# On $k$-colorability of (*bull*, $H$)-free graphs

Nadzieja Hodur [a], Monika Pilśniak [a,*], Magdalena Prorok [a], Ingo Schiermeyer [a,b]

[a] *AGH University of Krakow, al. Mickiewicza 30, 30-059 Kraków, Poland*
[b] *TU Bergakademie Freiberg, 09596 Freiberg, Germany*

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**ABSTRACT**

The 3-colorability problem is a well-known NP-complete problem and it remains NP-complete for *bull*-free graphs, where a *bull* is the graph consisting of a $K_3$ with two pendant edges attached to two of its vertices. In this paper, for $k \geq 3$, we characterize all $k$-colorable (*bull*, *claw*)-free graphs containing an induced cycle of length at least 6. Moreover, we present the full characterization of all non 4-colorable connected (*bull*, *claw*)-free graphs and (*bull*, *chair*, $C_5$)-free graphs, and all non 5-colorable connected (*bull*, *claw*, $C_5$)-free graphs.

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## 1. Introduction

We consider finite, simple, and undirected graphs. For terminology and notations not defined here, we refer to [2].

An *induced subgraph* of a graph $G$ is a graph on a vertex set $S \subseteq V(G)$ for which two vertices are adjacent if and only if they are adjacent in $G$. In particular, we say that the subgraph is *induced by* $S$. We also say that a graph $H$ is an *induced subgraph* of $G$ if $H$ is isomorphic to an induced subgraph of $G$.

Given a family $\mathcal{H}$ of graphs and a graph $G$, we say that $G$ is $\mathcal{H}$-*free* if $G$ contains no graph from $\mathcal{H}$ as an induced subgraph. In this context, the graphs of $\mathcal{H}$ are referred to as *forbidden induced subgraphs*.

A vertex-coloring of a graph $G$ is *proper*, if adjacent vertices obtain distinct colors. A graph is $k$-*colorable*, if it can be properly colored using $k$ colors. The smallest integer $k$ such that a given graph $G$ is $k$-colorable is called the *chromatic number* of $G$, denoted by $\chi(G)$. Clearly, $\chi(G) \geq \omega(G)$ for every graph $G$, where $\omega(G)$ denotes the *clique number* of $G$, that is, the order of a maximum complete subgraph of $G$. Furthermore, a graph $G$ is *perfect* if $\chi(G') = \omega(G')$ for every induced subgraph $G'$ of $G$. For a subgraph $H$ and a vertex $v$, let $d_H(v) = |N(v) \cap V(H)|$.

For an induced cycle $C_p$ with $p \geq 3$ let $C_p[k_1, k_2, \ldots, k_p]$ denote the *clique expansion* of an induced cycle $C_p$, where its vertices $v_1, v_2, \ldots, v_p$ are replaced by complete graphs $K_{k_i}$ for $1 \leq i \leq p$ and additional edges between all pairs of vertices from consecutive cliques. (see Fig. 1). By $G \oplus H$ we denote a graph with set of vertices $V(G) \cup V(H)$ and set of edges $E(G) \cup E(H) \cup \{vw : v \in V(G), w \in V(H)\}$.

Let $G$ be a clique expansion $C[k_1, \ldots, k_n]$ of a cycle $C_n$ and let $k \in \mathbb{N}$. Let us label the vertices of the first clique with numbers $1, \ldots, k_1$, vertices of the second clique with numbers $k_1 + 1, \ldots, k_1 + k_2$ and so on. Then the *circular $k$-coloring algorithm*, called also $k$-CC algorithm, is an algorithm assigning to $m$-th vertex of $G$ the color $((m-1) \bmod k) + 1$ for $m = 1, \ldots, k_1 + \ldots + k_n$.

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\* Corresponding author.
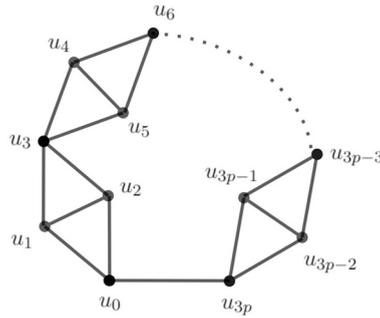*E-mail address:* pilsniak@agh.edu.pl (M. Pilśniak).

**Fig. 1.** The spindle graph $M_{3p+1}$. It could be also considered as the clique expansion $C_{2p+1}[2, 1, 2, 1, ..., 2, 1, 1]$ for $p \geq 2$.

The graph on five vertices $v_1$, $v_2$, $v_3$, $v_4$, $v_5$ and with the edges $v_1v_2$, $v_2v_3$, $v_3v_4$, $v_4v_5$, $v_2v_4$ is called a *bull*. Let $S_{i,j,k}$ be a 3-star with edges subdivided respectively $i - 1$, $j - 1$ and $k - 1$ times. The graph $S_{1,1,1}$ is called a *claw* and $S_{1,1,2}$ is called a *chair*.

