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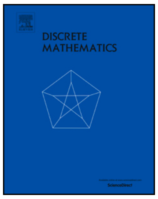

Quasi-majority neighbor sum distinguishing edge-colorings

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In this paper, a k-edge-coloring of G is any mapping $c : E(G) \longrightarrow [k]$. The edge-coloring c of G naturally defines a vertex-coloring $\sigma_c : V(G) \to \mathbb{N}$, where $\sigma_c(v) = \sum_{u \in N_G(v)} c(vu)$ for every vertex $v \in V(G)$. The edge-coloring c is said to be neighbor sum distinguishing if it results in a proper vertex-coloring σ_c, that is, $\sigma_c(u) \neq \sigma_c(v)$ for every edge uv in G.

We investigate neighbor sum distinguishing edge-colorings with local constraints, where the edge-coloring is quasi-majority at each vertex. Specifically, every vertex v is incident to at most $\lceil d(v)/2 \rceil$ edges of one color. This type of coloring is referred to as quasi-majority neighbor sum distinguishing edge-coloring. The minimum number of colors required for a graph to have a quasi-majority neighbor sum distinguishing edge-coloring is called the quasi-majority neighbor sum distinguishing index. A graph is nice if it has no component isomorphic to K_2. We prove that any nice graph admits a quasi-majority neighbor sum distinguishing edge-coloring using at most 12 colors. This bound can be improved for bipartite graphs and graphs with a maximum degree of at most 4. Specifically, we show that every nice bipartite graph can be colored with 6 colors, and every nice graph with a maximum degree of at most 4 can be colored with 7 colors. Additionally, we determine the exact value of the quasi-majority neighbor sum distinguishing index for complete graphs, complete bipartite graphs, and trees.

We also consider majority neighbor sum distinguishing edge-colorings, that is, when each vertex is incident to at most $d(v)/2$ edges with the same color.

1. Introduction

We focus on simple, finite graphs. The sets of vertices and edges of a graph G are denoted by $V(G)$ and $E(G)$, respectively. The term *order* of G refers to $|V(G)|$, while *size* of G refers to $|E(G)|$. $N_G(v)$ ($N(v)$ for short) denotes the neighborhood of a vertex v in a graph G and $d_G(v)$ ($d(v)$ for short) the degree of a vertex v in a graph G. For any set $S \subseteq V(G)$, the symbol $G[S]$ denotes the subgraph of G induced by S. Let G_1, G_2 be graphs such that $V(G_1) \cap V(G_2)$ can be nonempty. By $G_1 \cup G_2$ we mean a graph with the vertex set $V(G_1) \cup V(G_2)$ and the edge set $E(G_1) \cup E(G_2)$.

In this paper, a *k-edge-coloring* of a graph G is any mapping $c : E(G) \to [k]$. Any edge-coloring c induces a vertex-coloring $\sigma_c : V(G) \to \mathbb{N}$ given by

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$$\sigma_c(v) = \sum_{u\in N(v)} c(vu),$$

for every $v \in V(G)$. We say that a k-edge-coloring c *distinguishes* vertices $u, v \in V(G)$ if $\sigma_c(u) \neq \sigma_c(v)$. A k-edge-coloring of a graph G is termed *neighbor sum distinguishing* (NSD for short) if it distinguishes every pair of adjacent vertices, i.e. σ_c is a proper vertex-coloring of G. The smallest k for which there exists an NSD k-edge-coloring of a graph G is called the *neighbor sum distinguishing index* and is denoted by $\chi^e_{\sum}(G)$.

Observe that a graph G always admits an edge-coloring that induces a proper vertex-coloring, except when it includes K_2 as a component. By assigning a different power of 2 to each edge, we ensure that every vertex receives a distinct sum of colors of incident edges. However, the two vertices of K_2 cannot be distinguished in this way. Therefore, we call G *nice* whenever it lacks K_2 as a component.

The concept of coloring edges so that it generates a proper vertex-coloring has been frequently considered, particularly with the emergence of the 1-2-3 Conjecture posed in 2004 by Karoński, Łuczak and Thomason [8]. To be specific, this hypothesis states that the smallest value of k for which every nice graph G has a k-edge-coloring c such that $\sigma_c(u) \neq \sigma_c(v)$ for every edge $uv \in E(G)$ is equal to 3. After a series of tens of papers, this conjecture was finally proven by Keusch [9].

Theorem 1.1. (Keusch [9]) *Every nice graph G satisfies $\chi^e_{\sum}(G) \leq 3$.*

When we assume the additional restriction that the k-edge-coloring must be proper, then we obtain another version of NSD edge-coloring which we call *neighbor sum distinguishing proper edge-coloring*. The smallest value of k such that such a coloring exists is denoted by $\chi'_{\sum}(G)$. This value is related to a conjecture proposed by Flandrin, Marczyk, Przybyło, Saclé, and Woźniak in [5].

Conjecture 1.2. (Flandrin et al. [5]) *Every nice graph $G \neq C_5$ satisfies $\chi'_{\sum}(G) \leq \Delta(G) + 2$.*

This conjecture remains unresolved. Wang and Yan [13] showed that $\chi'_{\sum}(G) \leq \lceil (10\Delta(G)+2)/3 \rceil$ for graphs with $\Delta(G) \geq 18$. Additionally, Przybyło [11] established that $\chi'_{\sum}(G) \leq \Delta + O(\sqrt{\Delta})$, where $\Delta = \Delta(G)$. Let col(G) be the *coloring number* of G, defined as the smallest k such that there is a vertex ordering of G in which each vertex is preceded by at most $k-1$ of its neighbors. It is known that $\chi'_{\sum}(G) \leq 2\Delta(G) + \mathrm{col}(G) - 1$ [10] and $\chi'\sum(G) \leq \Delta(G) + 3\mathrm{col}(G) - 4$ [12].

Dailly, Duchéne, Parreau, and Sidorowicz in [4] generalized these conjectures by defining the concept of *neighbor sum distinguishing d-relaxed edge-coloring*. A k-edge-coloring is *d-relaxed* if each monochromatic set of edges induces a subgraph with maximum degree at most d. If a d-relaxed k-edge-coloring is distinguishing, then it is called a *neighbor sum distinguishing d-relaxed k-edge-coloring*. The smallest k for which there is a neighbor sum distinguishing d-relaxed k-edge-coloring of G is denoted by $\chi'^{d}_{\sum}(G)$. Consequently, $\chi'^{1}_{\sum}(G) = \chi'_{\sum}(G)$, and $\chi'^{\Delta(G)}_{\sum}(G)$ correspond to the 1-2-3 Conjecture.

A typical approach in studying graph parameters is to start with graphs of bounded maximum degree. While graphs with a maximum degree 2 (which are essentially forests of paths and cycles) are typically straightforward to analyze, the case of subcubic graphs (graphs with a maximum degree 3) often presents more complexity. For example, $\chi'_{\sum}(G) \leq 6$ for every nice subcubic graph G [6] (the conjecture posits that the bound should be 5). In [4], the authors examined the 2-relaxed case for subcubic graphs and proved that $\chi'^{2}_{\sum}(G) \leq 4$ for every nice graph G with maximum degree at most 3.

The paper [3] by Dailly and Sidorowicz investigates edge-colorings that distinguish vertices and allow adjacent edges to have the same color, but with an additional restriction. It is required that the set of edges incident to a vertex is not monochromatic when the degree of the vertex is large enough. It is proven that every nice graph has a neighbor sum distinguishing 7-edge-coloring such that the set of edges incident to a vertex of degree at least 6 is not monochromatic.

Inspired by these results, we explore an edge-coloring approach that allows each vertex v to be incident with at most $\lceil d(v)/2 \rceil$ edges of one color. We refer to an edge-coloring where every vertex v has at most $\lceil d(v)/2 \rceil$ incident edges of one color as to a quasi-majority edge-coloring. Thus, the aim of our work is to merge two types of coloring, a quasi-majority edge-coloring and a neighbor sum distinguishing edge-coloring, resulting in a new variant called quasi-majority neighbor sum distinguishing edge-coloring.

Our paper is organized as follows. In Section 2, we present the definition and some properties of quasi-majority edge-coloring. In Section 3, we define a quasi-majority neighbor sum distinguishing edge-coloring and outline its basic properties. Section 4 focuses on special classes of graphs, where we determine the exact value of the quasi-majority neighbor sum distinguishing index for complete graphs, complete bipartite graphs and trees. Additionally, we prove that the quasi-majority neighbor sum distinguishing index of graphs with maximum degree 4 is at most 7. In Section 5, we establish a constant upper bound on the quasi-majority neighbor sum distinguishing index of every nice graph. We demonstrate that every graph has a quasi-majority neighbor sum distinguishing 12-edge-coloring, and provide an upper bound on the quasi-majority neighbor sum distinguishing index in terms of the maximum degree of a graph. This result offers an upper bound better than 12 for graphs with small maximum degree. Bock, Kalinowski, Pardey, Pilśniak, Rautenbach, and Woźniak [2] introduced a majority edge-coloring, where each vertex v has at most $d(v)/2$ edges incident in one color. By merging majority edge-coloring and neighbor sum distinguishing edge-coloring, we derive majority neighbor sum distinguishing edge-coloring. In

Section 6, we discuss the impact of our results on majority neighbor sum distinguishing edge-coloring. In Section, 7 we present some open problems.

2. Quasi-majority edge-coloring

A k-edge-coloring c of a graph G is called *quasi-majority at a vertex* $v \in V(G)$ if v is incident to at most $\lceil d(v)/2 \rceil$ edges with color α, for every color $\alpha \in [k]$. If c is quasi-majority at every vertex of G, then it is called *quasi-majority* (QM for short). The minimum value of k for which there exists a QM k-edge-coloring of a graph G is called *quasi-majority index* and is denoted by $\chi^{QM}(G)$.

An edge-coloring of a graph G is called *majority* if every vertex $v \in V(G)$ is incident to at most $d(v)/2$ edges in one color. This concept was introduced in [2], where the following theorems were proven.

Theorem 2.1. (Bock et al. [2]) *Every finite graph of minimum degree at least* 2 *admits a majority* 4*-edge-coloring.*

Theorem 2.2. (Bock et al. [2]) *Let G be a connected graph.*

1. *If G has an even number of edges or G contains vertices of odd degrees, then G has a* 2*-edge-coloring such that, for every vertex v of G, at most* $\left\lceil \frac{d(v)}{2} \right\rceil$ *of the edges incident to v have the same color.*
2. *If G has an odd number of edges, all the vertices of G have even degree, and u is any vertex of G, then G has a* 2*-edge-coloring such that, for every vertex v of G distinct from u, exactly* $\frac{d(v)}{2}$ *of the edges incident to v have the same color, and exactly* $\frac{d(u)}{2} + 1$ *of the edges incident to u have the same color.*

From Theorem 2.2 we immediately derive that if G has an even size or G contains vertices of odd degrees, then there is a QM 2-edge-coloring of G. In turn, we see that if G has an odd size, all vertices of G have even degrees and u is any vertex in G, then there is a 2-edge-coloring of G such that at any vertex of G distinct from u this edge-coloring is QM, while at u exactly $\frac{d(u)}{2} + 1$ edges have the same color. It is enough to recolor one edge at u with a third color and then the edge-coloring at u is also QM. From this we get the following important fact that we use in this paper.

Corollary 2.3. *Every graph G satisfies* $\chi^{QM}(G) \le 3$.

Another consequence of Theorem 2.2 can be easily justified.

Proposition 2.4. *G has no quasi-majority* 2*-edge-coloring if and only if G has an odd number of edges and all vertices of G have even degrees.*

Observe that there does not exist a bipartite graph with an odd size where all the vertices have even degrees. Therefore, the following corollary is true.

Corollary 2.5. *For every bipartite graph G with* $\Delta(G) \ge 2$ *we have* $\chi^{QM}(G) = 2$.

