Systems Theory

Laboratory 4: *Selected continuous-time and discrete-time control algorithms. Optimal control.*

Purpose of the exercise:
Design, analysis, and verification of continuous- and discrete-time optimal control solutions for complex system, using MATLAB/Simulink environment.

1. Introduction

Most of real-time vibration control approaches are based on direct two-level (bang-bang) control such as displacement or velocity ground-hook, sliding mode control, fuzzy logic, heuristic algorithms, or, most widely, two-stage cascade algorithms with the calculation of an actuator required force using e.g. adaptive (positive or negative) stiffness and damping, control Lyapunov function based methods, or Pontryagin maximum principle based optimal control methods, including Linear Quadratic (Gaussian) Regulators (LQR/LQG), being a 1st stage, and force tracking algorithms using active or semiactive actuators, e.g. MR devices (being a 2nd stage); for MR dampers, a force tracking algorithm may include a feed-forward loop with an MR damper forward or inverse model, and a feedback loop with damper force sensor signal and e.g. PI control algorithm. Force tracking using semiactive actuators is compromised by their force value limitations, including impossibility to generate active forces. Another problem is that some advanced 1st stage algorithms need real-time oscillation frequency determination which may be an issue for polyperiodic or random (e.g. seismic) excitations – for such situations these algorithms switch to the passive operation mode.

2. A regarded system

A vibration reduction system that comprises a spring (of stiffness k_2) and an MR damper, built in parallel, with an additional stiff body of mass m_2, operating all together as an MR TVA system, is regarded. Alternatively, an active element is considered in place of an MR damper.

The analysed vibrating system/structure with an MR tuned vibration absorber (TVA) may be regarded as a two mass-spring-damper system, subjected to an external excitation force P ($P(t)$) (see Fig. 1):

$$\begin{cases} m_1\ddot{q}_1(t) = -k_1 q_1(t) - c_1\dot{q}_1(t) - k_2\left(q_1(t) - q_2(t)\right) - P_{MR}(t) + P(t) \\ m_2\ddot{q}_2(t) = k_2\left(q_1(t) - q_2(t)\right) + P_{MR}(t) \end{cases}$$

where $q_1(t)$ is a horizontal displacement of a protected system/structure (e.g. corresponding to a tower-nacelle system 1st bending mode of vibration), while $q_2(t)$ is an absorber absolute displacement. Designations m_1, c_1, k_1 state for (modal) mass, damping, and stiffness of the primary system/structure, according to dependencies (3) from *Lab. 1*. A vibration reduction system comprises a spring of stiffness k_2 and an MR damper, built in parallel, with an additional stiff body of mass m_2. $P_{MR}(t)$ is a force produced by the MR damper (alternatively by an active cylinder), see Fig. 1.

To determine the MR damper resistance force $P_{MR}(t)$, hyperbolic tangent model given during *Lab. 2*, will be used. Two approaches of reproducing the MR damper force may be used, as developed during *Lab. 3*.

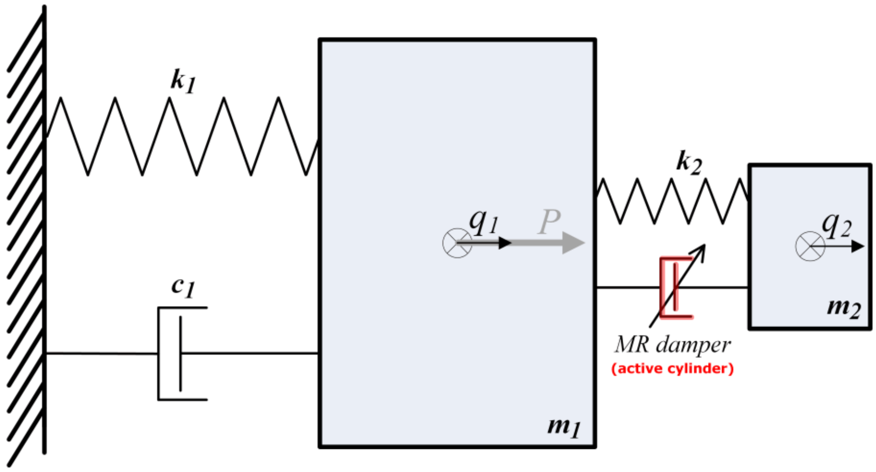

Fig. 1. A regarded system diagram

3. Optimal control [9,10]

(a) Pontryagin *maximum principle*

Assume state equation of the regarded system:

$$\dot{\boldsymbol{z}}(t) = \boldsymbol{f}(\boldsymbol{z}(t), u(t), t), \quad t \in [t_0, t_1] \tag{1}$$

where $\boldsymbol{z}(t)$ is the state vector with the initial value $\boldsymbol{z}(t_0) = \boldsymbol{z}_0$, $u(t)$ is piecewise-continuous control vector with constraints, $u(t) \in U$, and quality index to be minimised is:

$$G(\boldsymbol{z}, u) = \int_{t_0}^{t_1} g(\boldsymbol{z}(t), u(t), t)\, dt \tag{2}$$

Functions f and g are assumed to be continuously differentiable with respect to the state and continuous with respect to time and control. Let us define Hamiltonian in the form:

$$H(\boldsymbol{\xi}(t), \boldsymbol{z}(t), u(t), t) = -g(\boldsymbol{z}(t), u(t), t) + \boldsymbol{\xi}^T(t)\boldsymbol{f}(\boldsymbol{z}(t), u(t), t) \tag{3}$$