The independence number $\alpha(G)$ of the graph $G$ is the largest $k \in \mathbb{N}$ such that there exists $S \subset V(G)$, satisfying $|S| = k$ and $S$ is a set of independent vertices.

The 3-colorability problem is a well-known NP-complete problem and it remains NP-complete for *claw*-free graphs and $K_3$-free graphs. In the last two decades, a large number of results of colorings of graphs with forbidden subgraphs have been shown (cf. [3], [4], [5], [10], [11], [14], [15] and cf. [7], [12], [13] for three surveys). Further discussion on the structure of chair-free graphs may be found in [16].

Our research has been motivated by [5] and we use some definitions and notations from it. In particular, a non-3-colorable graph $C_{2n+1} \oplus K_1$ we call an *odd wheel* $W_{2n+1}$. A graph $G$ of order $3p + 1$, $p \geq 1$ we call a *spindle graph* $M_{3p+1}$ if it contains a cycle $C$: $u_0u_1 \ldots u_{3p}u_0$, where $\{u_{3i-2}, u_{3i-1}, u_{3i+1}, u_{3i+2}\} = N_G(u_{3i})$ and $\{u_{3i-3}, u_{3i}\} = N_G(u_{3i-1}) \cap N_G(u_{3i-2})$ for each $i \in [p]$, where $[p] := \{1, 2, \ldots, p\}$.

Observe that $M_4 \cong K_4$ and $M_7$ is known as the Moser spindle.

**Proposition 1** ([5]). *The graph $M_{3p+1}$ is not 3-colorable for every $p \geq 1$.*

Since the 3-colorability problem is NP-complete for claw-free graphs and $K_3$-free graphs (cf. [7]), it is also NP-complete for *bull*-free graphs. The following theorem in [5] and [9] have motivated our research.

**Theorem 2** ([5]). *Let $G$ be a connected (bull, claw)-free graph. Then one of the following holds*

  *(i)  $G$ contains $W_5$ or*
  *(ii) $G$ contains a spindle graph $M_{3i+1}$ for some $i \geq 1$ or*
  *(iii) $G$ is 3-colorable.*

**Theorem 3** ([9]). *Let $G$ be a connected (bull, chair)-free graph. Then*

  *(i)  $G$ contains an odd wheel or*
  *(ii) $G$ contains a spindle graph $M_{3i+1}$ for some $i \geq 1$ or*
  *(iii) $G$ is 3-colorable.*

The goal of this paper is to study $k$-colorability of (bull, $H$)-free graphs. For $k \geq 4$, we will assume without loosing generality that $\delta(G) \geq 4$, since otherwise $G$ can be reduced by removing vertices of degree less than 4 (its coloring is trivial). The following are our main results. The first theorem provides necessary and sufficient conditions for any clique expansion of an odd cycle to be $k$-colorable.

**Theorem 4.** *Let $n \geq 1$, $k \geq 3$ and $G$ be a clique expansion $C_{2n+1}[k_1, \ldots, k_{2n+1}]$. Then $G$ is $k$-colorable if and only if the following two conditions are satisfied (all indices are taken modulo $k$):*

  *(i)  $\forall i \in [2n + 1]$ $k_i + k_{i+1} \leq k$;*
  *(ii) $\sum_{i=1}^{2n+1} k_i \leq nk$.*

Using Theorem 10 and Theorem 11, we can observe an easy corollary of Theorem 4.

**Corollary 5.** *Let $G$ be a connected* (*bull*, *claw*)*-free graph containing an induced cycle of length $p \geq 7$. If $\alpha(G) \geq 3$, then $G$ is $k$-colorable or $G$ contains $K_{k+1}$ or $G$ is a clique expansion $C_p[k_1, \ldots, k_p]$ such that there exists $i \in [p]$ such that $k_i + k_{i+1} > k$ or $\sum_{i=1}^{p} k_i > nk$, where all indices are taken modulo $k$.*

Next, we obtain the full characterization of all non 4-colorable connected (*bull*, *claw*)-free graphs, and (*bull*, *chair*, $C_5$)-free graphs.