3. Definition and basic properties

A k-edge-coloring of a graph G is termed *quasi-majority neighbor sum distinguishing* (QM NSD for short) if it is quasi-majority and neighbor sum distinguishing. The minimum value of k for which there exists a QM NSD k-edge-coloring of a graph G is called the *quasi-majority neighbor sum distinguishing index* and is denoted by $\chi_{\sum}^{QM}(G)$. Observe that only nice graphs admit a QM NSD edge-coloring, and there is no graph with the QM NSD index equal to 1, so $\chi_{\sum}^{QM}(G) \ge 2$ for every nice graph G.

Let G be a nice graph. It is easy to see that

$$\chi_{\sum}^{e}(G) \le \chi_{\sum}^{QM}(G) \le \chi_{\sum}'(G).$$

The equality in the right-hand side inequality is achieved, e.g. for every graph G with $\Delta(G) = 2$. This means that if a nice graph G is a path or a cycle, then $\chi_{\sum}^{QM}(G) = \chi_{\sum}'(G)$, so according to the propositions included in [5], the following two propositions are true.

Proposition 3.1. *We have* $\chi_{\sum}^{QM}(P_3) = 2$*, and for every* $n \ge 4$ *we have* $\chi_{\sum}^{QM}(P_n) = 3$.

Proposition 3.2. *We have $\chi_{\sum}^{QM}(C_5)=5$, and for every $n\geq 3$ and $n\neq 5$ we have*

$$\chi_{\sum}^{QM}(C_n)=\begin{cases}3, & \text{if } n\equiv 0 \pmod 3,\\ 4, & \text{otherwise.}\end{cases}$$

Furthermore, in [4] subcubic graphs were considered and the following result has been proven.

Theorem 3.3. (Dailly et al. [4]) *If G is a nice subcubic graph with no component isomorphic to C_5, then it admits an* NSD 4*-edge-coloring such that every vertex of degree at least* 2 *is incident to at least two edges of different colors.*

If every vertex of degree at least 2 in a subcubic graph is incident to at least two edges of different colors, then the edge-coloring is quasi-majority at every vertex and so the edge-coloring is quasi-majority. Thus, we obtain the following.

Proposition 3.4. *If G is a nice subcubic graph with no component isomorphic to C_5, then $\chi_{\sum}^{QM}(G)\leq 4$.*

The upper bound in Proposition 3.4 is sharp, since cycles C_n for $n\equiv 1,2 \pmod 3$ require 4 colors for a QM NSD edge-coloring.

Proposition 3.5. *Let G be a nice graph without two adjacent vertices of the same degree. If $\chi^{QM}(G)=2$, then $\chi_{\sum}^{QM}(G)=2$.*

Proof. Let c be a QM 2-edge-coloring of G. If a vertex v has even degree $d(v)=2k$, then $\sigma_c(v)=3k$. If a vertex has odd degree $d(v)=2k+1$, then $\sigma_c(v)$ is either $3k+1$ or $3k+2$. Therefore, $\sigma_c(v)=\sigma_c(w)$ only if $d(v)=d(w)$. Since no two adjacent vertices have the same degree, the coloring c distinguishes adjacent vertices. □

We can also use interval colorings to find a QM NSD edge-coloring. A k-edge-coloring of a graph G is called an *interval coloring* if the colors of the edges incident to each vertex of G are distinct and form an interval of consecutive integers.

Proposition 3.6. *Let G be a nice graph without two adjacent vertices of the same degree. If G has an interval coloring, then $\chi_{\sum}^{QM}(G)=2$.*

Proof. Let c' be an interval coloring of G. We construct a new coloring c as follows: $c(e)=1$ if $c'(e)\equiv 1 \pmod 2$, and $c(e)=2$ if $c'(e)\equiv 0 \pmod 2$ for $e\in E(G)$. Therefore, a vertex v of even degree $d(v)=2k$ is incident to k edges of color 1 and k edges of color 2, resulting in $\sigma_c(v)=3k$. If a vertex has odd degree $d(v)=2k+1$, it is incident to k edges of color 1 and $k+1$ edges of color 2, or to $k+1$ edges of color 1 and k edges of color 2, so $\sigma_c(v)$ is either $3k+1$ or $3k+2$. Thus, the coloring is quasi-majority at every vertex, and $\sigma_c(v)=\sigma_c(w)$ only if $d(v)=d(w)$. □

Every bipartite graph G with $|V(G)|\leq 15$ admits an interval coloring, so every bipartite graph G with $|V(G)|\leq 15$ in which there are no two adjacent vertices with the same degree has a quasi-majority neighbor sum distinguishing 2-edge-coloring.

4. Special classes of graphs

In this section we study the QM NSD index of complete graphs, complete bipartite graphs, trees, and graphs with maximum degree at most 4.

4.1. Complete graphs

To determine the QM NSD index of complete graphs, we use the following two lemmas.

Lemma 4.1. *Every complete graph K_{2k+1} has a* QM NSD 3*-edge-coloring in which k vertices are incident to $k-1$ edges of color* 2 *and $k+1$ vertices are incident to k edges of color* 2*.*

Proof. We prove this by induction on the number of vertices. The lemma is true for $k=1$, that is, for the complete graph K_3. Assume that it is true for all complete graphs of odd order with fewer than $2k+1$ vertices. Let $V(K_{2k+1})=\{v_1,v_1,\dots,v_{2k+1}\}$. We decompose K_{2k+1} into two edge disjoint subgraphs G_1 and G_2 such that $K_{2k+1}=G_1\cup G_2$. Let $G_1=G[\{v_1,\dots,v_{2k-1}\}]$ and G_2 be a spanning subgraph of G that contains edges $E(G_2)=\{v_{2k}v_{2k+1}\}\cup\{v_{2k}v_i : i\in\{1,\dots,2k-1\}\}\cup\{v_{2k+1}v_i : i\in\{1,\dots,2k-1\}\}$. The subgraph G_1 is isomorphic to K_{2k-1}, so by the induction hypothesis there is a QM NSD 3-edge-coloring such that $k-1$ vertices are incident to $k-2$ edges in color 2 and k vertices are incident to $k-1$ edges of color 2. Let c_1 be such a coloring and $v_1,\dots,v_{k-1}$ be the vertices with $k-2$ incident edges in color 2. Let c_2 be an edge-coloring of G_2 such that

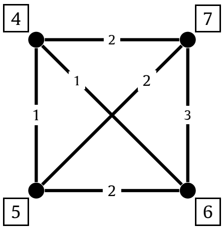

Fig. 1. A QM NSD 3-edge-coloring of K_4.

- $c_2(v_{2k}v_{2k+1}) = 2$;
- $c_2(v_{2k}v_i) = 2$ for $i \in \{1, \dots, k-1\}$;
- $c_2(v_{2k+1}v_i) = 2$ for $i \in \{1, \dots, k-1\}$;
- $c_2(v_{2k}v_i) = 1$ for $i \in \{k, \dots, 2k-1\}$;
- $c_2(v_{2k+1}v_i) = 3$ for $i \in \{k, \dots, 2k-1\}$.

Then, let c be the edge-coloring of K_{2k+1} such that $c(e) = c_i(e)$ if $e \in E(G_i)$ for $i = 1, 2$. We claim that c is the QM NSD 3-edge-coloring such that k vertices are incident to $k-1$ edges in color 2 and $k+1$ vertices are incident to k edges of color 2.

First, observe that c is a QM edge-coloring. By construction, c_2 is QM at v_{2k} and v_{2k+1}. The coloring c_1 implies that if $v_i \in \{v_1, \dots, v_{k-1}\}$, then it has $k-2$ edges colored with 2 in G_1. So, together with the two edges colored with 2 in G_2, every vertex v_i for $i \in \{1, \dots, k-1\}$ has k edges colored with 2. Since c_1 is a QM edge-coloring, every v_i for $i \in \{1, \dots, k-1\}$ has at most k edges colored with 1 and at most k edges colored with 3. This implies that c is QM at every vertex $v_i \in \{v_1, \dots, v_{k-1}\}$. Since every vertex v_j, for $j \in \{k, \dots, 2k-1\}$, is QM in (G_1, c_1) and we use two different colors at each vertex in c_2, it follows that it is also QM in (G, c). Thus, the edge-coloring c is QM.

Furthermore, vertices $v_1, \dots, v_{k-1}, v_{2k}, v_{2k+1}$ are incident to k edges colored with 2. Observe that every vertex v_j for $j \in \{k, \dots, 2k-1\}$ is adjacent to $k-1$ edges colored with 2, since edges incident to v_j in G_2 have colors 1 and 3.

Finally, we show that c is an NSD edge-coloring. $\sigma_c(v_{2k}) = \sigma_{c_2}(v_{2k}) = k + 2k = 3k$ and $\sigma_c(v_{2k+1}) = \sigma_{c_2}(v_{2k+1}) = 2k + 3k = 5k$, so $\sigma_c(v_{2k}) \neq \sigma_c(v_{2k+1})$. For every vertex v_i where $i \in \{1, \dots, 2k-1\}$, $\sigma_c(v_i) = \sigma_{c_1}(v_i) + 4$. Thus, since c_1 is an NSD edge-coloring, $\sigma_c(v_i) \neq \sigma_c(v_j)$ for $1 \le i < j \le 2k-1$. Observe that $3k - 3 = k - 1 + 2(k-1) \le \sigma_{c_1}(v_i) \le 2(k-1) + 3(k-1) = 5k - 5$ for $i \in \{1, \dots, 2k-1\}$. Thus, $\sigma_c(v_{2k}) \neq \sigma_c(v_i)$ and $\sigma_c(v_{2k+1}) \neq \sigma_c(v_i)$ for $i \in \{1, \dots, 2k-1\}$. Therefore, c is an NSD edge-coloring, which completes the proof. □

Lemma 4.2. *Let $k \ge 2$. Every complete graph K_{2k} has a* QM NSD 3*-edge-coloring in which at least $k-1$ vertices are incident to at most $k-1$ edges of color* 2*.*

Proof. Similarly as for the odd case, the proof goes by induction on the number of vertices. The lemma is true for $k = 2$, i.e. for the complete graph K_4 (see Fig. 1). Assume that it is true for all complete graphs of even order with fewer than $2k$ vertices. Let $V(K_{2k}) = \{v_1, v_2, \dots, v_{2k}\}$. We decompose K_{2k} into two edge disjoint subgraphs G_1 and G_2 such that $G = G_1 \cup G_2$. Let $G_1 = G[\{v_1, \dots, v_{2k-2}\}]$ and G_2 be a spanning subgraph of G that contains edges $E(G_2) = \{v_{2k-1}v_{2k}\} \cup \{v_{2k-1}v_i : i \in \{1, \dots, 2k-2\}\} \cup \{v_{2k}v_i : i \in \{1, \dots, 2k-2\}\}$. G_1 is isomorphic to K_{2k-2}, so by the induction hypothesis, a QM NSD 3-edge-coloring exists such that at least $k-2$ vertices are incident to at most $k-2$ edges in color 2. Let c_1 be such a coloring and $v_1, \dots, v_{k-2}$ be the vertices having at most $k-2$ edges in color 2. Let c_2 be an edge-coloring of G_2 such that

- $c_2(v_{2k-1}v_{2k}) = 2$;
- $c_2(v_{2k-1}v_i) = 2$ for $i \in \{1, \dots, k-2\}$;
- $c_2(v_{2k}v_i) = 2$ for $i \in \{1, \dots, k-2\}$;
- $c_2(v_{2k-1}v_i) = 1$ for $i \in \{k-1, \dots, 2k-2\}$;
- $c_2(v_{2k}v_i) = 3$ for $i \in \{k-1, \dots, 2k-2\}$.

