Fundamental Properties and Theorems of Operator Calculus Based on the Laplace Transform

1. *Linearity Theorem*

$$\mathcal{L}[\alpha_1 f_1(t)+\alpha_2 f_2(t)]=\alpha_1\mathcal{L}[f_1(t)]+\alpha_2\mathcal{L}[f_2(t)]$$

2. *Transformation of Function Derivatives*

$$\mathcal{L}[f^{(n)}(t)] = s^n \cdot F(s) - s^{n-1} \cdot f(0^+) - s^{n-2} \cdot f'(0^+) - \ldots - s \cdot f^{(n-2)}(0^+) - f^{(n-1)}(0^+)$$

$$\mathcal{L}[\ddot{f}(t)] = s^2 \cdot F(s) - s \cdot f(0^+) - \dot{f}(0^+)$$

$$\mathcal{L}[\dot{f}(t)] = s \cdot F(s) - f(0^+)$$

3. *Transformation of Function Integral*

$$\mathcal{L}\left[\int_0^t f(\tau)d\tau\right] = \frac{1}{s} \cdot F(s)$$

4. *Shift in the Real Domain (Time-Shift)*

$$\mathcal{L}[f(t-\tau)] = e^{-\tau s} \cdot F(s)$$

5. *Shift in the Complex Domain (Frequency-Shift)*

$$\mathcal{L}[e^{-at} \cdot f(t)] = F(s+a)$$

6. *Scaling Theorem*

$$\mathcal{L}[f(a \cdot t)] = \frac{1}{a} \cdot F\left(\frac{s}{a}\right)$$

7. *Complex Domain Differentiation*

$$\mathcal{L}[t \cdot f(t)] = (-1)\frac{dF(s)}{ds}$$

$$\mathcal{L}[t^n \cdot f(t)] = (-1)^n \frac{d^n F(s)}{ds^n}$$

8. *Final Value Theorem*

If exists: $\lim_{t\to\infty} f(t)$ *and* $\mathcal{L}[f(t)] = F(s)$ *then:*

$$\lim_{t\to\infty} f(t) = \lim_{s\to 0^+} s \cdot F(s)$$

9. *Convolution Theorem*

$$\mathcal{L}[f_1(t) * f_2(t)] = \mathcal{L}[f_1(t)] \cdot \mathcal{L}[f_2(t)] = F_1(s) \cdot F_2(s) \ ,$$

where the convolution of functions $f_1(t)$, $f_2(t)$.

is defined by the relation:

$$f_1(t) * f_2(t) = \int_0^t f_1(\tau) \cdot f_2(t-\tau) d\tau$$

Definitions

Let f(t) be a function of real t (i.e. time) that satisfies the following conditions:

1) f(t) = 0 for t < 0,

2) f(t) satisfies Dirichlet conditions:
   - The interval in which the function is defined can be divided into open subintervals where f(t) is monotonic.
   - At each point, the following holds: $f(t) = [\, f(t^-) + f(t^+) \,] / 2$.

3) f(t) is integrable over every finite interval, and the following inequality holds: $|\, f(t) \,| \le M e^{at}$ (M>0, a>0, for $t>t_0$).

Laplace Transform (for complex s, Re(s)≥0):

$$F(s) = \mathcal{L}\,[f(t)] = \int_0^\infty e^{-st} f(t) dt$$

Inverse Laplace Transform:

$$f(t) = \mathcal{L}^{-1}[F(s)] = \frac{1}{2\pi j} \int_{\alpha - j\infty}^{\alpha + j\infty} F(s) e^{st}\, ds$$

Laplace Transforms of Frequently Used Functions

Original f(t)	Transform F(s)
$\delta(t)$ - Dirac impulse	1
1(t) -unit step (Heaviside fcn)	$\frac{1}{s}$
t	$\frac{1}{s^2}$
$e^{\mp at}$	$\frac{1}{s \pm a}$
$\sin \omega t$	$\frac{\omega}{s^2 + \omega^2}$
$\cos \omega t$	$\frac{s}{s^2 + \omega^2}$