**Theorem 6.** *Let $G$ be a connected* (*bull*, *claw*)*-free graph and $i \geq 1$. Then*

(i) *$G$ contains $\overline{C}_7 \oplus K_1$ or*
(ii) *$G$ contains $C_5 \oplus K_2$ or*
(iii) *$G$ contains $M_4 \oplus K_1$ or $M_7 \oplus K_1$ or*
(iv) *$G$ contains $C_{2i+1}[2, 2, \ldots, 2, 1, 3, 1, 3, \ldots, 1]$ or $C_{2i+1}[2, 2, \ldots, 2, 1]$ or*
(v) *$G$ contains an induced cycle $C_5$ and $|V(G)| > 8$ or*
(vi) *$G$ is 4-colorable.*

**Theorem 7.** *Let $G$ be a connected* (*bull*, *chair*, $C_5$)*-free graph and $i \geq 1$. Then*

(i) *$G$ contains $\overline{C}_7 \oplus K_1$ or*
(ii) *$G$ contains $C_{2i+1} \oplus K_2$ or*
(iii) *$G$ contains $M_{3i+1} \oplus K_1$ or*
(iv) *$G$ contains $C_{2i+1}[2, 2, \ldots, 2, 1, 3, 1, 3, \ldots, 1]$ or $C_{2i+1}[2, 2, \ldots, 2, 1]$ or*
(v) *$G$ is 4-colorable.*

Finally, we present a full characterization of all non 5-colorable connected (*bull*, *claw*, $C_5$)-free graphs.

**Theorem 8.** *Let $G$ be a connected* (*bull*, *claw*, $C_5$)*-free graph. Then*

(i) *$G$ contains $K_6$ or*
(ii) *$G$ contains $\overline{C}_7 \oplus K_2$ or*
(iii) *$G$ contains $\overline{C}_9 \oplus K_1$ or*
(iv) *$\alpha(G) = 2$ and $|V(G)| \geq 11$ or*
(v) *$\alpha(G) = 2$ and $\Delta(G) \geq 9$ or*
(vi) *$G$ is a clique expansion $C_{2n+1}[k_1, \ldots, k_{2n+1}]$ with $k_1 + \ldots + k_{2n+1} - 5n > 0$ or*
(vii) *$G$ is 5-colorable.*

## 2. Preliminary results

A *hole* in a graph $G$ is an induced cycle of length at least 4, and an *antihole* in $G$ is an induced subgraph whose complement is a cycle of length at least 4. A hole (antihole) is *odd* if it has an odd number of vertices. As the main tool for proving Theorem 6 we will use the well-known Strong Perfect Graph Theorem shown by Chudnovsky et al. [6].

**Theorem 9** (*Chudnovsky et al. [6]*). *A graph is perfect if and only if it contains neither an odd hole nor an odd antihole as an induced subgraph.*

### 2.1. Independence number in claw-free graphs

The following two theorems have been shown in [4] and Lemma 12 is due to Ben Rebea.

**Theorem 10.** *[4] Every connected* (*bull*, *claw*)*-free graph $G$ such that $\alpha(G) \geq 3$ is perfect or is a clique expansion of an odd cycle of length at least 7.*

**Theorem 11.** *[4] Let $G$ be a connected* (*bull*, *claw*)*-free graph. Then*

(i) *if $G$ contains an independent set of size 3, then $G$ is $C_5$-free.*
(ii) *if $G$ contains a hole of length $k$ with $k \geq 6$, then $G$ is a clique expansion of $C_k$.*

**Lemma 12.** *[1] If $G$ is a claw-free graph such that $\alpha(G) \geq 3$ and $G$ contains an odd antihole, then $G$ contains a hole $C_5$.*
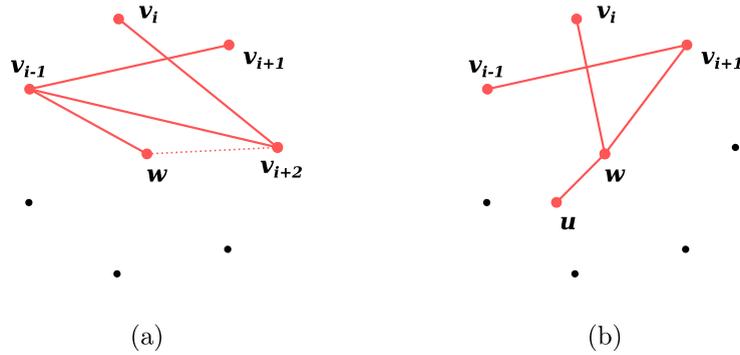
**Fig. 2.** Induced subgraphs constructed in the proofs of Lemma 14 and 15.

Combining Theorem 11 and Lemma 12 we obtain the following corollary.

**Corollary 13.** *Let $G$ be a connected (bull, claw)-free graph. If $G$ contains an odd antihole, then $\alpha(G) = 2$.* □

## 3. Lemmas

Let $G$ be a (bull, chair)-free graph such that $G$ contains an odd antihole $\overline{Q} = v_1 \ldots v_p$.