Then, let c be the edge-coloring of K_{2k} such that $c(e) = c_i(e)$ if $e \in E(G_i)$, for $i = 1, 2$. We claim that c is a QM edge-coloring. By construction, c_2 is QM at v_{2k-1} and v_{2k}. The coloring c_1 implies that if $v_i \in \{v_1, \dots, v_{k-2}\}$, then it has at most $k-2$ edges in G_1 colored with 2. So, together with the two edges colored with 2 in G_2, every vertex v_i, for $i \in \{1, \dots, k-2\}$, has at most k edges colored with 2. Since c_1 is a QM edge-coloring, every vertex $v_i \in \{v_1, \dots, v_{k-2}\}$ is incident to at most $k-1$ vertices in color 1 and at most $k-1$ vertices in color 3. Furthermore, since every vertex v_j, for $j \in \{k-1, \dots, 2k-2\}$, is QM in (G_1, c_1) and we use two different colors at each vertex in c_2, it follows that it is also QM in (G, c). Thus, the edge-coloring c is QM.

Each vertex v_j, for $j \in \{k-1, \dots, 2k-2\}$, is adjacent to at most $k-1$ edges colored with 2, since c_1 is a quasi-majority edge-coloring, and the edges incident to v_j in G_2 are colored 1 and 3. Moreover, both v_{2k-1} and v_{2k} are adjacent to at most

$k-1$ edges colored with 2. Therefore, there are at least $k+2$ vertices adjacent to at most $k-1$ edges of color 2. Hence, we can select $k-1$ vertices that satisfy the conditions of the lemma.

Finally, we show that c is an NSD edge-coloring. We have $\sigma_c(v_{2k-1}) = \sigma_{c_2}(v_{2k-1}) = 3k-2$ and $\sigma_c(v_{2k}) = \sigma_{c_2}(v_{2k}) = 5k-2$, so $\sigma_c(v_{2k-1}) \neq \sigma_c(v_{2k})$. For every vertex v_i where $i \in \{1,\dots,2k-2\}$, $\sigma_c(v_i) = \sigma_{c_1}(v_i)+4$. Thus, since c_1 is an NSD edge-coloring, $\sigma_c(v_i) \neq \sigma_c(v_j)$ for $1 \le i < j \le 2k-2$. Observe that $3k-5 \le \sigma_{c_1}(v_i) \le 5k-7$ for $i \in \{1,\dots,2k-2\}$. Thus, $\sigma_c(v_{2k}) \neq \sigma_c(v_i)$ and $\sigma_c(v_{2k+1}) \neq \sigma_c(v_i)$ for $i \in \{1,\dots,2k-2\}$. Therefore, c is an NSD edge-coloring. □

Theorem 4.3. *For $n \ge 3$, we have $\chi_{\sum}^{QM}(K_n) = 3$.*

Proof. From Lemmas 4.1 and 4.2 it follows that $\chi_{\sum}^{QM}(K_n) \le 3$. Let c be a QM edge-coloring of K_n with 2 colors. If n is odd, then $\sigma_c(v) = \frac{3n-3}{2}$ for every $v \in V(K_n)$, so c is not NSD. If n is even, then $\sigma_c(v) = \frac{3n}{2}-2$ or $\sigma_c(v) = \frac{3n}{2}-1$ for every $v \in V(K_n)$, so again c is not NSD. □

4.2. Bipartite graphs

To prove an upper bound on the QM NSD index for bipartite graphs, we use the result established in [8].

Theorem 4.4. (Karoński et al. [8]) *Let Γ be a finite abelian group of odd order and let G be a non-trivial $|\Gamma|$-colorable graph. Then, there is an edge-coloring of G with elements of Γ such that the resulting vertex-coloring is proper.*

In [3], it was noted that the proof of Theorem 4.4 leads to the following result.

Theorem 4.5. (Dailly, Sidorowicz [3]) *Let G be a connected, nice bipartite graph with a bipartition (V_1, V_2). Then G admits an* NSD *3-edge-coloring. Moreover, there is an* NSD *3-edge-coloring c of G such that $\sigma_c(v_1) \neq \sigma_c(v_2) (\text{mod } 3)$ for every $v_1 \in V_1$ and $v_2 \in V_2$.*

Theorem 4.6. *Every nice bipartite graph G satisfies $\chi_{\sum}^{QM}(G) \le 6$.*

Proof. Let (V_1, V_2) be a bipartition of G. By Theorem 4.5, there is an NSD 3-edge-coloring c of the graph G such that $\sigma_c(v_1) \neq \sigma_c(v_2) (\text{mod } 3)$ for $v_1 \in V_1, v_2 \in V_2$. By E_i we denote the set of edges with color i for $i \in [3]$. By Corollary 2.5, there is QM 2-edge-coloring of every subgraph induced by E_i. Thus, we recolor some edges of E_1 with color 4 in such a way that the coloring is QM at every vertex in subgraph induced by E_1. In a similar manner, we recolor certain edges of E_2 with the color 5 and some edges of E_3 with the color 6. Let c' be the obtained edge-coloring. Thus $\sigma_c(v) = \sigma_{c'} (\text{mod } 3)$ and so c' is an QM NSD 6-edge-coloring. □

For complete bipartite graphs, we have a strict result. It is easily seen that $\chi_{\sum}^{QM}(K_{2,2}) = 4$.

Theorem 4.7. *Every nice, complete bipartite graph $K_{n,m}$ such that $K_{n,m} \neq K_{2,2}$ satisfies*

$$\chi_{\sum}^{QM}(K_{n,m}) = \begin{cases} 2, & \text{if } n \neq m, \\ 3, & \text{otherwise.} \end{cases}$$

Proof. By Corollary 2.5, $K_{n,m}$ has a QM 2-edge-coloring. If $n \neq m$, then no two adjacent vertices have the same degree. Thus, by Proposition 3.5, we have $\chi_{\sum}^{QM}(K_{n,m}) = 2$ for $n \neq m$.

When $m = n$, two colors are not sufficient for a QM NSD edge-coloring of $K_{n,n}$. This is because the only possibility for a QM 2-edge-coloring of $K_{n,n}$ is to color $\lceil \frac{n}{2} \rceil$ edges at each vertex with one color and the remaining edges with the other color, but such an edge-coloring will not be NSD. We now show that if $m = n$, then there exists a QM NSD 3-edge-coloring of $K_{n,n}$.

Let $K_{n,n} = (V_1, V_2, E)$, where $V_1 = \{a_1,\dots,a_n\}$, $V_2 = \{b_1,\dots,b_n\}$. We decompose $K_{n,n}$ into the following three edge disjoint subgraphs G_1, G_2, and G_3. Let $G_1 = K_{n,n}\left[\left\{a_1,\dots,a_{\lceil\frac{n}{2}\rceil}, b_1,\dots,b_{\lceil\frac{n}{2}\rceil}\right\}\right]$, $G_2 = K_{n,n}\left[\left\{a_{\lceil\frac{n}{2}\rceil+1},\dots,a_n, b_{\lceil\frac{n}{2}\rceil+1},\dots,b_n\right\}\right]$ and let G_3 be a spanning subgraph of $K_{n,n}$ with $E(G_3) = E(K_{n,n}) \setminus (E(G_1) \cup E(G_2))$. Let c_1 be an edge-coloring of G_1 and G_2 such that every edge of G_1 and G_2 has color 1.

First, we consider the case when n is odd. Let c_2 be an edge-coloring of G_3 such that $c_2(a_i b_j) = 2$ for $i \le \lceil \frac{n}{2} \rceil$, $j \ge \lceil \frac{n}{2} \rceil + 1$, and $c_2(a_i b_j) = 3$ for $i \ge \lceil \frac{n}{2} \rceil + 1$, $j \le \lceil \frac{n}{2} \rceil$. By the construction of c_1 and c_2,

- at every a_i with $i \le \lceil \frac{n}{2} \rceil$, we have $\lceil \frac{n}{2} \rceil$ edges with color 1 and $\lfloor \frac{n}{2} \rfloor$ edges with color 2;
- at every a_i with $i \ge \lceil \frac{n}{2} \rceil + 1$, we have $\lceil \frac{n}{2} \rceil$ edges with color 3 and $\lfloor \frac{n}{2} \rfloor$ edges with 1;
- at every b_i with $i \le \lceil \frac{n}{2} \rceil$, we have $\lceil \frac{n}{2} \rceil$ edges with color 1 and $\lfloor \frac{n}{2} \rfloor$ edges with 3;

- at every b_i with $i \geq \lceil \frac{n}{2} \rceil + 1$, we have $\lceil \frac{n}{2} \rceil$ edges with color 2 and $\lfloor \frac{n}{2} \rfloor$ edges with 1.

The edge-coloring c is QM. We can also observe that $\sigma_c(a_i) = \frac{3n-1}{2}$ for every vertex a_i with $i \leq \lceil \frac{n}{2} \rceil$; $\sigma_c(a_i) = 2n+1$ for every vertex a_i with $i \geq \lceil \frac{n}{2} \rceil + 1$; $\sigma_c(b_i) = 2n-1$ for every vertex b_i with $i \leq \lceil \frac{n}{2} \rceil$; and $\sigma_c(b_i) = \frac{3n+1}{2}$ for every vertex b_i with $i \geq \lceil \frac{n}{2} \rceil + 1$. Hence, the edge-coloring c is NSD.

Let now n be even and let c_2 be an edge-coloring of G_3 such that $c_2(a_ib_j) = 2$ if i is odd, $j \in [n]$ and $c_2(a_ib_j) = 3$ if i is even, $j \in [n]$. By the construction of c_1 and c_2, we obtain:

- at every vertex a_i with odd i, we have $\frac{n}{2}$ edges with color 1 and $\frac{n}{2}$ edges with color 2;
- at every a_i with even i, we have $\frac{n}{2}$ edges with color 1 and $\frac{n}{2}$ edges with color 3;
- at every vertex b_i with $i \leq \frac{n}{2}$, we have $\frac{n}{2}$ edges with color 1, $\lceil \frac{n}{4} \rceil$ edges with color 3, and $\lfloor \frac{n}{4} \rfloor$ edges with color 2;
- at every b_i with $i \geq \frac{n}{2} + 1$, we have $\frac{n}{2}$ edges with color 1, $\lceil \frac{n}{4} \rceil$ edges with color 2, and $\lfloor \frac{n}{4} \rfloor$ edges with color 3.

It is easy to see that the edge-coloring c is QM. For every vertex a_i with odd i, we have $\sigma_c(a_i) = \frac{3n}{2}$; for every vertex a_i with even i, we have $\sigma_c(a_i) = 2n$. If $n \equiv 0 \pmod 4$, then $\sigma_c(b_i) = \frac{7n}{4}$ for every vertex b_i, $i = \{1, \dots, \frac{n}{2}\}$. If $n \equiv 2 \pmod 4$, then $\sigma_c(b_i) = \frac{7n+2}{4}$ for $i = \{1, \dots, \frac{n}{2}\}$ and $\sigma_c(b_i) = \frac{7n-2}{4}$ for $i = \{\frac{n}{2}+1, \dots, n\}$. Hence, the edge-coloring c is also NSD. □

4.3. Trees

Theorem 4.8. *For each tree T of order $n \geq 3$ we have*

$$\chi_{\Sigma}^{QM}(T) = \begin{cases} 2, & \text{if } T \text{ has no two adjacent vertices of equal even degree,} \\ 3, & \text{otherwise.} \end{cases}$$

Proof. Clearly, $\chi_{\Sigma}^{QM}(T) \geq 2$ for every tree with at least three vertices. Furthermore, if in T there are two adjacent vertices x, y such that $d(x) = d(y) = k$ and k is even, then to obtain a QM edge-coloring, we have only two possibilities. We can color xy with color 1 and assign color 2 to each of the $\frac{k}{2}$ edges incident to each vertex, with the remaining $\frac{k}{2} - 1$ edges receiving color 1. Alternatively, we can color xy with color 2 and assign color 1 to each of the $\frac{k}{2}$ edges incident to each vertex, with the remaining $\frac{k}{2} - 1$ edges receiving color 2. In both cases, $\sigma_c(x) = \sigma_c(y)$; thus, we have to use three colors.

