Laboratory 3

Mathematical modelling in MATLAB/Simulink using the example of a permanent magnet DC motor

1. Objectives of the exercise

- familiarization with the principle of operation of a DC electric motor,
- familiarization with the methods of creating electric motor models in the form of: differential equations, state and output equations, block diagrams, and operator transfer functions,
- determination of the step response of the motor in MATLAB/Simulink,
- determination of the motor response to rectangular signals in MATLAB/Simulink.

2. Theoretical introduction

2.1. Introduction

DC electric motors are very often used as executive elements in control systems. The main advantages of these motors are: high torque, good efficiency and small dimensions. The disadvantages are: sparking (industrial interference) and wear of commutator brushes. Over the last few decades, a number of motors with a special design have been introduced to the market, characterized by very good dynamic properties.

The schematic construction of a DC motor with a permanent magnet is shown in Figure 2.1.The **torque** in electric motors is created because of interaction between the external magnetic field and the magnetic field generated around the conductor through which the current flows. In low-power DC motors, the external magnetic field is usually generated by **permanent magnets** placed in a stationary motor housing called the **stator**. The rotor located in the stator magnetic field contains windings consisting of many wire frames connected to the commutator. Typically, these windings are wound on a core made of ferromagnetic material. As a result of the interaction of the stator flux and the current flowing in the rotor windings, the previously mentioned torque is created. In order for the torque acting on the rotor to be maximum, the magnetic flux vectors of the stator and rotor should be perpendicular to each other. This is ensured by the **commutator**, which switches subsequent frames of the rotor winding, causing appropriate changes in the direction of the flowing current. The voltage supplying the commutator is supplied by **brushes** made of specially prepared carbon. In motors of this type, the control circuit is always the rotor circuit. Changes in the voltage supplying the control circuit cause changes in the torque and thus, at a given rotor load torque, a change in the **angular velocity** of the rotor.

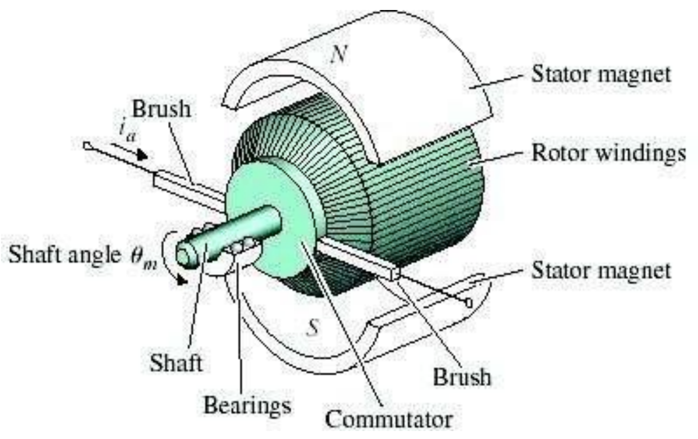

Fig. 2.1 Construction of a permanent magnet DC motor

2.2. Mathematical model of the motor written in the form of differential equations

When creating a motor model, attention should be paid to finding the relationship between the voltage supplying the motor (U_z) and the angular velocity of the motor (ω_s). The equivalent circuit of a DC motor, reduced to the rotor circuit, is shown in Fig. 2.2. Considering the electrical and mechanical parameters of the rotor circuit separately, two equations can be written to model its operation.

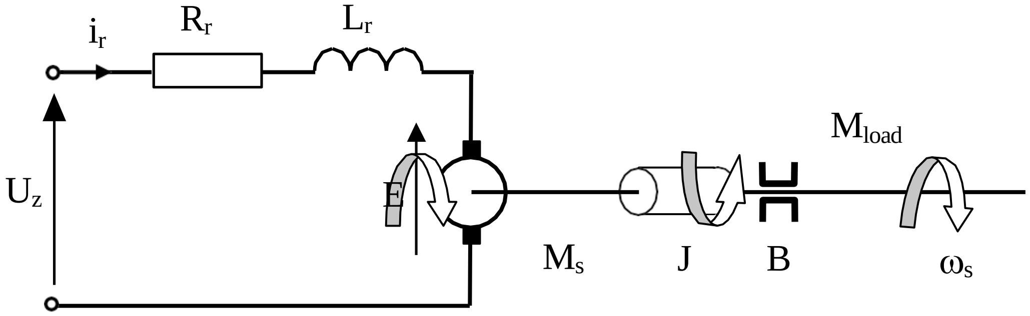

Fig. 2.2 DC Motor Rotor Circuit Equivalent Diagram

Electrical parameters

The electrical quantities appearing in the diagram characterize, respectively:

U_z – rotor supply voltage,
i_r – current flowing in the rotor windings,
R_r – equivalent resistance of the rotor windings,
Lr – equivalent inductance of the rotor windings,
E – electromotive force of induction,
ω_s – angular velocity of the rotor.

Based on the equivalent diagram and Kirchhoff's second law, the electrical equation of the motor can be written

$$U_z = U_{R_r} + U_{L_r} + E \tag{2.1}$$

The voltage across the rotor winding resistance is proportional to the current flowing through it

$$U_{R_r} = R_r i_r$$

The voltage referred to the rotor inductance is proportional to the change in the current flowing through it (losses in the magnetic circuit have been neglected here)

$$U_{L_r} = L_r \frac{di_r}{dt}$$

When the rotor rotates, an electromotive force (EMF) is induced in its windings, the value of which is proportional to the angular velocity of the rotor

$E = k_e \omega_s,$ where k_e is an electrical constant, that depends on, among other things, the stator magnetic flux and the number of turns in the rotor windings.

Substituting the subsequent components of the voltage U_z into equation (2.1), we obtain

$$U_z = R_r i_r + L_r \frac{di_r}{dt} + k_e \omega_s \tag{2.2}$$

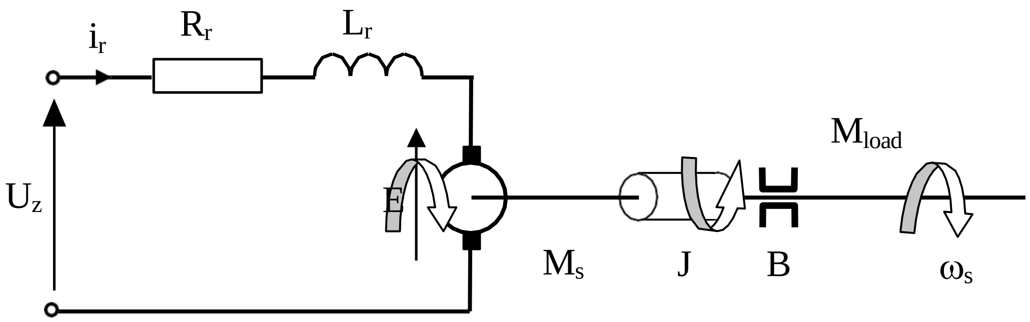

Fig. 2.2 DC Motor Rotor Circuit Equivalent Diagram

Mechanical parameters

The mechanical quantities appearing in the diagram characterize respectively:

M_s – rotor torque,
ω_s – rotor angular velocity,
B – viscous friction coefficient reduced to the rotor shaft,
J – moment of inertia reduced to the rotor shaft,
i_r – current flowing in the rotor windings,
M_{load} – constant motor load torque.

The rotor torque used to overcome the torques opposing its motion can be written as

$$M_s = M_a + M_v + M_{load} \tag{2.3}$$

Assuming that the stator magnetic flux is constant, the rotor torque, proportional to the current flowing through the rotor, can be written as

$$M_s = k_m i_r$$

where k_m is a mechanical constant, that depends on, among other things, the stator magnetic flux and the number of turns in the rotor windings.