**Lemma 14.** *If a vertex $w \in G \setminus \overline{Q}$ is adjacent to $\overline{Q}$, then $w$ has no two consecutive non-neighbors in $\overline{Q}$.*

**Proof.** Suppose $w \in N(\overline{Q})$ and $w$ has $\ell$ consecutive non-neighbors $v_i, \ldots, v_{i+\ell-1}$, where $wv_{i-1} \in E(G)$. Then, the set $\{w, v_{i-1}, v_i, v_{i+1}, v_{i+2}\}$ induces a *chair*, if $w$ is not adjacent to $v_{i+2}$, or a *bull*, if it is (see Fig. 2a). □

**Lemma 15.** *(Rev1 1(b)(c)(d))Let a vertex $u \in N_2(\overline{Q})$ be adjacent to a vertex $w \in N(\overline{Q})$. Then $w$ is adjacent to all vertices of $\overline{Q}$.*

**Proof.** By Lemma 14, $w$ must have at least two consecutive neighbors $v_i, v_{i+1}$ on $\overline{Q}$ (since $\overline{Q}$ is odd). Without loss of generality $v_{i-1}w \notin E(G)$. Then the set $\{u, w, v_{i-1}, v_i, v_{i+1}\}$ induces a *chair* (see Fig. 2b). □

**Lemma 16.** *Let $w \in N(\overline{Q})$ be a vertex such that $d_{\overline{Q}}(w) = \left\lceil \frac{|\overline{Q}|}{2} \right\rceil$. Then $G$ contains a hole $C_5$.*

**Proof.** Suppose that $\overline{Q} \neq \overline{C_5} = C_5$ and $d_{\overline{Q}}(w) = \left\lceil \frac{|\overline{Q}|}{2} \right\rceil$. By Lemma 14, $w$ has exactly two neighbors $v_i, v_{i+1}$ such that $wv_{i-1}, wv_{i+2} \notin E(G)$. Then the set of vertices $\{w, v_i, v_{i+2}, v_{i-1}, v_{i+1}\}$ induces a cycle $C_5$. □

## 4. Proof of Theorem 4

By the definition of the clique expansion, the condition (i) is necessary. It is not difficult to see that condition (ii) is necessary as well. Namely, let us denote by $K^i$ the $i$-th clique of the expansion and assume, without loosing generality, that $K^1$ is colored with colors $1, \ldots, k_1$. Of course, $G$ is $k$-colorable if and only if there exists such a coloring of $G - K^{2n+1}$ that at least $s$ colors from $\{1, \ldots, k_1\}$ are repeated on $K^{2n}$, where

$$k_1 + k_{2n} - s + k_{2n+1} = k$$

and therefore $s = k_1 + k_{2n} + k_{2n+1} - k$. Let us say that if $s \leq 0$, then our task becomes trivial - the $CC$ algorithm starting from $K^1$ and cut at $K^{2n}$ can be completed to a proper coloring. Thus, we can assume that $k_i + k_{i+1} + k_{i+2} > k$ for any $i$.

Now an easy calculation shows that if $S \subset c(K^i)$ is a set of colors, then $K^{i+2}$ contains always at least $|S| + k_{i+2} - (k - k_{i+1})$ colors from $S$. Therefore, $K^{i+3}$ contains no more than $|S| - (|S| + k_{i+2} - (k - k_{i+1})) = k - k_{i+1} - k_{i+2}$ colors from $S$. Thus, on $K^4$ we can repeat at most $k - k_2 - k_3$ colors of $c(K^1)$, on $K^6$ additionally no more than $k - k_4 - k_5$, and so on. Finally, on $K^{2n}$ we can repeat no more than $(n-1)k - k_2 - \ldots - k_{2n-1}$ colors from $\{1, \ldots, k_1\}$. In order to obtain a proper coloring, we need at least $s$ colors of $c(K_1)$ repeated on $K^{2n}$, so we get a necessary condition

$$s = k_1 + k_{2n} + k_{2n+1} - k \leq (n-1)k - k_2 - \ldots - k_{2n-1}$$

equivalent to (ii).

Having conditions (i) and (ii) satisfied, we can construct an easy coloring as follows. Let us start coloring cliques with $k$-CC algorithm and let $l \in \{1, \ldots, n\}$ be the smallest index such that $c(K^{2l})$ contains colors $m+1, \ldots, m+p$, where $m$ and $p$ are integers such that $p \geq s$ and $m + p \leq k_1$. Such an index exists by the condition (ii). Then, we can color every clique $K^{2j+1}$, $l \leq j < n$, with consecutively increasing colors $1, 2, \ldots, m$ and then (if $k_{2j+1} > m$) with consecutively decreasing colors $k, k-1, \ldots$. Every clique $K^{2j}$, $l < j \leq n$, we color with consecutively increasing colors $m+1, m+2, \ldots$. It is easy to see that $K^{2n}$ has at least $p$ common colors with $K^1$, thus, we can always color $K^{2n+1}$ with remaining colors.

## 5. Proof of Theorem 6

Let us point out that for perfect graphs Theorem 6 is trivially true. Therefore, we can restrict our attention to non-perfect graphs. Let $G$ be a (*bull, claw*)-free graph. If $G$ is non-perfect, it must contain an odd antihole or an odd cycle of length at least 5.