Pick a vertex v_0 as a root of the tree T. Let $l(v)$ be the distance of a vertex v to the root v_0, and let $h(T) = \max_{v \in V(T)} l(v)$. Each value of $l(v)$ we call a *level* of T.

We show that there exists a QM NSD 2-edge-coloring of T if T has no two adjacent vertices of equal even degree, and a QM NSD 3-edge-coloring, otherwise. In both cases, we start the edge-coloring from the edges incident to the root v_0 and color these edges such that the edge-coloring at v_0 is QM. Then we consider consecutive vertices in the BFS ordering until we color all edges in T.

Suppose that y is the first vertex in the BFS coloring with an uncolored edge. Let x be its parent. Denote by c the partial coloring obtained so far. Then c is QM at x. We show how to extend the edge-coloring c to uncolored edges incident to y such that $\sigma_c(x) \neq \sigma_c(y)$ and c is QM at y. Observe that if y is a leaf, then $\sigma_c(x) \neq \sigma_c(y)$, and c already is a QM edge-coloring at y, thus we can assume that y is a non-leaf child of x.

Case 1. T has no two adjacent vertices of equal even degree.

We prove that we can always color edges incident to y with two colors such that the edge-coloring is QM at y and distinguishes x and y by sums. At the beginning note that if we want to color the edges incident to any vertex $v \in V(T)$ with a QM 2-edge-coloring, we have the following three types of possible colorings.

(1) $d(v)$ is odd, $\frac{d(v)+1}{2}$ edges are colored with 1, and $\frac{d(v)-1}{2}$ edges with 2.
(2) $d(v)$ is odd, $\frac{d(v)+1}{2}$ edges are colored with 2, and $\frac{d(v)-1}{2}$ edges with 1.
(3) $d(v)$ is even, $\frac{d(v)}{2}$ edges are colored with 1, and $\frac{d(v)}{2}$ edges with 2.

If the edge-coloring is of type (i) at v, then $\sigma_c(v) \equiv i \pmod 3$, for $i \in \{1, 2\}$. If the edge-coloring is type (3) at v, then $\sigma_c(v) \equiv 0 \pmod 3$.

Observe that if $d(y)$ is even and since $d(y) \geq 2$, then it follows that irrespective of the color of xy, we have the opportunity to color the uncolored edges at y so that we achieve at y an edge-coloring of type (3). On the other hand, when $d(y)$ is odd and since $d(y) \geq 3$, regardless of a color of the edge xy, we can extend the edge-coloring at y to an edge-coloring of type 1 and type 2.

We color the edges incident to y in the following way.

- If $d(y)$ is even, then we apply such an edge-coloring to ensure that we have a coloring of type (3) at y.

Table 1
The types of colorings to which the coloring at y is extended.

		The type of coloring at x					
		E1	E2	E3	O1	O2	O3
$c(xy)$	1	E3, O3	E1/E3, O1/O3	E1, O1	E3, O3	E1/E3, O1/O3	E1, O1
	2	E2/E3, O2/O3	E3, O3	E2, O2	E2/E3, O2	E3, O3	E2, O2
	3	E2, O2/O3	E1, O1/O3		E2, O2/O3	E1, O1/O3	E1/E2, O1/O2

- If $d(y)$ is odd, then the coloring of the edges incident to y depends on the coloring of edges incident to x. When $d(x)$ is even or $d(x)$ is odd and the edge-coloring at x is of type (2), then we color the edges so that we obtain at y the edge-coloring of type (1). When $d(x)$ is odd and the edge-coloring at x is of type (1), then we color the edges so that we obtain at y the edge-coloring of type (2).

Obviously, the edge-coloring is QM at y. We claim that it distinguishes x and y. If $d(x)$ is odd and $d(y)$ is even, then $\sigma_c(x) \equiv 1 \text{ or } 2 \pmod 3$ and $\sigma_c(y) \equiv 0 \pmod 3$, therefore $\sigma_c(x) \neq \sigma_c(y)$. If $d(x)$ and $d(y)$ are both odd, then we have chosen the type of edge-coloring for y such that $\sigma_c(x) \not\equiv \sigma_c(y) \pmod 3$. If $d(x)$ is even and $d(y)$ is odd, then $\sigma_c(x) \equiv 0 \pmod 3$ and $\sigma_c(y) \equiv 1 \text{ or } 2 \pmod 3$, hence $\sigma_c(x) \neq \sigma_c(y)$. If both $d(y), d(x)$ are even, then $\sigma_c(y) \neq \sigma_c(x)$, since $d(y) \neq d(x)$.

Case 2. T has two adjacent vertices of equal even degree.

Now we prove that we can always color the edges incident to y with three colors so that the edge-coloring is QM at y and distinguishes x and y by sums. First, observe that, in contrast to Case 1, there are many possibilities for coloring edges incident to a vertex that result in a QM edge-coloring, as we can use three colors. We define specific types of edge-colorings at the vertex, distinguishing between vertices of even degree and vertices of odd degree. We choose colors for edges in such a way that each type results in a sum at the vertex that is different modulo 3.

Let $v \in V(T)$ be a vertex of even degree. We use the following three types of coloring of the edges incident to v.

E1: $\frac{d(v)}{2}$ edges have color 1, $\frac{d(v)}{2} - 1$ edges have color 2, and one edge has color 3; so $\sigma_c(v) \equiv 1 \pmod 3$;
E2: $\frac{d(v)}{2} - 1$ edges have color with 1, $\frac{d(v)}{2}$ edges have color 2, and one edge has color 3; so $\sigma_c(v) \equiv 2 \pmod 3$;
E3: $\frac{d(v)}{2}$ edges have color 1 and $\frac{d(v)}{2}$ edges have color 2; so $\sigma_c(v) \equiv 0 \pmod 3$.

If $v \in V(T)$ has odd degree, then we use the following three types of coloring of the edges incident to v.

O1: $\frac{d(v)-1}{2}$ edges we have color 1, $\frac{d(v)-3}{2}$ edges have color 2, and two edges have color 3; so $\sigma_c(v) \equiv 1 \pmod 3$;
O2: $\frac{d(v)-3}{2}$ edges we have color 1, $\frac{d(v)-1}{2}$ edges have color 2, and two edges have color 3; so $\sigma_c(v) \equiv 2 \pmod 3$;
O3: if $\frac{d(v)-1}{2}$ edges we color with 1, $\frac{d(v)-1}{2}$ edges have color 2 and one edge has color 3; so $\sigma_c(v) \equiv 0 \pmod 3$.

Observe that depending on the color of the edge xy and the parity of $d(y)$, one type of edge-coloring is available for the edges incident to y so that $\sigma_c(x) \not\equiv \sigma_c(y) \pmod 3$. Then, the edge-coloring distinguishes vertices x and y, and is QM at y.

Table 1 shows the types of coloring to which we extend the coloring c to the edges incident to y depending on $c(xy)$ and the type of coloring at x. One cell in the table is empty because in the coloring of type E3 there is no edge with color 3. □

4.4. Graphs with maximum degree at most 4

If G is connected and $\Delta(G) \leq 2$, then G is either a cycle or a path. Thus, Propositions 3.1 and 3.2 give the value of QM NSD index for graphs with maximum degree at most 2. If $\Delta(G) \leq 3$ and G has no component isomorphic to C_5, then the QM NSD index is at most 4, by Theorem 3.4. In this subsection, we prove that there is a QM NSD 7-edge-coloring of every nice graph with maximum degree at most 4. The proof is based on the Combinatorial Nullstellensatz of Alon [1].

Theorem 4.9 *(Alon [1]). Let $\mathbb{F}$ be an arbitrary field, and let $P = P(x_1, \dots, x_n)$ be a polynomial in $\mathbb{F}[x_1, \dots, x_n]$. Suppose the degree of P equals $\sum_{i=1}^{n} k_i$, where each k_i is a nonnegative integer, and suppose the coefficient of $x_1^{k_1} \cdots x_n^{k_n}$ in P is nonzero. Then if $S_1, \dots, S_n$ are subsets of $\mathbb{F}$ with $|S_i| > k_i$, there are $s_1 \in S_1, \dots, s_n \in S_n$ such that $P(s_1, \dots, s_n) \neq 0$.*

Theorem 4.10. *For every nice graph G with maximum degree at most 4, we have*

$$\chi_{\sum}^{QM}(G) \leq 7.$$

Proof. We proceed by induction on the number of edges. By Theorem 3.4, the result is true for all nice subcubic graphs. Thus, the result is also true for graphs of size two or three. Assume that the result is true for graphs of size at most $m - 1$.

Let G be a nice graph with $|E(G)| = m$ and $\Delta(G) = 4$. We may assume that G is connected since otherwise, by induction, every component has a QM NSD 7-edge-coloring. Let $v \in V(G)$ and $d(v) = \Delta(G) = 4$.

Case 1. In $N(u)$, there are two adjacent vertices.

Let $N(v) = \{v_1, v_2, v_3, v_4\}$ and $v_1v_2 \in E(G)$. Let $G' = G - \{vv_1, vv_2\}$. Each component of G' different from K_2 satisfies the claim. Observe that G' has at most one component isomorphic to K_2. By the induction hypothesis, there is a QM NSD 7-edge-coloring of the components of G' that are different from K_2. Let c be such a coloring. Additionally, if G' contains a K_2, we extend c by coloring it with any color from [7].

To complete the edge-coloring, it remains to assign colors to the two edges vv_1 and vv_2 in such a way that all vertices in v, v_1, v_2 are distinguished from their neighbors. The coloring should be QM at v, v_1, v_2, v_3, v_4.

First, we count how many colors are forbidden for the edges vv_1 and vv_2 to obtain an edge-coloring ensuring that: (1) all neighbors, except for the pairs (v_1, v_2), (v, v_3), and (v, v_4), are distinguished; and (2) the coloring is QM at each vertex, except v.

Consider the edge vv_1. Next, we analyze the situation based on the degree of v_1. If v_2 is the only neighbor of v_1 in G', then the color of vv_1 must be different from $c(v_1v_2)$, and therefore there is at most one forbidden color for vv_1. If v_1 has exactly two neighbors v_2 and u_1 in G', then v_1 must be distinguished from u_1. However, regardless of a color we use for vv_1, the coloring will be QM at v_1 since $c(v_1u_1) \neq c(v_1v_2)$. Thus, again, at most one color is forbidden. If v_1 has three neighbors in G', then at most one color must be forbidden to ensure that the coloring will be QM at v_1, and at most two additional colors must be forbidden to ensure that v_1 is distinguished from the two neighbors (except v_2). So, at most three colors are forbidden. In summary, the edge vv_1 can have at most three forbidden colors. Similarly, for the edge vv_2, there are also at most three forbidden colors.

We denote by F_1 the set of admissible colors for the edge vv_1, and by F_2 be the set of admissible colors for the edge vv_2, so $|F_1| \geq 4$ and $|F_2| \geq 4$. By selecting a color for vv_1 from F_1 and a color for vv_2 from F_2, we obtain a 7-edge-coloring of G in which all neighbors, except (v_1, v_2), (v, v_3), and (v, v_4), are distinguished. Additionally, the coloring is QM at each vertex, except v.

Let $x_1 \in F_1$ and $x_2 \in F_2$ be the colors assigned to the edges vv_1 and vv_2, respectively. To ensure a QM NSD edge-coloring with the colors x_1 and x_2, the following conditions must be satisfied:

- $x_1 + x_2 + \sigma_c(v) \neq \sigma_c(v_i)$ – we need to ensure that v is distinguished from each v_i for $i \in \{3, 4\}$;
- $x_2 + \sigma_c(v) \neq \sigma_c(v_1)$ – we need to ensure that v and v_1 are distinguished from each other;
- $x_1 + \sigma_c(v) \neq \sigma_c(v_2)$ – we need to ensure that v and v_2 are distinguished from each other;
- $x_1 + \sigma_c(v_1) \neq x_2 + \sigma_c(v_2)$ – we need to ensure that v_1 and v_2 are distinguished from each other;
- $x_1 \neq x_2$ – we need to ensure that the coloring is QM at v.