The torque related to the angular acceleration of the rotor can be written as

$$M_a = J\frac{d\omega_s}{dt}$$

The torque related with the resistance to rotor movement can be written as

$$M_v = B\omega_s$$

Substituting the subsequent components of the torque M_s into equation (2.3), we obtain

$$k_m i_r = J\frac{d\omega_s}{dt} + B\omega_s + M_{load} \tag{2.4}$$

By transforming equations (2.2) and (2.4) we obtain a system of differential equations which is the motor model:

(2.5)

……………………………………………

2.3 Mathematical model of the motor written in the form of state and output equations

Assuming the current flowing in the rotor windings (i_r) and the rotor angular velocity (ω_s) as state variables, we can write the motor model in the form of state and output equations. We perform the change of variables

$$x_1 = i_r$$

$$x_2 = \omega_s$$

$$u_1 = U_z$$

$$u_2 = M_{load}$$

$$y = \omega_s$$

obtaining a system of equations

(2.6)

Equations (2.6) are written in matrix form

$$\begin{cases} \dot{x} = Ax + Bu \\ y = Cx + Du \end{cases} \tag{2.7}$$

or after writing out

That is:

Mathematical model of the motor in the form of a block diagram

Applying the Laplace transform to equations (2.5) we get

By transforming the obtained equations, assuming zero initial conditions, we obtain

Based on the above equations, the motor block diagram can be drawn, as shown in Fig. 2.3.

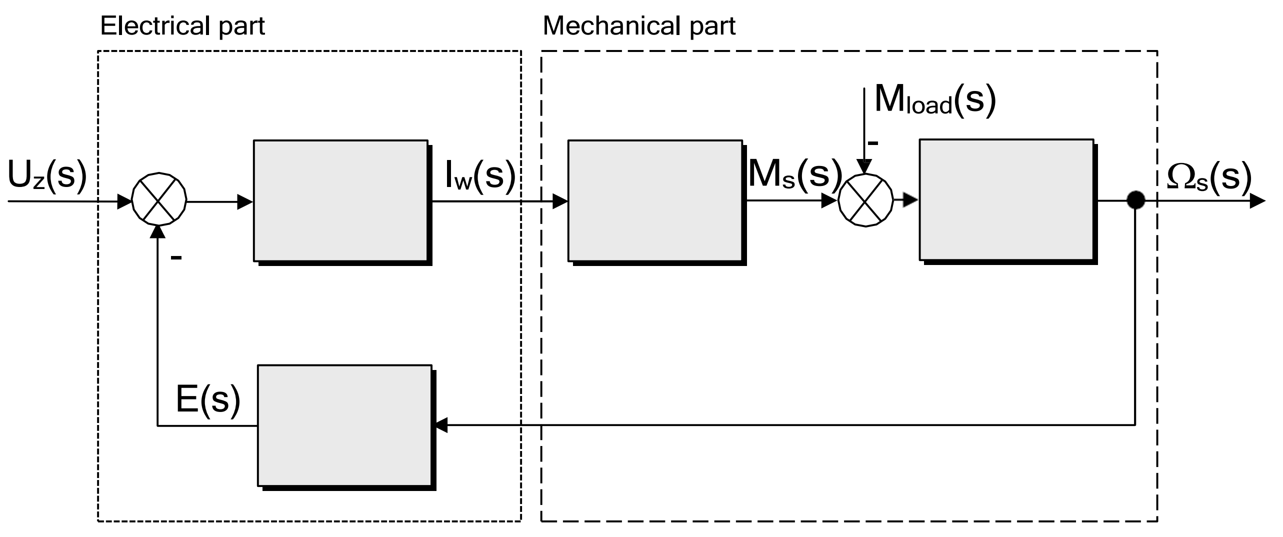

Fig. 2.3 DC motor block diagram

Mathematical model of the motor in the form of the transfer function

Assuming the rotor angular velocity (ω_s) as the output quantity and the rotor supply voltage (U_z) as the input quantity and considering the motor without load ($M_{load} = 0$), we can determine the transfer function of this system by making subsequent modifications to the above block diagram (Fig. 2.4).