Note that by Corollary 13 this leaves us with only two possible cases.

### 5.1. G contains an odd hole of length at least 7 and $\alpha(G) \geq 3$

Let us recall that by Theorem 10, the graph $G$ is a clique expansion of the cycle. The easy corollary of Theorem 4 is that if $G^* = C[k_1, \ldots, k_p]$ is a clique expansion of an odd cycle of length at least 7, then either $G^*$ contains $K_5 = M_4 \oplus K_1$ or $G^* = C[2, 2, \ldots, 2, 1, 3, 1, 3, \ldots, 1]$ or $G^* = C[2, \ldots, 2, 1]$, or $G^*$ is 4-colorable.

### 5.2. G contains an odd antihole, so $\alpha(G) = 2$

Let $\overline{Q} = v_1 v_2 \ldots v_p$ be an odd antihole and $N_2(\overline{Q})$ be a set of all the vertices of distance 2 from $\overline{Q}$. Let us point out two simple facts.

**Fact 17.** $\overline{Q}$ is a dominating set in $G$.

**Proof.** Suppose there exists $w \in N_2(\overline{Q})$. Then we have an independent set $\{w, v_1, v_2\}$ of 3 vertices, which contradicts Corollary 13. □

Now, using similar argument as above we can prove as follows.

**Fact 18.** Let $w, w' \in N(\overline{Q})$ and $N(w) \cup N(w') \neq \overline{Q}$. Then $ww' \in E(G)$. □

Note that if $p \geq 11$, then $\overline{Q}$ contains $K_5$, but case (*iii*) of Theorem 6. Next, if $p = 9$, then the graph contains $C[1, 3, 1, 3, 1]$, but then case (*iv*) of Theorem 6. So, we assume that $p = 5$ or $p = 7$ and we show the coloring of the graph $G$.

First, assume $p = 7$. We will show that if the graph $G$ is not 4-colorable, then it does contain one of the exceptional subgraphs.

By Lemma 14 we know that for any $w \in N(\overline{Q})$ it holds $d_{\overline{Q}}(w) \geq 4$. Of course, if there is a vertex $w \in V(G)$ such that $N_{\overline{Q}}(w) = \overline{Q}$, then we have exceptional graph $\overline{C}_7 \oplus K_1$. Moreover, if there is a vertex $w$ with $d_{\overline{Q}}(w) = 4$, then by Lemma 16 we have a hole $C_5$, which was considered in the previous case.

Thus, assume $d_{\overline{Q}}(w) \in \{5, 6\}$. If $N(\overline{Q}) = \{w\}$, then $G$ is obviously 4-colorable. Let $w'$ be another vertex in $N(\overline{Q})$. If $N_{\overline{Q}}(w) \cup N_{\overline{Q}}(w') = \overline{Q}$, then $G$ contains $C[1, 3, 1, 3, 1]$. Otherwise we must have $ww' \in E(G)$, and $G$ contains a complete graph $K_5$.

Finally let $p = 5$. We can assume that $|V(G)| \leq 8$, otherwise we have point (*v*) of Theorem 6. By Lemma 14 we know that for any $w \in N(\overline{Q})$ we have $d_{\overline{Q}}(w) \in \{3, 4, 5\}$ and if $d_{\overline{Q}}(w) = 3$, then non-neighbors of $w$ are non-consecutive. Let us define the following sets:

$$A_i = \{v \in N(\overline{Q}) : N_{\overline{Q}}(v) = \{v_i, v_{i+1}, v_{i+3}\}\}.$$
$$B_i = \{v \in N(\overline{Q}) : N_{\overline{Q}}(v) = \{v_i, v_{i+1}, v_{i+2}, v_{i+3}\}\}.$$
$$C = \{v \in N(\overline{Q}) : N_{\overline{Q}}(v) = \overline{Q}\}.$$

Let also $A = \bigcup_{i=1}^{p} A_i$ and $B = \bigcup_{i=1}^{p} B_i$. By Fact 17, we have $V(G) = \overline{Q} \cup A \cup B \cup C$.