To demonstrate the existence of colors x_1 and x_2 that satisfy all the aforementioned conditions, we define the polynomial:

$$\begin{aligned} P(x_1, x_2) = (&x_1 + x_2 + \sigma_c(v) - \sigma_c(v_3)) \\ &(x_1 + x_2 + \sigma_c(v) - \sigma_c(v_4) \\ &(x_2 + \sigma_c(v) - \sigma_c(v_1)) \\ &(x_1 + \sigma_c(v) - \sigma_c(v_2)) \\ &(x_1 - x_2 + \sigma_c(v_1) - \sigma_c(v_2)) \\ &(x_1 - x_2). \end{aligned}$$

If there exist values of x_1 and x_2 such that $P(x_1, x_2) \neq 0$ and $x_i \in F_i$ for $i \in [2]$, then the x_i's satisfy all the conditions. By coloring vv_1 and vv_2 with x_1 and x_2, respectively, we can extend the edge-coloring c to a QM NSD edge-coloring.

We apply Theorem 4.9 to prove that x_1 and x_2 exist. First, we assert that the coefficient of the monomial $x_1^3x_2^3$ is non-zero. Note that this coefficient in P is identical to the one in the following polynomial:

$$P_1(x_1, x_2) = (x_1 + x_2)^2(x_1 - x_2)^2x_1x_2.$$

The coefficient of the monomial $x_1^3x_2^3$ is -2. Since $|F_1| > 3$ and $|F_2| > 3$, Theorem 4.9 implies that there are $x_1 \in F_1$ and $x_2 \in F_2$ such that $P(x_1, x_2) \neq 0$. Therefore, a QM NSD 7-edge-coloring of G exists.

Case 2. There is no edge in $N(v)$

Let $N(v) = \{v_1, v_2, v_3, v_4\}$ and $G' = G - v$. Each component of G' different from K_2 admits a QM NSD 7-edge-coloring. Let c be a QM NSD 7-edge-coloring of components of G' and additionally extend c to the components isomorphic to K_2, which we color with any color from [7].

To complete the edge-coloring, we only need to color the edges vv_i for $i \in [4]$. We select a color for each edge vv_i in such a way that each v_i is distinguished from its neighbors in G', and the coloring is QM at each v_i for $i \in [4]$. Additionally, after coloring the four edges vv_1, vv_2, vv_3, and vv_4, the vertex v must be distinguished from its neighbors, and the coloring must be QM at v.

First, we count how many colors we need to forbid for the edges vv_i to obtain an edge-coloring in which v_i is distinguished from its neighbors in G, and the coloring is QM at v_i for $i \in [4]$. We analyze the situation based on the degree of v_i in G'.

If $d_{G'}(v_i) = 0$, then we can use for vv_i any color from [7].

If $d_{G'}(v_i) = 1$, then to distinguish v_i from its neighbor, at most one color is forbidden. Furthermore, the color of vv_i has to be different from the color of the edge incident to v_i in G', resulting in at most two forbidden colors in total.

If $d_{G'}(v_i) = 2$, then to distinguish v_i from its neighbors, at most two colors are forbidden. However, regardless of which color we use for vv_i, the coloring will still be QM at v_i, giving us at most two forbidden colors in total.

If $d_{G'}(v_i) = 3$, then to distinguish v_i from its neighbors, at most three colors are forbidden. Furthermore, if v_i is incident to two edges of one color, then this color is also forbidden for vv_i, so we have at most four forbidden colors in this case.

In summary, there are at most four forbidden colors for vv_i. We denote by F_i the set of admissible colors for the edge vv_i, so $|F_i| \geq 3$ for $i \in [4]$, so $|F_i| \geq 3$ for $i \in [4]$. After coloring each edge vv_i with a color $x_i \in F_i$ for $i \in [4]$, we obtain a 7-edge-coloring of G in which all neighbors, except (v, v_i) for $i \in [4]$, are distinguished. In addition, the coloring is QM at each vertex, except u.

Let $x_i \in F_i$ be the colors assigned to vv_i for $i \in [4]$. To ensure a QM NSD edge-coloring, for the colors x_i, the following conditions must also be met:

- $x_1 + x_2 + x_3 + x_4 - x_i \neq \sigma_c(v_i)$ – we need to ensure that v and v_i are distinguished for $i \in [4]$;
- $x_1 \neq x_2$ and $x_3 \neq x_4$ – we need to ensure that the coloring is QM at v.

We consider the polynomial

$$\begin{aligned} P(x_1, x_2, x_3, x_4) = &(x_2 + x_3 + x_4 - \sigma_c(v_1)) \\ &(x_1 + x_3 + x_4 - \sigma_c(v_2)) \\ &(x_1 + x_2 + x_4 - \sigma_c(v_3)) \\ &(x_1 + x_2 + x_3 - \sigma_c(v_4)) \\ &(x_1 - x_2)(x_3 - x_4). \end{aligned}$$

If there are values x_1, x_2, x_3, x_4 such that $P(x_1, x_2, x_3, x_4) \neq 0$ and $x_i \in F_i$ for $i \in [4]$, then the values x_i meet all the conditions. By coloring the edges vv_i with the colors x_i, we can extend the edge-coloring c to a QM NSD 7-edge-coloring.

We utilize Theorem 4.9 again at this step to show that such x_i exist. We examine the coefficient of the monomial $x_1^2 x_2 x_3^2 x_4$. Note that this coefficient in P is identical to the one in the following polynomial:

$$P_1(x_1, x_2, x_3, x_4) = (x_2 + x_3 + x_4)(x_1 + x_3 + x_4)(x_1 + x_2 + x_4)(x_1 + x_2 + x_3)(x_1 - x_2)(x_3 - x_4).$$

The coefficient of the monomial $x_1^2 x_2 x_3^2 x_4$ is equal to 3. Since $|F_1| > 2$, $|F_2| > 1$, $|F_3| > 2$ and $|F_4| \geq 1$, Theorem 4.9 guarantees that there exist values $x_i \in F_i, i \in [4]$ such that $P(x_1, x_2, x_3, x_4) \neq 0$ and therefore a QM NSD 7-edge-coloring of G exists. □

5. A general upper bound

In [7], Kalkowski, Karoński and Pfender proved that $\chi^e_\Sigma(G) \leq 5$ for every nice graph G. Using a modification of their proof, we obtain the following upper bound for $\chi^{QM}_\Sigma(G)$ of an arbitrary nice graph G.

Theorem 5.1. *For every nice graph G,*

$$\chi^{QM}_\Sigma(G) \leq 12.$$

Proof. Without loss of generality, we may assume that G is connected since otherwise we could consider connected components separately. Moreover, we may assume that $|V(G)| \geq 3$ and $\Delta(G) \geq 2$ since G is a nice graph, and there is nothing to prove for $G = K_1$. We present a construction of a QM NSD coloring $c : E(G) \to [12]$.

Order the vertex set $V(G) = \{v_1, \ldots, v_n\}$ such that $d(v_n) \geq 2$, and each vertex v_i, for $i < n$, has a neighbor v_j with $j > i$. Take any QM 3-edge-coloring c_0 of G with colors 5, 6, 7 and put an initial value $c(e) = c_0(e)$ for each edge $e \in E(G)$. In our algorithm, we can change the color $c(e)$ of each edge e at most twice. To every vertex v_i with $i < n$, we will assign a set $W(v_i) = \{w(v_i), w(v_i) + 4\}$ of possible final values of $\sigma_c(v_i)$ such that $w(v_i) \in \{0, 1, 2, 3\} \pmod 8$, and $W(v_i) \cap W(v_j) = \emptyset$ for every $v_i v_j \in E(G)$. In the final step of our algorithm, we will adjust the colors of the edges incident to v_n to guarantee that $\sigma_c(v_n) \neq \sigma_c(v_i)$ whenever $v_i v_n \in E(G)$.

In the first step, we count $\sigma_c(v_1) = \sum_{u \in N(v_1)} c_0(v_1 u)$, and define the sets $W(v_1) = \{w(v_1), w(v_1)+4\}$ by putting $w(v_1) = \sigma_c(v_1)$ if $\sigma_c(v_1) \in \{0, 1, 2, 3\} \pmod 8$, or $w(v_1) = \sigma_c(v_1) - 4$ otherwise.

Let $2 \le k \le n-1$, and assume that we have already established the set $W(v_i)$ for each $i < k$, and

(1) $c(v_i v_j) \in [12]$ for $1 \le i < j \le n$,
(2) $\sigma_c(v_i) \in W(v_i)$ for $i < k$,
(3) $c(v_k v_j) = c_0(v_j v_k)$ for $j > k$,
(4) $c(v_i v_k) \in \{5, 6, 7, 8\}$ for $i < k$.

In view of condition (4), if $v_i v_k \in E(G)$, then we can either add or subtract 4 to $c(v_i v_k)$, so that the resulting coloring c is still quasi-majority and $\sigma_c(v_i) \in W(v_i)$. If v_k has d neighbors v_i with $i < k$, then this gives us $d+1$ choices for $\sigma_c(v_k)$. Additionally, we can change $c(v_k v_{j_0})$, where j_0 is the smallest $j > k$ with $v_k v_j \in E(G)$, using the following rule. It is easy to see that if we want to change the color of $c(v_k v_{j_0})$, then for each of its end-vertices v_k, v_{j_0}, there is at most one color violating the quasi-majority coloring of that vertex. Hence, we can choose a new color $c(v_k v_{j_0}) \in \{5, 6, 7, 8\}$ such that c is still a quasi-majority edge-coloring of G. This gives us additional d choices for $\sigma_c(v_k)$. In total, we have $2d+1$ possible values for $\sigma_c(v_k)$, at least one of them does not belong to any $W(v_i)$ for $i < k$. Therefore, a possible recoloring of some edges $v_i v_k$ with $i < k$, and an edge $v_k v_{j_0}$, results in an edge-coloring c satisfying the conditions:

(1′) $c(v_i v_j) \in [12]$ for $1 \le i < j \le n$,
(2′) $\sigma_c(v_i) \in W(v_i)$ for $i \le k$,
(3′) $c(v_k v_j) = c_0(v_j v_k)$ for $j > k$, except for $j = j_0$,
(4′) $c(v_i v_n) \in \{5, 6, 7, 8\}$ for $i < n$.

This way, we successively assign pairwise disjoint sets $W(v_k)$ for all $k \le n-1$.

In the final step, we have to determine $\sigma_c(v_n)$. If $v_i v_n \in E(G)$ for some $i < n$, then condition (4′) allows us to subtract or add 4 to $c(v_i v_n)$ such that $\sigma_c(v_i) \in W(v_i)$. Hence, we have $d(v_n) + 1 \ge 3$ possible options for $\sigma_c(v_n)$. Let s be the smallest such possible option, which we obtain by subtracting 4 from $c(v_i)$ for all $v_i \in N(v_n)$ with $\sigma_c(v_i) = w(v_i) + 4$, and adding 4 nowhere. If $s \in \{4, 5, 6, 7\} \pmod 8$, then s cannot be equal to $\sigma_c(v_i)$ for any neighbor v_i of v_n since otherwise we could increase s by subtracting 4 from $c(v_i v_n)$. Hence, we can put $\sigma_c(v_n) = s$. Let then $s \in \{0, 1, 2, 3\} \pmod 8$. If there exists a $v_i \in N(v_n)$ with $\sigma_c(v_i) \neq s$, then we keep $c(v_i)$ unchanged and subtract 4 everywhere else, thus obtaining a suitable value $\sigma_c(v_n) = s + 4$. If $\sigma_c(v_i) = w(v_i)$ for all $v_i \in N(v_n)$ then we subtract 4 from all edges incident to v_n except two of them.

The resulting values of σ_c yield a proper vertex-coloring of G. □

For $\Delta(G) = 5$ and $\Delta(G) = 6$, we get a better upper bound for $\chi_{\sum}^{QM}(G)$ than in the results given previously.