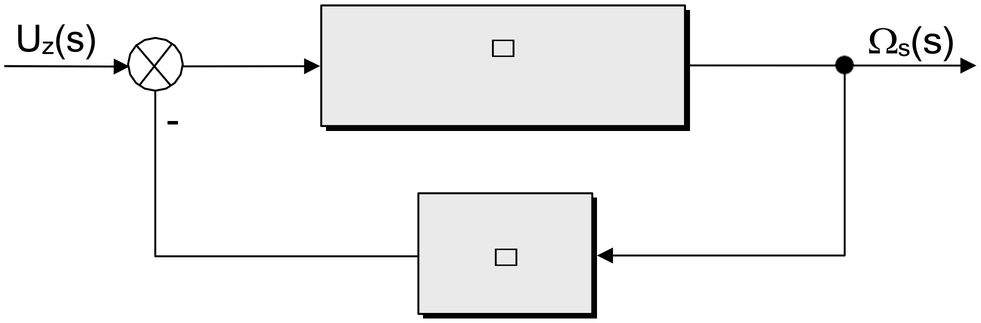

Fig. 2.4 Converted DC Motor Block Diagram

Based on the block diagram in Fig. 2.4, the transfer function $G(s)$ can be written as

$$G(s) = \frac{\Omega_s(s)}{U_z(s)} = \frac{\dfrac{k_m}{(sL_r + R_r)(sJ + B)}}{1 + \dfrac{k_m k_e}{(sL_r + R_r)(sJ + B)}}$$

Multiplying numerator and denominator by $(sL_r + R_r)(sJ + B)$ we receive

$$G(s) = \frac{k_m}{(sL_r + R_r)(sJ + B) + k_m k_e}$$

Transforming further we get

$$G(s) = \frac{k_m}{JL_r s^2 + (R_r J + BL_r)s + R_r B + k_m k_e} \tag{2.8}$$

Mathematical model of the motor in the form of the transfer function

Usually, the viscous friction coefficient B is small, because of which we assume that $R_r J \gg B L_r$ and $k_e k_m \gg R_r B$. The motor transfer function $G(s)$ is then written in a simplified form

$$G(s) = \frac{k_m}{J L_r s^2 + R_r J s + k_m k_e}$$

Dividing the numerator and denominator by $k_m k_e$ we get 1

$$G(s) = \frac{\frac{1}{k_e}}{\frac{J R_r}{k_m k_e}\frac{L_r}{R_r}s^2 + \frac{R_r J}{k_m k_e}s + 1}$$

Substituting $T_m = \frac{J R_r}{k_m k_e}$, $T_e = \frac{L_r}{R_r}$, $K = \frac{1}{k_e}$ we receive

$$G(s) = \frac{\Omega_s(s)}{U_z(s)} = \frac{K}{T_m T_e s^2 + T_m s + 1}$$

Therefore, assuming the angular velocity (ω_s) as the output quantity, the motor transfer function was obtained in the form of a **second-order term**. The mechanical time constant T_m is usually at least an order of magnitude greater than the electrical time constant T_e. In such a case, the constant T_e can be omitted, and the motor becomes a **first-order inertial term**.

$$G(s) = \frac{\Omega_s(s)}{U_z(s)} = \frac{K}{T_m s + 1}$$

If the output quantity is the **angular displacement of the rotor shaft** (α_s), which can be determined after integrating the angular velocity of the rotor ($\alpha(s) = \Omega(s)/s$), the transfer function $G(s)$ will take the form

$$G(s) = \frac{\alpha_s(s)}{U_z(s)} = \frac{K}{s(T_m s + 1)}$$

In this case the motor is a **real integrating term** (i.e. a series connection of an integrating and a first-order inertial term).

Determining the motor step response in MATLAB/Simulink

The step response of the motor was determined based on two methods. The first method used the motor transfer function (2.8):

$$G(s) = \frac{k_m}{JL_r s^2 + (R_r J + BL_r)s + R_r B + k_m k_e}$$

Below is the source of the program written in MATLAB, in which the transfer function $G(s)$ is modeled and the step response of the motor is determined using the step function.

```
% Defining model parameters
    Rr = 2;     Lr = 0.1;     ke = 0.1;
    J = 0.1;   B = 0.5;    km = 0.1;

% Determination of the motor's transfer function
    licz = km;
    mian = [J*Lr  Rr*J + B*Lr  Rr*B + km*ke];
    system = tf(licz,mian);

% Determination of step response parameters
    t = 0:0.02:1.4;
    resp=step(system,t);

% Plotting the step characteristic
    plot(t,resp,'ro'); grid
    xlabel('time (s)'),ylabel('angular velocity ωs (rad/s)')
    title('Step Response of a DC Motor')
```

In the second method, based on the motor block diagram shown in Fig. 2.3, the corresponding diagram in Simulink, shown in Fig. 2.5, was built, assuming that $M_{load} = 0$. The input signal is the unit step signal.