**Case 1.** $C \neq \emptyset$. Let $w \in C$. If there is another vertex $w' \in C$, then $ww' \notin E(G)$ (otherwise we have an exceptional graph $C_5 \oplus K_2$). Thus, we can give $w$ and $w'$ the same color 1. If there is a third vertex $w'' \in N(\overline{Q})$, then note that $w'' \notin C$ (otherwise we have a *claw*, if $w, w', w''$ are not adjacent, or $C_5 \oplus K_2$, if they are). So we can color with 2 the vertex $w''$ and one of its non-neighbors on the cycle $\overline{Q}$. The rest of the cycle we color with 3 and 4. If $w', w'' \in A \cup B$, then note that either $ww' \notin E(G)$ or $ww'' \notin E(G)$ or $N_{\overline{Q}}(w') \cup N_{\overline{Q}}(w'') \neq \overline{Q}$ (otherwise the graph $G$ contains $M_7 \oplus K_1$). Assume
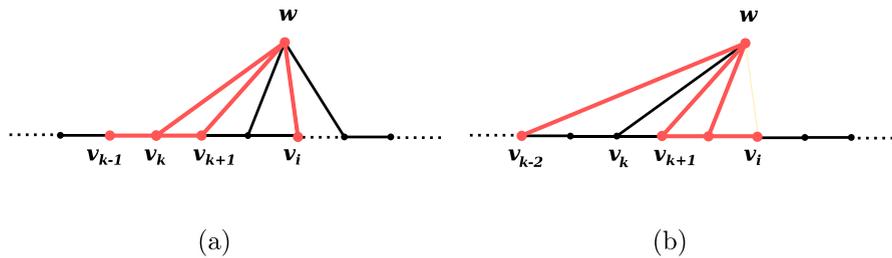
**Fig. 3.** Induced subgraphs constructed in the proof of Fact 3.

$ww' \notin E(G)$. Then we color $w, w'$ with 1, $w''$ and one of its non-neighbors on the cycle with 2 and the rest of the cycle with 3, 4. Analogously for $ww'' \notin E(G)$. Assume $ww', ww'' \in E(G)$. As we said before, we have $N_{\overline{Q}}(w') \cup N_{\overline{Q}}(w'') \neq \overline{Q}$. But then the vertices $w', w''$ must be adjacent and the graph $G$ contains $K_5$.

**Case 2.** $C = \emptyset$. Let $w, w', w'' \in A \cup B$. Then each of those vertices has at least 1 non-neighbor on $\overline{Q}$. Due to our assumption that $\delta(G) \geq 4$, the non-neighborhoods are disjoint. If possible, we take three vertices $u, u', u''$, non-neighbors of $w, w', w'$ respectively, such that only two of $u, u', u''$ are adjacent. We color $w, w', w''$ and their non-neighbors with colors 1, 2, 3, respectively. The remained vertices of $\overline{Q}$ are non-adjacent, so we can color them with 4. If such a triple of non-neighbors does not exist, it means $w, w', w'' \in B$ and their two common neighbors are adjacent. Thus, without loss of generality, $ww' \notin E(G)$ (or we have $K_5$). Then we color $w, w'$ with 1, $w''$ and its non-neighbor with 2 and the rest of the cycle with 3 and 4, what finishes the proof.

## 6. Proof of Theorem 7

As before, the result is obvious for perfect graphs, so using Theorem 9 we can split the proof into two cases – when the graph $G$ contains an odd antihole and when it contains an odd hole. In this theorem, we also forbid $C_5$, so both hole and antihole must be of length at least 7.

### 6.1. G contains an odd antihole

Let us assume the graph $G$ contains an odd antihole $\overline{Q} = v_1 \ldots v_p$. As we have mentioned in the proof of Theorem 6, if $p \geq 11$, then $G$ contains $K_5$, and if $p = 9$, then $G$ contains $C[1, 3, 1, 3, 1]$. So let $p = 7$. We will see that our conclusions will be identical as in the respective part of the proof of Theorem 6.

By Lemma 15, if there is a vertex $u \in N_2(\overline{Q})$, then it must have a neighbor $w$ such that $N_{\overline{Q}}(w) = \overline{Q}$. But if such a $w$ exists, then $G$ contains an exceptional subgraph $\overline{C}_7 \oplus K_1$. Moreover, by Lemma 16, if $d_{\overline{Q}}(w) = 4$, then $G$ contains $C_5$. So we can assume $N_2(\overline{Q}) = \emptyset$ and $d_{\overline{Q}}(w) \in \{5, 6\}$ for any $w \in N(\overline{Q})$ and we finish the coloring as in the proof of Theorem 6.

### 6.2. G contains an odd hole of length at least 7

Let $Q = v_1 \ldots v_p$ be an odd hole of length at least 7. We will prove that if a (bull, chair)-free graph $G$ does contain such a cycle, then it satisfies some useful structural properties. A similar, but more general structural analysis of (bull, chair)-free graphs can be found in [8].

All indices will be taken modulo $p$.

**Fact 19.** Let $w \in N(Q)$. Then either $N_Q(w) = \{v_{i-1}, v_i, v_{i+1}\}$ for some $i \in \{1, \ldots, p\}$ or $N_Q(w) = Q$.

**Proof.** Let $v_k, \ldots, v_{k+\ell-1}$ be the longest sequence of consecutive neighbors of $w$ on the cycle and suppose $\ell < p$. We will show that in this case $\ell = 3$ and $v_k, v_{k+1}, v_{k+2}$ are the only neighbors of $w$ on the cycle.