Theorem 5.2. *For every nice graph G,*

$$\chi_{\sum}^{QM}(G) \le \left\lceil \frac{3\Delta + 4}{2} \right\rceil .$$

Proof. We can assume that G is connected, as otherwise, we could analyze its connected components independently. Furthermore, we may take $|V(G)| \ge 3$ and $\Delta(G) \ge 2$, given that G is a nice graph, and the case $G = K_1$ does not require any further examination.

Let $m = |E(G)|$. We employ induction with respect to m. For $m = 2$, the claim is obvious, so assume that $m \ge 3$, and that a suitable edge-coloring exists for every graph with less than m edges. Let G be a graph with m edges. Take any vertex y of degree $d(y) \ge 2$ and edges xy and yz incident to it. Assume that x and z are not adjacent; otherwise, G is a complete graph, and the assertion follows directly from Theorem 4.3. Let H denote the graph obtained by deleting the edges xy and yx from G. Under the induction hypothesis, we know that each component of H with at least two edges can be colored with a QM NSD $\left\lceil \frac{3\Delta+4}{2} \right\rceil$-edge-coloring. For the rest of the components, which consist of isolated edges or vertices, we assign color 1. We demonstrate that this coloring can be extended to the required edge-coloring of G.

Suppose that the neighborhood sets of the vertices x, y, and z in graph H are non-empty. Let $x_1, \dots, x_p$ be the vertices adjacent to x in H, $y_1, \dots, y_q$ be the vertices adjacent to y in H, and $z_1, \dots, z_r$ be the vertices adjacent to z in H. Then p, r do not exceed $\Delta - 1$ and q does not exceed $\Delta - 2$. We begin by determining the number of colors available to color the edge xy in order to extend edge-coloring to the desired one.

- We cannot use one color to avoid an edge-coloring not QM at x.
- We may have to exclude at most $\Delta - 1$ colors because, while choosing the color of xy we fix the sum at x, which may be the same as the sums already determined at $x_1, \dots, x_p$.
- Moreover, one color can produce the same sums at y and z (regardless of the color chosen for the edge yz, which contributes to the sums at both y and z).

In total, we have at least $\left\lceil \frac{3\Delta+4}{2} \right\rceil - 1 - (\Delta - 1) - 1 = \left\lceil \frac{\Delta+2}{2} \right\rceil$ free choices of colors for xy. Analogously, we also have at least $\left\lceil \frac{\Delta+2}{2} \right\rceil$ free colors for xy, which is not colored yet.

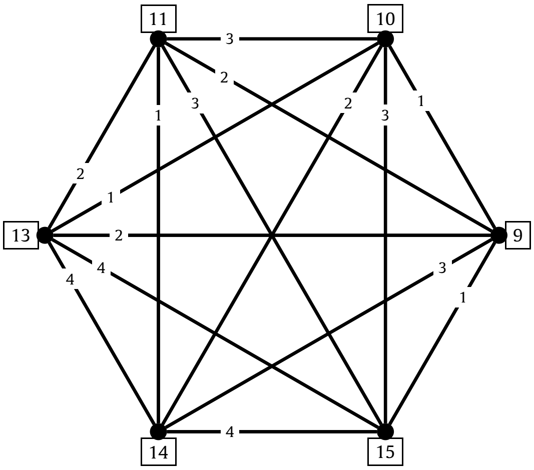

Fig. 2. A majority NSD 4-edge-coloring of K_6.

All we have to do now is to choose the colors for xy and yz. From lists of lengths at least $\left\lceil \frac{\Delta+2}{2} \right\rceil$ we have to choose two colors so that they are different (otherwise the edge-coloring at y will not be QM) and such that the sum at y is distinct from the at most $\Delta - 2$ sums already determined at $y_1, \dots, y_q$. To demonstrate that this is always achievable, it is enough to observe that for any two lists of numbers A, B, each containing s elements, there are at least $2s-3$ pairs $(a_i, b_i) \in A \times B$ such that $a_i \neq b_i$, $i = 1, \dots, 2s-3$, for which every sums $a_i + b_i$ are all different from each other. Specifically, for $s = \left\lceil \frac{\Delta+2}{2} \right\rceil$ we obtain $2s - 3 \geq (\Delta - 2) + 1$. In fact, these are, for example, the pairs from $(\{a\} \times (B \setminus \{a\})) \cup ((A \setminus \{a, b\}) \times \{b\})$, where $a = \min A$ and $b = \max B$.

The resulting edge-coloring meets our conditions, even if there are isolated edges or vertices within the components of H. Note also, if some of the neighborhood sets of the vertices x, y, and z in H are empty, the proof is analogous. □

6. Majority neighbor sum distinguishing edge-colorings

A k-edge-coloring of a graph G is called *majority neighbor sum distinguishing* if it is neighbor sum distinguishing and is majority. The minimum value of k for which there exists a majority neighbor sum distinguishing k-edge-coloring is denoted by $\chi^{M}_{\sum}(G)$. Recall that majority edge-colorings exist only for graphs with minimum degree at least 2. Observe, when $\Delta(G) \leq 3$, then $\chi^{M}_{\sum}(G) = \chi'_{\sum}(G)$, because the majority edge-coloring must then be proper. If every vertex of the graph G has even degree, then $\chi^{M}_{\sum}(G) = \chi^{QM}_{\sum}(G)$. Thus, our results on $\chi^{QM}_{\sum}(G)$ presented in the previous sections can be transferred to $\chi^{M}_{\sum}(G)$.

First, we determine the majority neighbor sum distinguishing index of complete graphs. We require the following lemma.

Lemma 6.1. *If $k \geq 3$, there exists a majority* NSD 4*-edge-coloring c of K_{2k} such that v_k has at most $k-2$ edges of color* 2*, or v_{k+1} has at most $k-2$ edges of color* 3*. Here $(v_1, \dots, v_{2k})$ denotes the ordering of vertices of K_{2k} with $\sigma_c(v_i) < \sigma_c(v_j)$ for $i < j$.*

Proof. We prove this by induction on the number of vertices. The lemma is true for $k = 3$, see the majority NSD 4-edge-coloring of K_6 in Fig. 2.

Assume that this is true for all complete graphs of even order with fewer than $2k$ vertices. We decompose K_{2k} into two edge disjoint subgraphs G_1 and G_2 such that $G = G_1 \cup G_2$. Let G_1 be a complete graph K_{2k-2} and let $V(G_1) = \{v_1, \dots, v_{2k-2}\}$. Let x, y be remaining two vertices of K_{2k} and G_2 be a spanning subgraph of K_{2k} that contains edges $E(G_2) = \{xy\} \cup \{xv_i : i \in \{1, \dots, 2k-2\}\} \cup \{yv_i : i \in \{1, \dots, 2k-2\}\}$. By the induction hypothesis G_1 has a majority NSD 4-edge-coloring c_1 such that $\sigma_{c_1}(v_i) < \sigma_{c_1}(v_j)$ for $i < j$ and v_{k-1} has at most $k-3$ edges of color 2, or v_k has at most $k-3$ edges of color 3. The coloring of G_2 depends on whether v_{k-1} has at most $k-3$ edges of color 2, or v_k has at most $k-3$ edges of color 3. Therefore, we consider two cases.

Case 1. v_{k-1} has at most $k-3$ edges of color 2 in the coloring c_1.

Let c_2 be an edge-coloring of G_2 such that

- $c_2(xy) = 3$;
- $c_2(xv_i) = 1$ for $i \in \{1, \dots, k-2\}$;
- $c_2(xv_i) = 2$ for $i \in \{k+1, \dots, 2k-2\}$;

- $c_2(yv_i)=3$ for $i\in\{1,\dots,k-2\}$;
- $c_2(yv_i)=4$ for $i\in\{k+1,\dots,2k-2\}$;
- $c_2(xv_{k-1})=2$, $c_2(yv_{k-1})=2$;
- $c_2(xv_k)=1$, $c_2(yv_k)=4$.

Let c be the edge-coloring of K_{2k} such that $c(e)=c_i(e)$ if $e\in E(G_i)$, for $i=1,2$. We claim that c is a majority edge-coloring. By construction, c_2 is majority at x and y. The vertex v_{k-1} has at most $k-3$ edges in G_1 colored with 2. So, together with the two edges in G_2 colored with 2, v_{k-1} has at most $k-1$ edges colored with 2. Since c_1 is a majority edge-coloring, v_{k-1} is incident to at most $k-2$ vertices in color 1 and at most $k-2$ vertices in color 3. Furthermore, since every vertex v_j, for $j\in\{1,\dots,k-2,k,k+1\dots,2k-2\}$, is majority in (G_1,c_1) and is majority in (G_2,c_2), it follows that it is also majority in (G,c). Thus, the edge-coloring c is majority.

We show that c is an NSD edge-coloring. For each vertex v_i, where $i\in\{1,\dots,k-1\}$, we have $\sigma_c(v_i)=\sigma_{c_1}(v_i)+4$, and for each vertex v_i where $i\in\{k+1,\dots,2k-2\}$, we have $\sigma_c(v_i)=\sigma_{c_1}(v_i)+6$. Additionally, $\sigma_c(v_k)=\sigma_{c_1}(v_k)+5$. Thus, the coloring distinguishes the vertices $\{v_1,\dots,v_{2k-2}\}$, with $\sigma_c(v_i)<\sigma_c(v_j)$ for $i<j$. As $\sigma_c(x)=3k$ and $\sigma_c(v_1)\geq 3(k-1)+4$, we have $\sigma_c(x)<\sigma_c(v_1)$. Similarly, since $\sigma_c(y)=7k-5$ and $\sigma_c(v_{2k-2})\geq 7(k-1)-5+6$, we have $\sigma_c(y)>\sigma_c(v_{2k-2})$. Therefore, c is indeed an NSD edge-coloring.

Since c_1 is a majority coloring, v_k is incident to at most $k-2$ edges of color 3. In c_2, we avoid using color 3 for the edges incident to v_k, so v_k has at most $k-2$ edges of color 3. Consequently, the ordering $(x,v_1,\dots,v_{k-2},y)$ satisfies the assumptions of the lemma.

Case 2. v_k has at most $k-3$ edges of color 3 in the coloring c_1.

Let c_2 be an edge-coloring of G_2 such that

- $c_2(xy)=2$;
- $c_2(xv_i)=1$ for $i\in\{1,\dots,k-2\}$;
- $c_2(xv_i)=2$ for $i\in\{k+1,\dots,2k-2\}$;
- $c_2(yv_i)=3$ for $i\in\{1,\dots,k-2\}$;
- $c_2(yv_i)=4$ for $i\in\{k+1,\dots,2k-2\}$;
- $c_2(xv_{k-1})=1$, $c_2(yv_{k-1})=4$;
- $c_2(xv_k)=3$, $c_2(yv_k)=3$.

