \subseteq B_i$ is contained in $\mathcal{S}$. Pick $B \in \mathcal{S}$ with $B \subseteq B_i$ and let $\mathcal{S}' = \mathcal{S} \setminus \{B\}$. Minimality of $\mathcal{S}$ implies that there is some $\gamma \in (\tilde{\Gamma}_{k,i})_{(\mathcal{S}')}$ which acts non-trivially on $\mathcal{B}_{i+1}$. Since (iii) holds for $\tilde{\Gamma}_{k,i}$ we have that $\gamma B \subseteq B_i$. If $\gamma B = B$, then $\gamma \in (\tilde{\Gamma}_{k,i})_{(\mathcal{S})}$ which is impossible. Otherwise, $\gamma B \in \mathcal{S}'$, and therefore γB is fixed by $\gamma \in (\tilde{\Gamma}_{k,i})_{(\mathcal{S}')}$ which is also impossible.

Note that by minimality of $\mathcal{S}$, if $B \in \mathcal{S}$, then B is not stabilised by $\tilde{\Gamma}_{k,i}$. In particular if $B_i \in \mathcal{B}_i$ with $B_i \supseteq B \in \mathcal{S}$, then there is some $\gamma \in \tilde{\Gamma}_{k,i}$ such that $B \neq \gamma B \subseteq B_i$ and hence $|B| \leq \frac{|B_i|}{2}$. This fact will be used later to bound the number of used colours.

For the proof of (iii), note that by (i) we have $(\Gamma_k)_{c_{k,i}[B_j]} = (\Gamma_k)_{c_{k,i+1}[B_j]}$ for every $j \leq i$, and consequently

$$\tilde{\Gamma}_{k,i+1} = \tilde{\Gamma}_{k,i} \cap (\Gamma_k)_{c_{k,i+1}[\mathcal{B}_{i+1}]} = (\tilde{\Gamma}_{k,i})_{c_{k,i+1}[\mathcal{B}_{i+1}]},$$

where the second equality simply uses the fact that $\tilde{\Gamma}_{k,i} \leq \Gamma_k$. Further note that by definition of $\mathcal{S}$ we have $(\tilde{\Gamma}_{k,i})_{(\mathcal{S})} \leq (\Gamma_k)_{(\mathcal{B}_{i+1})} = \Gamma_{k,i+1}$. So all we need to show is that

$$(\tilde{\Gamma}_{k,i})_{c_{k,i+1}[\mathcal{B}_{i+1}]} \leq (\tilde{\Gamma}_{k,i})_{(\mathcal{S})},$$

that is, every element of $\tilde{\Gamma}_{k,i}$ which preserves $c_{k,i+1}[\mathcal{B}_{i+1}]$ must fix $\mathcal{S}$ pointwise.

Let $B \in \mathcal{S}$, let $\gamma \in (\tilde{\Gamma}_{k,i})_{c_{k,i+1}[\mathcal{B}_{i+1}]}$ and let $B_i \in \mathcal{B}_i$ such that $B \subseteq B_i$. Since $c_{k,i}$ satisfies (iii) we know that $\gamma \in \tilde{\Gamma}_{k,i}$ and thus $\gamma B \subseteq B_i$. Since $B \in \mathcal{S}$ we have (by (ii) for $c_{k,i}$ and the definition of $c_{k,i+1}$) that the colour of the vertices of B is different from the colour with respect to $c_{k,i+1}$ of the vertices in every other subclass of B_i , and hence $\gamma B = B$.

We obtain c_{k+1} from $c_{k,n}$ as follows. Split every equivalence class in $\mathcal{B}_n$ into at most $\lceil\sqrt{\Delta}\rceil$ subsets of size at most $\lceil\sqrt{\Delta}\rceil$. This is possible because every equivalence class with respect to R_n contains at most Δ vertices. Now for each $B \in \mathcal{B}_n$, one of the subclasses keeps the same colour as in $c_{k,n}$, while the others are recoloured with distinct colours in $\overline{1}, \overline{2}, \dots, \overline{\lceil\sqrt{\Delta}\rceil}$.

Finally, we need to check that c_{k+1} has the desired properties. By construction, it satisfies (a), (b), and (c). These properties also immediately imply that $\Gamma_{k+1} \leq \Gamma_k$, or more precisely that $\Gamma_{k+1} = (\Gamma_k)_{c_{k+1}}$. The colouring c_{k+1} was constructed from $c_{k,n}$ in a way that every $B \in \mathcal{B}_n$ still contains some vertices with the same colour as in $c_{k,n}$. Hence, every element of Γ_k which preserves c_{k+1} must also preserve $c_{k,n}$, whence $\Gamma_{k+1} \leq \Gamma_{k,n}$ by (iii) for $c_{k,n}$.