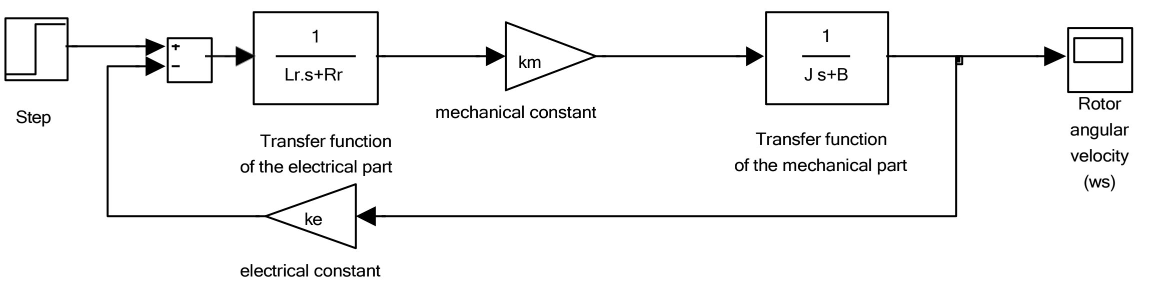

Fig. 2.5 Motor block diagram built in Simulink

Determining the motor step response in MATLAB/Simulink

In order to numerically simulate the motor operation, its parameters (coefficients and constants) must be defined. Let us assume that:

- $R_r = 2\ \Omega$,
- $L_r = 0.1$ H,
- $k_e = 0.1$ Vs/rad,
- $J = 0.1\ \mathrm{kgm}^2$,
- B = 0.5 Nms/rad,
- $k_m = 0.1$ Nm/A,

Before starting the simulation, the above parameters must be entered into the MATLAB workspace by typing:

>> Rr=2; Lr=0.1; ke=0.1; J=0.1; B=0.5; km=0.1;

As a result of executing the program in MATLAB and running the simulation in Simulink, the waveforms shown in Fig. 2.6 were obtained.

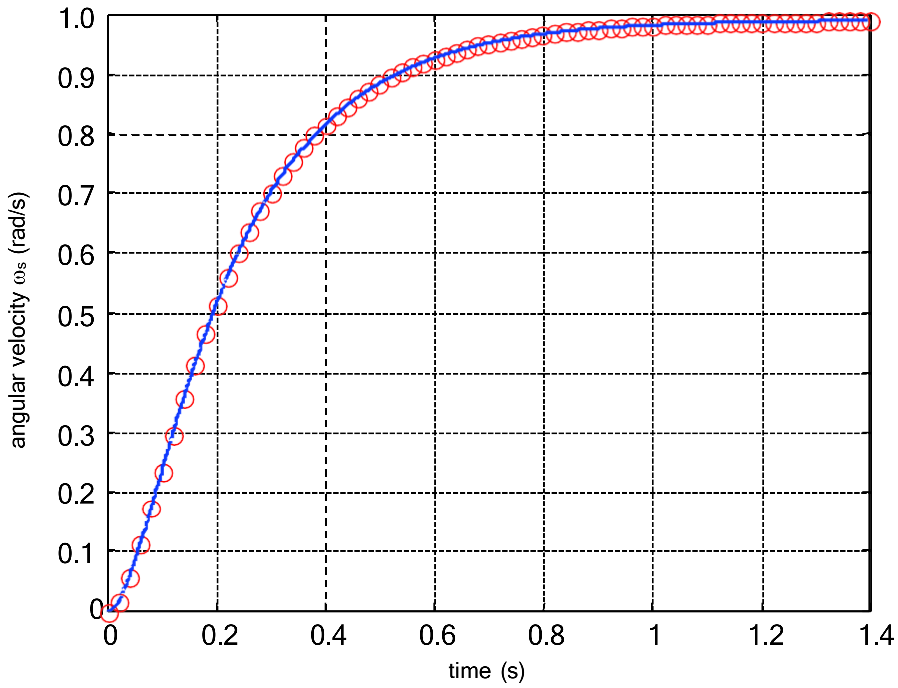

Fig. 2.6 Step response of a DC motor, obtained using:
a) the motor transfer function – circle
b) the motor model determined in Simulink – line

As can be seen, the waveforms obtained using the presented methods are identical, which confirms the usefulness of both MATLAB and Simulink for simulating the operation of systems. The nature of the waveforms obtained indicates that the motor, with such an assumed model, is indeed a second-order member.