Let c be the edge-coloring of K_{2k} defined by setting $c(e)=c_i(e)$ if $e\in E(G_i)$, for $i=1,2$. As in Case 1, we can show that c is a majority edge-coloring. For each vertex v_i, where $i\in\{1,\dots,k-2\}$, we have $\sigma_c(v_i)=\sigma_{c_1}(v_i)+4$, and for each vertex v_i where $i\in\{k,\dots,2k-2\}$, we have $\sigma_c(v_i)=\sigma_{c_1}(v_i)+6$. Additionally, $\sigma_c(v_{k-1})=\sigma_{c_1}(v_{k-1})+5$. Thus, this coloring distinguishes the vertices $\{v_1,\dots,v_{2k-2}\}$, with $\sigma_c(v_i)<\sigma_c(v_j)$ for $i<j$. Since $\sigma_c(x)=3k$ and $\sigma_c(v_1)\geq 3(k-1)+4$, we have $\sigma_c(x)<\sigma_c(v_1)$. Similarly, because $\sigma_c(y)=7k-5$ and $\sigma_c(v_{2k-2})\geq 7(k-1)-5+6$, we find out that $\sigma_c(y)>\sigma_c(v_{2k})$. Therefore, c is an NSD edge-coloring. Finally, the ordering $(x,v_1,\dots,v_{k-2},y)$ satisfies the assumptions of the lemma, since v_{k-1} is adjacent to at most $k-2$ edges of color 2. □

Theorem 6.2. *For $n\geq 3$, we have*

$$\chi_{\sum}^{M}(K_n)=\begin{cases}3, & \text{if} \quad n \text{ is odd},\\ 4, & \text{if} \quad n \text{ is even and } n\geq 6,\\ 5, & \text{if}\, n=4.\end{cases}$$

Proof. As each vertex of K_{2k+1} has even degree $2k$, it follows from Theorem 4.3 that $\chi_{\sum}^{M}(K_{2k+1})=\chi_{\sum}^{QM}(K_{2k+1})=3$. For a majority NSD edge-coloring of the complete graph K_{2k} of even order, at least four colors are required. This is because no vertex can have more than $k-1$ edges of a single color incident to it. Consequently, for colors from [3], the smallest possible sum at a vertex is $k-1+2(k-1)+3=3k$, and the largest possible sum is $3(k-1)+2(k-1)+1=5k-4$. Since we can only achieve $2k-3$ distinct sums and each vertex must have a unique sum, a minimum of four colors are necessary for a majority NSD edge-coloring. It is easy to verify that K_4 requires five colors. However, by Lemma 6.1 four colors suffice for a majority NSD edge-coloring of any complete graph with at least 6 vertices and of even order. □

Theorem 6.3. *Let both m,n be even. Then*

$$\chi_{\sum}^{M}\left(K_{n,m}\right)=\begin{cases}4, & \text{if } n=m=2,\\ 3, & \text{if } n=m\geq 4,\\ 2, & \text{if } n\neq m.\end{cases}$$

If at least one of the integers n,m is odd and $n,m\geq 2$, then

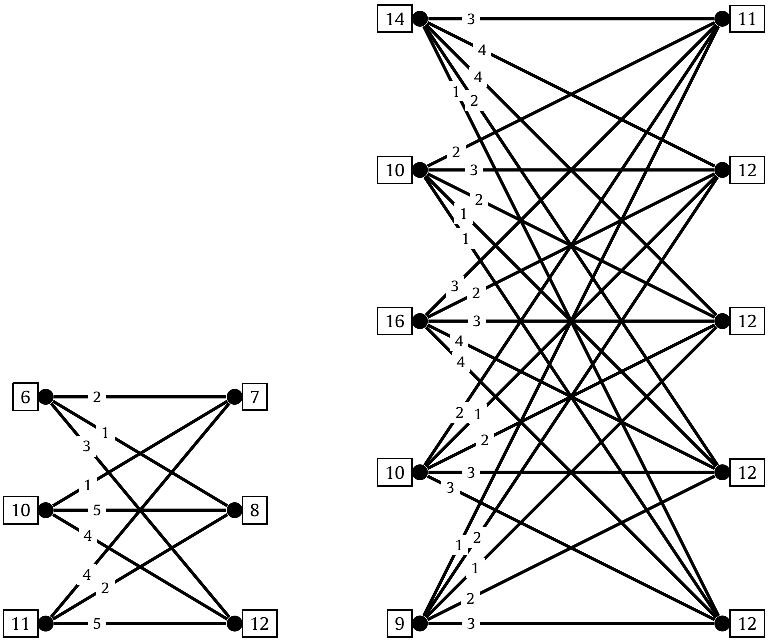

Fig. 3. A majority NSD 5-edge-coloring of $K_{3,3}$ and 4-edge-coloring of $K_{5,5}$.

$$\chi_{\sum}^{M}\left(K_{n,m}\right)=\begin{cases}5, & \textit{if } n=m=3,\\ 4, & \textit{if } n=m=5,\\ 3, & \textit{otherwise.}\end{cases}$$

Proof. Theorem 4.7 implies that when both n and m are even and $K_{n,m}\neq K_{2,2}$, we have $\chi_{\sum}^{M}\left(K_{n,m}\right)=2$ if $n\neq m$, and $\chi_{\sum}^{M}\left(K_{n,n}\right)=3$ for $n\geq 4$. Otherwise, if $K_{n,m}$ contains vertices of odd degrees at least 3, that is, for odd m or n, then at least three colors are necessary for a majority neighbor sum distinguishing edge-coloring. However, in the case of $K_{3,3}$, even four colors are insufficient to distinguish adjacent vertices. To see this, observe that in any majority NSD 4-edge-coloring c of $K_{3,3}$, we have $\sigma_c(v)\in\{6,7,8,9\}$ for every $v\in V(K_{3,3})$. If $\sigma_c(v)=9$, then there is no edge colored with 1 incident to v. On the other hand, if $\sigma_c(v)\in\{6,7,8\}$, then v must be incident to an edge colored with 1. Thus, to achieve a majority NSD edge-coloring, no vertex can have the sum of 9. However, it can be easily verified that if $\sigma_c(v)\in\{6,7,8\}$ for every vertex $v\in V(K_{3,3})$, it becomes impossible to distinguish between the vertices of different sets in the bipartition. For $K_{5,5}$, we need at least four colors for a majority NSD edge-coloring. To see that there is no majority NSD 3-edge-coloring of $K_{5,5}$, observe that in any such coloring c, we have $\sigma_c(v)\in\{9,10,11\}$ for every $v\in V(K_{5,5})$. This makes it impossible to distinguish between vertices from the two different sets of the bipartition. Fig. 3 presents a majority NSD 5-edge-coloring of $K_{3,3}$ and a majority NSD 4-edge-coloring of $K_{5,5}$. Thus, we have $\chi_{\sum}^{M}(K_{3,3})=5$ and $\chi_{\sum}^{M}(K_{5,5})=4$.

Now, consider the remaining cases, that is, when at least one of the integers n,m is odd.

Let $K_{n,m}=(V_1,V_2,E)$, where $V_1=\{a_1,\dots,a_n\}$, $V_2=\{b_1,\dots,b_m\}$. Let $c':E(K_{n,m})\to[n+m-1]$ be the interval coloring of $K_{n,m}$ such that $c'(a_ib_j)=i+j-1$. First, we consider all cases except when $n=m$. In these cases, using the coloring c', we construct a new edge-coloring $c:E(K_{n,m})\to[3]$ by setting $c(e)\equiv c'(e)\pmod 3$. It is straightforward to see that c is a majority coloring, as $\lceil d/3\rceil\leq\lfloor d/2\rfloor$ for $d\geq 2$. Now, we claim that c is also distinguishing.

If $n=3k$ and $m=3\ell$, then $\sigma_c(a_i)=6\ell$ and $\sigma_c(b_j)=6k$ for $i\in[n]$ and $j\in[m]$. By our assumption, $k\neq\ell$, so $\sigma_c(a_i)\neq\sigma_c(b_j)$ for all $i\in[n]$ and $j\in[m]$, which means that c is a majority NSD edge-coloring. If $n=3k$ and $m=3\ell+1$ (or $m=3\ell+2$), then $\sigma_c(a_i)\in\{6\ell+1,6\ell+2,6\ell+3\}$ (or $\sigma_c(a_i)\in\{6\ell+3,6\ell+4,6\ell+5\}$) and $\sigma_c(b_j)=6k$ for $i\in[n]$ and $j\in[m]$. Therefore, $\sigma_c(a_i)\neq\sigma_c(b_j)$ for all $i\in[n]$ and $j\in[m]$. Thus, c is a majority NSD edge-coloring. If $n=3k+1$ and $m=3\ell+1$, then $\sigma_c(a_i)\in\{6\ell+1,6\ell+2,6\ell+3\}$ and $\sigma_c(b_j)\in\{6k+1,6k+2,6k+3\}$ for $i\in[n]$ and $j\in[m]$. Since $k\neq\ell$, we have $\sigma_c(a_i)\neq\sigma_c(b_j)$ for all $i\in[n]$ and $j\in[m]$. Similarly, if $n=3k+2$ and $m=3\ell+2$, then $\sigma_c(a_i)\in\{6\ell+3,6\ell+4,6\ell+5\}$ and $\sigma_c(b_j)\in\{6k+3,6k+4,6k+5\}$ for $i\in[n]$ and $j\in[m]$. As in the previous case, since $k\neq\ell$, we have $\sigma_c(a_i)\neq\sigma_c(b_j)$ for all $i\in[n]$ and $j\in[m]$.

Suppose $n=3k+1$ and $m=3\ell+2$. In this case, we have $\sigma_c(a_i)\in\{6\ell+3,6\ell+4,6\ell+5\}$ and $\sigma_c(b_j)\in\{6k+1,6k+2,6k+3\}$ for $i\in[n]$ and $j\in[m]$. If $k\neq\ell$, then $\sigma_c(a_i)\neq\sigma_c(b_j)$ for all $i\in[n]$ and $j\in[m]$, resulting in a majority NSD edge-coloring. However, if $k=\ell$ (i.e., for $K_{3k+1,3k+2}$), we need to recolor some edges. Specifically, we recolor the edges a_ib_i for $i\in\{1,4,\dots,3k+1\}$, where initially $c(a_ib_i)=1$. We recolor these edges to 3. After recoloring, we have $\sigma_c(a_i)=6k+5$ and $\sigma_c(b_j)=6k+3$ for $i,j\in\{1,4,\dots,3k+1\}$. Thus, after recoloring, $\sigma_c(a_i)\in\{6k+4,6k+5\}$ and $\sigma_c(b_j)\in\{6k+2,6k+3\}$ for all

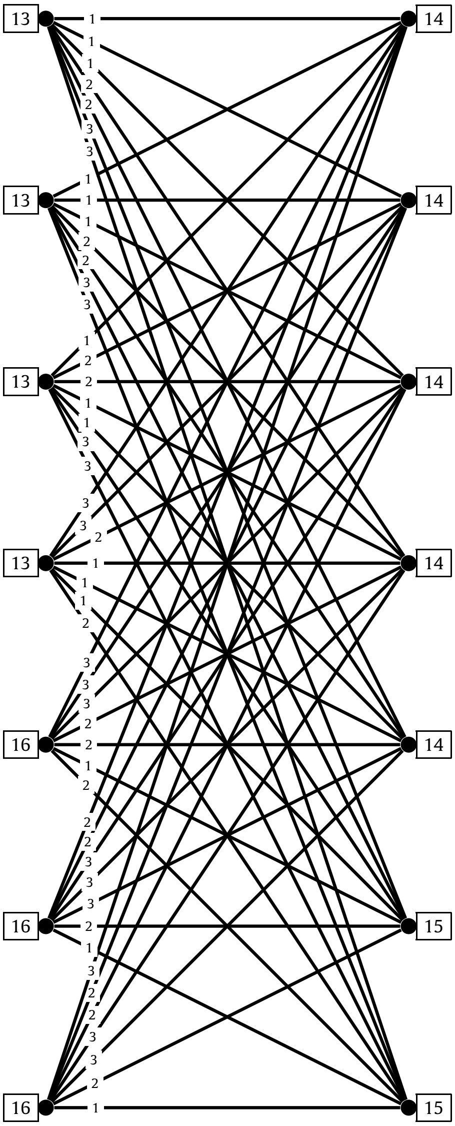

Fig. 4. A majority NSD 3-edge-coloring of $K_{7,7}$.

$i \in [3k+1]$, $j \in [3k+2]$, ensuring a NSD edge-coloring. Furthermore, for $i \in \{1, 4, \dots, 3k+1\}$, vertex a_i has k edges in color 1, $k+1$ edges in color 2, and $k+1$ edges in color 3. Similarly, for $j \in \{1, 4, \dots, 3k+1\}$, vertex b_j has k edges in color 1, k edges in color 2, and $k+1$ edges in color 3. Therefore, the coloring remains a majority edge-coloring.