In particular, Γ_{k+1} setwise fixes all equivalence classes with respect to R_n , and by construction of c_{k+1} from $c_{k,n}$ we conclude that there are no orbits with respect to Γ_{k+1} of size more than $\lceil \sqrt{\Delta} \rceil$ in $S(v_0, k+1)$. The second part of (d) follows by induction.

For the first part of (d), assume for a contradiction that there is $\gamma \in \Gamma_{k+1}$ fixing setwise every $B \in \mathcal{B}_n$ but not fixing $B(v_0, k)$ pointwise. Thus the neighbours of v and γv in $B(v_0, k)$ coincide for every $v \in B(v_0, k+1)$. Hence, we can define an automorphism γ' of G by

$$\gamma'v = \begin{cases} v & \text{if } v \in B(v_0, k), \\ \gamma v & \text{otherwise.} \end{cases}$$

This contradicts infinite motion, as $\gamma^{-1}\gamma'$ only moves vertices inside $B(v_0, k)$. Thus, we have proved that every element of Γ_{k+1} fixes $B(v_0, k)$ pointwise.

Note that in this last argument finite motion (no matter how large) is not enough since the number of vertices in $B(v_0, k)$ is unbounded. Indeed, this is the only part of the proof requiring infinite motion; in particular, replacing this argument by something that works under the assumption of large finite motion would lead to an answer to Question 1.2.

3.3 Counting the colours

To determine the number of colours used in the colouring procedure, first note that at most $\lceil \sqrt{\Delta} \rceil$ colours from the set $\{\bar{1}, \bar{2}, \dots\}$ are used. It only remains to determine the number of colours from $\mathbb{N}$ used throughout the procedure.

Let v be a vertex with $c_\infty(v) \in \mathbb{N}$. Assume that $v \in S(v_0, k+1)$, and as before, let $A_1, \dots, A_n$ be the orbits of Γ_k on $S(v_0, k)$. By construction, $c_\infty(v) = c_{i,n} = \max_{i \leq n} (c_{k,i}(v))$, so it suffices to check how large $c_{k,i}(v)$ can get.

By definition $c_{k,0}(v) = 1$. For $0 \leq i \leq n$, let $B_i \in \mathcal{B}_i$ be the equivalence class with respect to R_i containing v , and let $\mathcal{S}_i$ be the set of blocks in $\mathcal{B}_{i+1}$ whose colour with respect to $c_{k,i+1}$ differs from their colour with respect to $c_{k,i}$, that is,

$$\mathcal{S}_i = \{B \in \mathcal{B}_{i+1} \mid c_{k,i}[\mathcal{B}_{i+1}](B) \neq c_{k,i+1}[\mathcal{B}_{i+1}](B)\}.$$

By construction,

$$c_{k,i+1}[\mathcal{B}_{i+1}](B) - c_{k,i}[\mathcal{B}_{i+1}](B) \leq |\mathcal{S}_i| \leq 2\lceil \sqrt{\Delta} \rceil < 2\sqrt{\Delta} + 1$$

for every $B \in \mathcal{S}_i$. So if $c_{k,i}(v) \neq c_{k,i+1}(v)$, then $B_{i+1} \in \mathcal{S}_i$, and

$$c_{k,i+1}(v) < c_{k,i}(v) + 2\sqrt{\Delta} + 1.$$

It only remains to determine an upper bound for the number of steps i for which $B_{i+1} \in \mathcal{S}_i$. Note that for each such i , there must be some $\gamma \in \Gamma_{k,i}$ such that $B_{i+1} \neq \gamma B_{i+1} \in \mathcal{B}_{i+1}$.

For the minimal i such that $B_{i+1} \in \mathcal{S}_i$, this implies that B_{i+1} is not the equivalence class with no neighbours in A_j for any $j \leq i$. Thus $|B_{i+1}| \leq \Delta$. For every subsequent

such i , we have that B_{i+1} and γB_{i+1} are subclasses of B_i with $|B_{i+1}| = |\gamma B_{i+1}|$, so $|B_{i+1}| \leq \frac{|B_i|}{2}$. This implies that $B_{i+1} \in \mathcal{S}$ for at most $(1 + \log_2 \Delta)$ many values of i .

Summing up, we get that

$$c_{k+1}(v) = c_{k,n}(v) \leq 1 + (2\sqrt{\Delta} + 1)(1 + \log_2 \Delta) < 2(\sqrt{\Delta} + 1)(\log_2 \Delta + 1)$$

Together with the additional $\lceil \sqrt{\Delta} \rceil < \sqrt{\Delta} + 1$ colours from the set $\{\bar{1}, \bar{2}, \dots\}$ we have thus used no more than $(\sqrt{\Delta} + 1)(2 \log_2 \Delta + 3)$ colours as claimed. $\square$

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