Firstly, suppose that $\ell = 1$, that is, $w$ has no consecutive neighbors on $Q$. Then, since $Q$ is odd, the graph $G$ must contain an induced *chair*.

Suppose now that $\ell = 2$. Then the set $\{v_{k-1}, v_k, w, v_{k+1}, v_{k+2}\}$ induces a *bull*.

Finally, suppose $\ell \geq 3$ and $w$ has a neighbor $v_i$ among $\{v_4, \ldots, v_{k-2}\}$. Then the set $\{v_{k-1}, v_k, w, v_{k+1}, v_i\}$ (if there is $v_i$ such that $i \neq k - 2$) – see Fig. 3a) or the set $\{v_{k-2}, w, v_{k+1}, v_{k+2}, v_{k+3}\}$ (otherwise) induces a *bull* – see Fig. 3b. □

Now we can define the following sets:

$A_i = \{v \in N(Q) : N_Q(v) = \{v_{i-1}, v_i, v_{i+1}\}\}$, and $A = \bigcup_{i=1}^{p} A_i$.
$D = \{v \in N(Q) : N_Q(v) = Q\}$.

**Fact 20.** $G[Q \cup A]$ is a clique expansion of the cycle $Q$.

**Proof.** Let $w, w' \in A_i \setminus \{v_i\}$. Of course, if $ww' \notin E(G)$, then the set $\{v_{i-3}, v_{i-2}, v_{i-1}, w, w'\}$ induces a *chair*.

Let now $w' \in A_{i+1} \setminus \{v_{i+1}\}$ and let $ww' \notin E(G)$. Then the set $\{v_{i-2}, v_{i-1}, w, v_i, w'\}$ induces a *bull*.

Finally, let $w' \in A_j$, where $|j - i| \geq 2$, and let $ww' \in E(G)$. By symmetry we can assume $j \leq i - 4$. Then the set $\{v_{i-2}, v_{i-1}, v_i, w, w'\}$ induces a *bull*. $\square$

**Fact 21.** There are no edges between $Q$ and $G \setminus (Q \cup A \cup D)$. Moreover, if $w \in D$ is adjacent to a connected component $C$ of $G \setminus (Q \cup A \cup D)$, then $w$ dominates $C$.

**Proof.** The first part of the claim is obvious, since if a vertex $v \in A_i$ has a neighbor $u \in N_2(Q)$, then $\{v_{i-2}, v_{i-1}, v_i, v, u\}$ induces a *bull*.

Suppose now $w \in D$ is adjacent to a connected component $C$ of $G \setminus (Q \cup A)$, but does not dominate $C$. Thus, there exist $u, u' \in C$ such that $wu, uu' \in E(G)$ but $wu' \notin E(G)$. But then the set $\{u', u, w, v_1, v_3\}$ induces a *chair*. $\square$

Note that we can assume, without loosing generality, that $D$ is an independent set (otherwise we have an exceptional subgraph $C_{2k+1} \oplus K_2$). Thus, the graph $G$ is 4-colorable if and only if

(i) $G[Q \cup A]$ is 3-colorable and
(ii) $G \setminus (Q \cup A \cup D)$ is 3-colorable.

By Theorem 3 we know that $G \setminus D$ is 3-colorable if and only if it does not contain an odd wheel or a spindle graph $M_{3i+1}$. Every connected component $C$ of $G \setminus D$ is dominated by some vertex $w$ of $D$, so if $C$ does contain odd wheel, then $G$ contains $C_{2k+1} \oplus K_2$, and if $C$ contains $M_{3i+1}$, then $G$ contains $M_{3i+1} \oplus K_1$.

This completes the proof of Theorem 7.

## 7. Proof of Theorem 8

The proof is again similar to the proof of Theorem 6. Let us point out that for perfect graphs the theorem is trivially true. Therefore, we can restrict our attention to non-perfect graphs. A non-perfect (*bull*, *claw*, $C_5$)-free graph $G$ must contain an odd antihole or a hole of length $p \geq 7$.

With Corollary 13 this leaves us with only two possible cases.

### 7.1. $G$ contains an odd hole of length at least 7 and $\alpha \geq 3$

Let us recall that by Theorem 10 our graph $G$ is a clique expansion of the cycle. Now by Theorem 4 $G$ is 5-colorable or $k_i + k_{i+1} \geq 6$ for some integer $i \in \{1, \ldots, 2n+1\}$ and $G$ contains $K_6$ or $k_1 + \ldots + k_{2n+1} - 5n > 0$. This is statement $(i)$ and $(vi)$.

### 7.2. $G$ contains an odd antihole and $\alpha(G) = 2$

For a graph $G$ let $\beta_0(G)$ denote its matching number. The following fact is well known.