Determining the motor response to rectangular signals in MATLAB/Simulink

Based on the motor block diagram shown in Figure 2.3, a corresponding diagram was built in Simulink, shown in Fig. 2.7. In order to numerically simulate the motor operation, its parameters (coefficients and constants) must be defined. Assume as before that:

- $R_r = 2\ \Omega$,
- $L_r = 0.1$ H,
- $k_e = 0.1$ Vs/rad,
- $J = 0.1$ kgm^2,
- $B = 0.5$ Nms/rad,
- $k_m = 0.1$ Nm/A,

and that both the input signal and the load torque are rectangular signals with appropriate parameters

- $U_z = 10$ V,
- $M_{load} = 0.2$ Nm

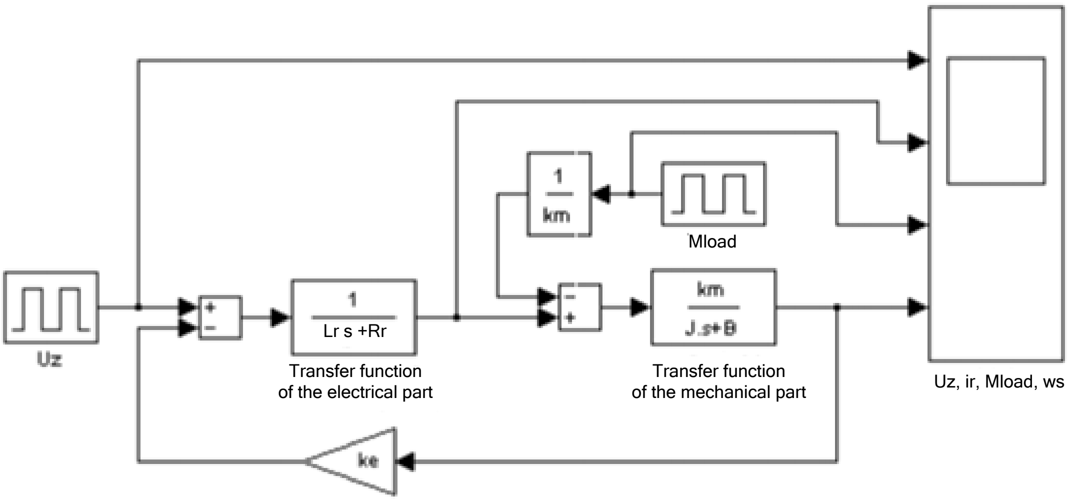

Fig. 2.7 Motor block diagram built in Simulink

Before running the simulation, the above parameters must be entered into the MATLAB workspace by typing:

```
>> Rr=2; Lr=0.1; ke=0.1; J=0.1; B=0.5; km=0.1;
```

After running the simulation, we obtain the following graphs shown in Fig.2.8: rotor supply voltage (U_z), rotor current (i_r), motor load torque (M_{load}) and rotor angular velocity (ω_s) as a function of time.

Determining the motor response to rectangular signals in MATLAB/Simulink

After running the simulation, we obtain the following graphs shown in Fig.2.8: rotor supply voltage (U_z), rotor current (i_r), motor load torque (M_{load}) and rotor angular velocity (ω_s) as a function of time.