Finally, suppose $n = m = 2k+1$. Fig. 4 presents a majority NSD 3-edge-coloring of $K_{7,7}$. Thus, we may assume that $k \geq 4$. In this case, using the coloring c', we first construct a new edge-coloring $c : E(K_{n,m}) \to [2]$ by setting $c(e) \equiv c'(e) \pmod 2$. As a result, we have $\sigma_c(a_i) = \sigma_c(b_i) = 3k+1$ for $i \in \{1, 3, \dots, 2k+1\}$ and $\sigma_c(a_i) = \sigma_c(b_i) = 3k+2$ for $i \in \{2, 4, \dots, 2k\}$. In the next step, we recolor certain edges with color 3. Specifically, we assign:

1. $c(a_ib_i) = 3$, $c(a_ib_{i+1}) = 3$, $c(a_{i+1}b_i) = 3$, $c(a_{i+1}b_{i+1}) = 3$ for $i \in \{1, 3, 5, \dots, 2k-1\}$,
2. $c(a_{2k+1}b_{2k+1}) = 3$,
3. $c(a_ib_{i+1}) = 3$, $c(a_ib_{i+2}) = 3$ for $i \in \{2, 4, \dots, 2k-2\}$,
4. $c(a_{2k}b_1) = 3$, $c(a_{2k}b_2) = 3$,
5. $c(a_1b_{2k+1}) = 3$.

After steps 1 and 2 we have $\sigma_c(a_i) = \sigma_c(b_i) = 3k+4$ for $i \in \{1, 3, \dots, 2k-1\}$ and $\sigma_c(a_i) = \sigma_c(b_i) = 3k+5$ for $i \in \{2, 4, \dots, 2k\}$ and $\sigma_c(a_{2k+1}) = \sigma_c(b_{2k+1}) = 3k+3$.

After steps 3, 4, 5 of recoloring, as a result, we have:

- $\sigma_c(a_i) = 3k+8$ for $i \in \{2, 4, \dots, 2k\}$,
- $\sigma_c(a_i) = 3k+4$ for $i \in \{3, 5, 7, \dots 2k-1\}$,
- $\sigma_c(a_1) = 3k+6$, $\sigma_c(a_{2k+1}) = 3k+3$,
- $\sigma_c(b_i) = 3k+7$ for $i \in \{2, 4, \dots, 2k\}$,
- $\sigma_c(bi) = 3k+5$ for $i \in \{1, 3, \dots 2k+1\}$.

Thus, c is an NSD edge-coloring. Since at each vertex (except a_{2k+1}) at least one edge was recolored from 1 to 3 and at least one edge from 2 to 3, each vertex (except a_{2k+1}) has at most k edges in color 1 and at most k edges in color 2. At vertex a_{2k+1}, the edge previously colored 1 was recolored to 3. Therefore, even at a_{2k+1}, there are at most k edges colored 1 and at most k edges colored 2. Additionally, there are at most four edges in color 3 at each vertex. Hence, the coloring is a majority NSD edge-coloring. □

The following result provides a general upper bound for $\chi^M_\sum(G)$. Although its proof is analogous to that of Theorem 5.1, we provide it for convenience.

Theorem 6.4. *For every graph G with $\delta(G) \geq 2$,*

$$\chi^M_\sum(G) \leq 18.$$

Proof. By Theorem 2.1, G has a majority 4-edge coloring. Again, we may assume that G is connected. We will construct a majority NSD coloring $c : E(G) \to [18]$.

Order the vertex set $V(G) = \{v_1, \dots, v_n\}$ such that each vertex v_i, for $i < n$, has a neighbor v_j with $j > i$. Let c_0 be a majority 4-edge-coloring of G with colors 7, 8, 9, 10. Set an initial value $c(e) = c_0(e)$ for each edge $e \in E(G)$. To every vertex v_i with $i \leq n-1$, we will assign a set $W(v_i) = \{w(v_i), w(v_i)+6\}$ of possible final values of $\sigma_c(v_i)$ such that $w(v_i) \in \{0, \dots 5\} \pmod{12}$, and $W(v_i) \cap W(v_j) = \emptyset$ for every $v_i v_j \in E(G)$.

In the first step, we count $\sigma_c(v_1) = \sum_{u \in N(v_1)} c_0(v_1 u)$, and define the sets $W(v_1) = \{w(v_1), w(v_1)+6\}$ by putting $w(v_1) = \sigma_c(v_1)$ if $\sigma_c(v_1) \in \{0, \dots, 5\} \pmod{12}$, or $w(v_1) = \sigma_c(v_1) - 6$ otherwise.

Let $2 \leq k \leq n-1$, and assume that we have already established the set $W(v_i)$ for each $i < k$, and

(1) $c(v_i v_j) \in [18]$ for $1 \leq i < j \leq n$,
(2) $\sigma_c(v_i) \in W(v_i)$ for $i < k$,
(3) $c(v_k v_j) = c_0(v_j v_k)$ for $j > k$,
(4) $c(v_i v_k) \in \{7, \dots 12\}$ for $i < k$.

In view of condition (4), if $v_i v_k \in E(G)$, then we can either add or subtract 6 to $c(v_i v_k)$, so that the resulting c is still a majority coloring, and $\sigma_c(v_i) \in W(v_i)$. If v_k has d neighbors v_i with $i < k$, then this gives us $d+1$ choices for $\sigma_c(v_k)$. Furthermore, we can change $c(v_k v_{j_0})$, where j_0 is the smallest $j > k$ with $v_k v_j \in E(G)$, using the following rule. It is not difficult to see that if we want to change the color of $c(v_k v_{j_0})$, then for each of its end-vertices v_k, v_{j_0}, there are at most two colors violating the majority coloring of that vertex. Hence, we can choose a new color $c(v_k v_{j_0}) \in \{7, \dots, 12\}$ such that c is still a majority edge-coloring of G. This gives us additional d choices for $\sigma_c(v_k)$. In total, we have $2d+1$ possible values for $\sigma_c(v_k)$, at least one of them does not belong to any $W(v_i)$ for $i < k$. Therefore, a possible recoloring of some edges $v_i v_k$ with $i < k$ and an edge $v_k v_{j_0}$ results in an edge-coloring c satisfying the conditions:

(1′) $c(v_i v_j) \in [18]$ for $1 \leq i < j \leq n$,
(2′) $\sigma_c(v_i) \in W(v_i)$ for $i \leq k$,
(3′) $c(v_k v_j) = c_0(v_j v_k)$ for $j > k$, except for $j = j_0$,
(4′) $c(v_i v_n) \in \{7, \dots, 12\}$ for $i < n$.

This way, we successively assign pairwise disjoint sets $W(v_k)$ to all $k \leq n-1$.

In the final step, we have to determine $\sigma_c(v_n)$. If $v_i v_n \in E(G)$ for some $i < n$, then condition (4′) allows us to subtract or add 6 to $c(v_i v_n)$ such that $\sigma_c(v_i) \in W(v_i)$. Hence, we have $d(v_n) + 1 \geq 3$ possible options for $\sigma_c(v_n)$. Let s be the smallest such possible option, which we obtain by subtracting 6 from $c(v_i)$ for all $v_i \in N(v_n)$ with $\sigma_c(v_i) = w(v_i) + 6$, and adding 6 nowhere. If $s \in \{6, \dots, 11\} \pmod{12}$, then s cannot be equal to $\sigma_c(v_i)$ for any neighbor v_i of v_n since otherwise we could increase s by subtracting 6 from $c(v_i v_n)$. Hence, we can put $\sigma_c(v_n) = s$. Let then $s \in \{0, \dots, 5\} \pmod{12}$. If there exists a $v_i \in N(v_n)$ with $\sigma_c(v_i) \neq s$, then we keep $c(v_i)$ unchanged and subtract 6 everywhere else, thus obtaining a suitable value $\sigma_c(v_n) = s + 6$. If $\sigma_c(v_i) = w(v_i)$ for all $v_i \in N(v_n)$ then we subtract 6 from all edges incident to v_n except two of them.

The resulting values of σ_c yield a proper vertex-coloring of G. □

It is not difficult to modify the proof of Theorem 6.4 to justify the following two results.

Proposition 6.5. *If G is a graph with minimum degree at least 4, then $\chi^M_\sum(G) \leq 15$.*

Proposition 6.6. *If G is a graph of even size with all vertices of even degrees, then $\chi^M_\sum(G) \leq 12$.*

Indeed, Proposition 6.5 follows from the fact proven in [2] that every graph G with $\delta(G) \geq 4$ admits a majority 3-edge-coloring. To prove Proposition 6.6, we apply item 1. of Theorem 2.2, which implies that a graph of even size with all vertices of even degrees has a majority 2-edge-coloring.

7. Conclusion and open problems

In this paper, we explore the quasi-majority neighbor sum distinguishing edge-coloring of graphs. We provide general upper bounds and determine exact values or upper bounds for the QM NSD index of specific graph classes.

A key question arising from our results is to determine the minimum integer k such that every graph admits a quasi-majority neighbor sum distinguishing k-edge-coloring. We prove that 12 colors suffice for any graph. However, our findings for specific graph classes suggest that this number could be much smaller. The graph with the highest known QM NSD index 5 is C_5. Thus, the primary open problem is to identify an infinite family of graphs with a QM NSD index of at least 5. Nevertheless, we believe that such a family might not exist and propose the following conjecture.

Conjecture 7.1. *Every nice graph* $G \neq C_5$ *satisfies* $\chi_{\sum}^{QM}(G) \leq 4$.

For majority colorings, we suppose the following.

Conjecture 7.2. *Every graph* G *with* $\delta(G) \geq 2$ *satisfies* $\chi_{\sum}^{M}(G) \leq 5$.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

References

[1] N. Alon, Combinatorial Nullstellensatz, Comb. Probab. Comput. 8 (1999) 7–29.
[2] F. Bock, R. Kalinowski, J. Pardey, M. Pilśniak, D. Rautenbach, M. Woźniak, Majority edge-colorings of graphs, Electron. J. Comb. 30 (1) (2023), #P1.42.
[3] A. Dailly, E. Sidorowicz, Neighbor sum distinguishing edge-weightings with local constraints, Discrete Appl. Math. 336 (2023) 109–124.
[4] A. Dailly, É. Duchéne, A. Parreau, E. Sidorowicz, The neighbor sum distinguishing relaxed edge colouring, Appl. Math. Comput. 419 (2022) 126864.
[5] E. Flandrin, A. Marczyk, J. Przybyło, J.-F. Saclé, M. Woźniak, Neighbor sum distinguishing index, Graphs Comb. 29 (2013) 1329–1336.
[6] J. Huo, Y. Wang, W. Wang, Neighbour-sum-distinguishing edge choosability of subcubic graphs, J. Comb. Optim. 34 (3) (2017) 742–759.
[7] M. Kalkowski, M. Karoński, F. Pfender, Vertex-coloring edge-weightings: towards the 1-2-3-conjecture, J. Comb. Theory, Ser. B 100 (2010) 34–349.
[8] M. Karoński, T. Łuczak, A. Thomason, Edge weights and vertex colours, J. Comb. Theory, Ser. B 91 (2004) 151–157.
[9] R. Keusch, A solution to the 1-2-3 Conjecture, J. Comb. Theory, Ser. B 166 (2024) 183–202.
[10] J. Przybyło, Neighbor distinguishing edge colorings via the combinatorial nullstellensatz, SIAM J. Discrete Math. 27 (3) (2013) 1313–1322.
[11] J. Przybyło, A note on asymptotically optimal neighbour sum distinguishing colourings, Eur. J. Comb. 77 (2019) 49–56.
[12] J. Przybyło, T-L. Wong, Neighbour distinguishing edge colourings via the Combinatorial Nullstellensatz revisited, J. Graph Theory 80 (2015) 299–312.
[13] G. Wang, G. Yan, An improved upper bound for the neighbour sum distinguishing index of graphs, Discrete Appl. Math. 175 (2014) 126–128.