**Fact 22.** Let $G$ be a graph with $\alpha(G) = 2$. Then $\chi(G) = n - \beta_0(\overline{G}) \geq \frac{n}{2}$. $\square$

Let $\overline{Q} = v_1 v_2 \ldots v_p v_1$ be an odd antihole. Note that if $p \geq 13$ then $\overline{Q}$ contains $K_6$ and if $p = 11$, then the graph contains $C[1, 4, 1, 4, 1]$. So we may assume that $p \in \{7, 9\}$. If $|V(G)| \geq 11$, then $\chi(G) \geq 6$ by Fact 22. This is statement $(iv)$. So we may assume that $|V(G)| \leq 10$.

First we consider the case $p = 9$. If $G \cong \overline{C}_9 \oplus K_1$, then $\chi(G) = 6$. This is statement $(iii)$. If $G \not\cong \overline{C}_9 \oplus K_1$, then $\chi(G) = 5$.

Next we consider the case $p = 7$. If $7 \leq |V(G)| \leq 8$, then $G \subset W_7$, hence $G$ is 5-colorable. Now consider $|V(G)| = 9$. If $G \cong \overline{C}_7 \oplus K_2$, then $\chi(G) = 6$. This is statement $(ii)$. If $G \not\cong \overline{C}_7 \oplus K_2$, then $\chi(G) = 5$ by Fact 22.

Finally, consider $|V(G)| = 10$. If $\Delta(G) = 9$, then $G$ is not 5-colorable. This is statement $(v)$. If $\beta_0(\overline{G}) = 5$, then $G$ is 5-colorable by Fact 22. Hence we may assume that $\Delta(G) \leq 8$ and $\beta_0(\overline{G}) \leq 4$. Let $H := G - \overline{Q}$. We first show the following fact.

**Fact 23.** The graph $H$ is complete.

**Proof.** Let $C = \{v \in N(\overline{Q}) : N_{\overline{Q}}(v) = \overline{Q}\}$. Let $H = \{w_1, w_2, w_3\}$. Suppose $H$ is not complete and let $w_1 w_2 \notin E(G)$. Then $w_3 \in C$. Since $\Delta(G) \leq 8$, we may assume that $w_1 w_3 \notin E(G)$. Hence $w_2 \in C$ as well. Now $w_2 w_3 \in E(G)$ and so $G$ contains $\overline{C}_7 \oplus K_2$. This is statement $(ii)$. $\square$

Since $\Delta(G) \leq 8$ we obtain $d_{\overline{Q}}(w) \leq 6$ for any $w \in V(H)$. By Lemma 14 we know that for any $w \in V(H)$ we have $d_{\overline{Q}}(w) \geq 4$. Moreover, if $d_{\overline{Q}}(w) = 4$, then by Lemma 16 we have a hole $C_5$, a contradiction. Hence, we may assume that $5 \leq d_{\overline{Q}}(w) \leq 6$ for any $w \in V(H)$. Moreover, we obtain the following fact. It can be proven in exactly the same way as Lemma 16.

**Fact 24.** If $d_{\overline{Q}}(w) = 5$ for a vertex $w \in V(H)$, then there exists some integer $i$ such that $wv_i, wv_{i+2} \notin E(G)$.

**Fact 25.** If $w_1 v_i, w_2 v_{i+1}, w_3 v_{i+2} \notin E(G)$ or $w_1 v_i, w_2 v_{i+1}, w_3 v_{i+4} \notin E(G)$, then $G$ is 5-colorable.

**Proof.** Observe that in these two cases we have $\beta_0(\overline{G}) = 5$, and $G$ is 5-colorable by Fact 22. $\square$

Suppose first that $d_{\overline{Q}}(w_i) = 6$ for $i = 1, 2, 3$. Then $G$ contains $K_6$ or $G$ is 5-colorable by Fact 25. Suppose next that $d_{\overline{Q}}(w_1) = 5$. We may assume that $w_1 v_i, w_1 v_{i+2} \notin E(G)$. If $w_2 v_{i+1} \notin E(G)$ or $w_3 v_{i+1} \notin E(G)$, then $\overline{Q} - v_{i+1} + w_1$ is a $\overline{C_7}$ such that the corresponding subgraph $H$ is not complete, contradicting Fact 23. So we may assume that $w_2 v_{i+1}, w_3 v_{i+1} \in E(G)$. Now if $v_{i+3}, v_{i+6} \in N(w_2) \cap N(w_3)$, then there is $K_6$. Hence we may assume that $w_2 v_{i+3} \notin E(G)$. Then by Lemma 14 and Fact 24 we obtain $v_{i+4}, v_{i+6} \in N(w_2)$. Now by Fact 25 we conclude that $v_{i+4}, v_{i+6} \in N(w_3)$ and find $K_6$.

This completes the proof of Theorem 8.

## Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this article.

The corresponding author completed this form on behalf of all the authors of a submission.

## Data availability

No data was used for the research described in the article.

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