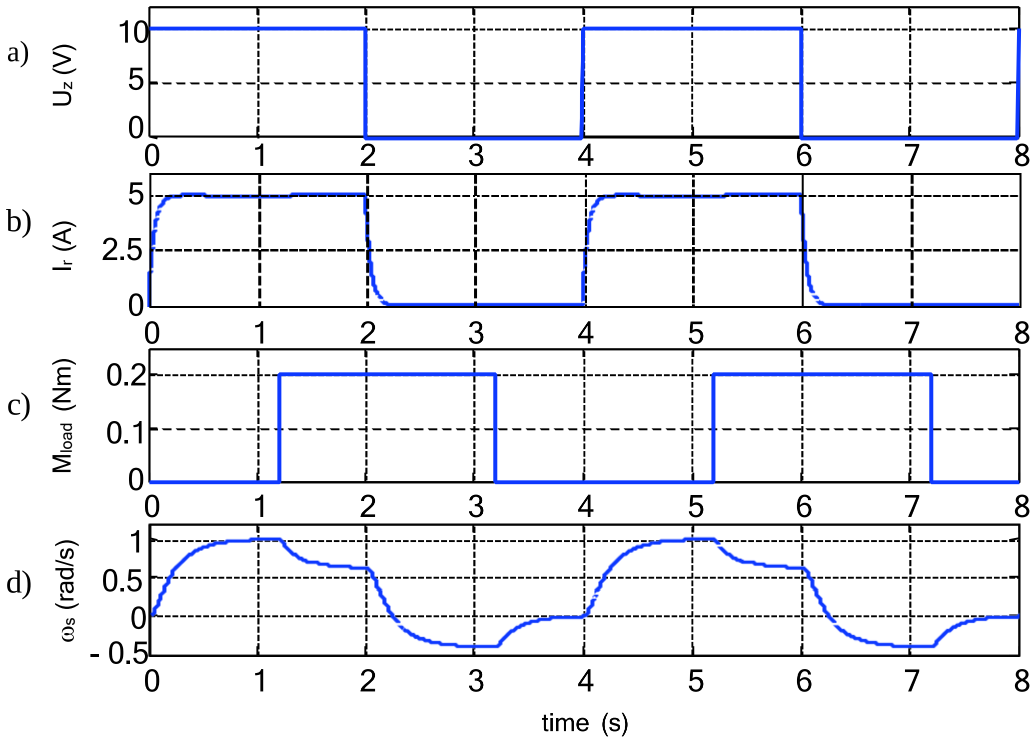

Fig. 2.8. Waveforms obtained during simulation: a) rotor supply voltage (U_z), b) rotor current (i_r), c) motor load torque (M_{load}), d) rotor angular velocity (ω_s)

The motor operation was also simulated using its model written in the form of state and output equations. Assuming that the load torque $M_{load} = 0$, and the initial conditions

$$\begin{bmatrix} i_{r0} \\ \omega_{s0} \end{bmatrix} = \begin{bmatrix} x_{10} \\ x_{20} \end{bmatrix}$$

and adopting the variables

$$x_1 = i_r$$
$$x_2 = \omega_s$$
$$u_1 = U_z$$
$$y = \omega_s$$

based on equations 2.7 we get

$$\begin{cases} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -\dfrac{R_r}{L_r} & -\dfrac{k_e}{L_r} \\ \dfrac{k_m}{J} & -\dfrac{B}{J} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} \dfrac{1}{L_r} \\ 0 \end{bmatrix} [u] \\ y = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \end{cases}$$

The block diagram of the system is shown in Fig. 2.9. The block "*Motor model in the form of state and output equations*", in which the parameters of the system model are recorded, is shown in Fig. 2.10.

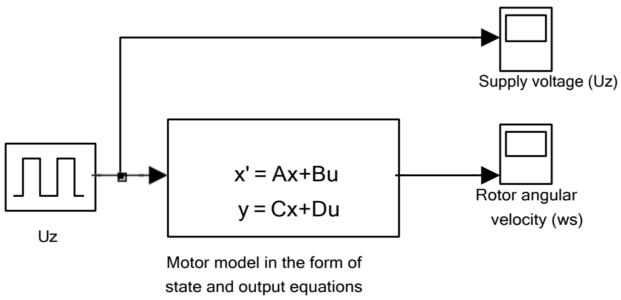

Fig. 2.9. Motor block diagram built in Simulink

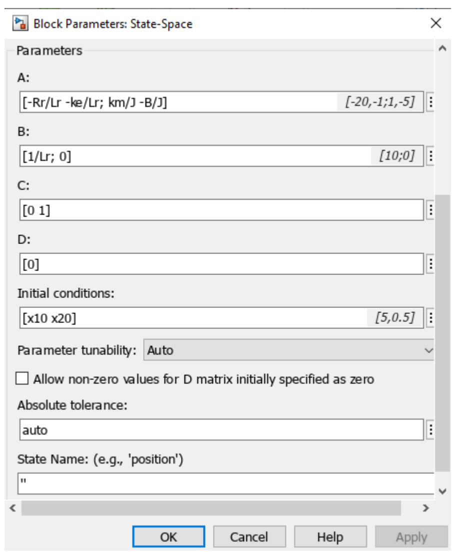

Fig. 2.10. Motor model parameters

Determining the motor response to rectangular signals in MATLAB/Simulink

In order to be able to numerically simulate the motor operation, its parameters (coefficients and constants) must be defined. Assume as before that:

- $R_r = 2\ \Omega$,
- $L_r = 0.1$ H,
- $k_e = 0.1$ Vs/rad,
- $U_z = 10$ V.
- $J = 0.1$ kgm^2,
- $B = 0.5$ Nms/rad,
- $k_m = 0.1$ Nm/A,

and that the initial conditions are

- x10 = 5;
- x20 = 0.5;

Before starting the simulation, the above parameters should be entered into the MATLAB workspace as before by typing

>> Rr=2; Lr=0.1; ke=0.1; J=0.1; B=0.5; km=0.1; x10=5; x20=0.5;

After running the simulation, we obtain the waveforms of the rotor supply voltage (U_z) and the rotor angular velocity (ω_s) as a function of time on the graphs shown in Fig. 2.11

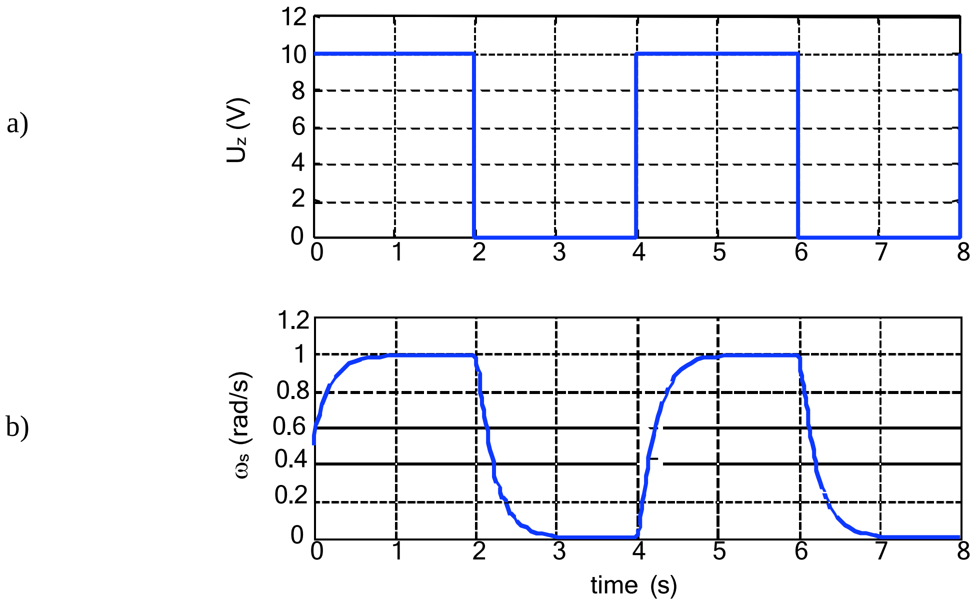

Fig. 2.11. Waveforms obtained during simulation:
a) rotor supply voltage (U_z), b) rotor angular velocity (ω_s)

These waveforms can also be obtained using the *plot* function by typing in MATLAB:

>> plot (Uz(:,1),Uz(:,2));
>> ylabel('Uz (V)');

>> plot (ws(:,1),ws(:,2));
>> xlabel('czas (s)'); ylabel ('ws (rad/s)');