If $(\boldsymbol{z}^*(t), u^*(t))$ is an optimal controlled process (optimal trajectory of state, and optimal control, respectively), there exist an adjoint (co-state) variable $\boldsymbol{\xi}$ satisfying the equation ($\boldsymbol{f}_z$ and g_z are $\boldsymbol{f}$ and g derivatives with respect to the state $\boldsymbol{z}$):

$$\dot{\boldsymbol{\xi}}(t) = -\boldsymbol{f}_z^{*T}(\boldsymbol{z}^*(t), u^*(t), t)\boldsymbol{\xi}(t) + g_z^T(\boldsymbol{z}^*(t), u^*(t), t), \quad t \in [t_0, t_1] \tag{4}$$

with a terminal (transversality) condition:

$$\boldsymbol{\xi}(t_1) = 0 \tag{5}$$

so that $u^*(t)$ maximises the Hamiltonian over the set U for almost all $t \in [t_0, t_1]$, i.e.:

$$u^*(t) = \arg \max_{\boldsymbol{u}(t) \in U} H(\boldsymbol{\xi}(t), \boldsymbol{z}^*(t), u(t), t)$$

thus:

$$H(\boldsymbol{\xi}(t), \boldsymbol{z}^*(t), u^*(t), t) \geq H(\boldsymbol{\xi}(t), \boldsymbol{z}^*(t), u(t), t) \quad \forall u(t) \in U \tag{6}$$

(b) Optimal control as *multiobjective optimisation* problem

A parametric scalarising approach (e.g. weighted sum approach) can be used to convert set of objectives (e.g. primary system displacement minimisation, actuator force and stroke minimisation) into a single parametric objective function. By varying the parameters (*weights*) and optimising so scalarised function, various Pareto-optimal solutions can be found.

Remark 1: Weight of each objective should be *proportional* to its *relative importance* in scalarised multiobjective optimisation problem; *inversely proportional* to the allowable range of the particular quantity squared.

(c) Optimal control for *linear* systems – infinite horizon *continuous*-time *LQR*

A continuous Linear-Quadratic Regulator (LQR) is a state-feedback controller defined for continuous-time state-space system. Its parameters are calculated by solving the optimal problem called the continuous LQR problem. The LQR control algorithm can be employed for semi-active control of the tower-nacelle system with MR TVA, assuming P_{MR} (a force produced by an MR damper) as a control input. Using this algorithm, the optimal control signal P_{MR} is obtained. To induce the MR damper to generate the desired optimal control force, the MR damper force tracking algorithm / inverse model is used (see Fig. 2 and Fig. 3).

Consider a controllable (stabilisable), *linear*, time-invariant (*LTI*), *continuous system*:

$$\dot{\boldsymbol{z}}(t) = \boldsymbol{A}\boldsymbol{z}(t) + \boldsymbol{B}u(t)$$

where $\boldsymbol{z}(t) = [z_1(t) \quad z_2(t) \quad z_3(t) \quad z_4(t)]^T$ is a state vector ($z_1(t) = q_1(t)$, $z_2(t) = \dot{q}_1(t)$, $z_3(t) = q_2(t)$, $z_4(t) = \dot{q}_2(t)$), and a (multiobjective) *quadratic cost functional* over infinite interval:

$$J = \frac{1}{2}\int_{t_0}^{\infty} \boldsymbol{z}^T(t)\boldsymbol{Q}\boldsymbol{z}(t) + u^T(t)Ru(t)\, dt$$

where: $\boldsymbol{Q} = \boldsymbol{Q}^T \geq 0, R = R^T > 0$ are matrices of weights, $u(t) = P_{MR}(t)$, and terminal cost is assumed zero. The *optimal control* $u^*(t)$ law is given by:

$$u^*(t) = -\boldsymbol{K}\boldsymbol{z}^*(t)$$

where:

$$\boldsymbol{K} = R^{-1}\boldsymbol{B}^T\boldsymbol{S}$$

while $\boldsymbol{S}$, a constant (for infinite horizon case), positive definite, symmetric matrix, is a solution of a nonlinear *Algebraic Riccati Equation*:

$$\boldsymbol{Q} + \boldsymbol{A}^T\boldsymbol{S} + \boldsymbol{S}\boldsymbol{A} - \boldsymbol{S}\boldsymbol{B}R^{-1}\boldsymbol{B}^T\boldsymbol{S} = \boldsymbol{0}$$

while the *optimal trajectory* is given by:

$$\dot{\boldsymbol{z}}^*(t) = (\boldsymbol{A} - \boldsymbol{B}R^{-1}\boldsymbol{B}^T\boldsymbol{S})\boldsymbol{z}^*(t)$$

The continuous-time LQR optimisation problem may be solved using *lqr* function from *MATLAB/Simulink Optimization toolbox*. That function calculates the optimal state-feedback gain that minimises the assumed quadratic cost functional.

(d) Optimal control for *linear* systems – infinite horizon *discrete*-time *LQR* (*DLQR*)

A discrete Linear-Quadratic Regulator (DLQR) is a state-feedback controller defined for discrete-time state-space system. Its parameters are calculated by solving the optimal problem called the discrete LQR (DLQR) problem. Using this algorithm, the optimal control signal P_{MR} is obtained. To induce the MR damper to generate the desired optimal control force, the MR damper force tracking algorithm / inverse model is used (see Fig. 2 and Fig. 3).

Consider *linear*, time-invariant, *discrete system*:

$$\boldsymbol{z}(k+1) = \boldsymbol{A}_d \boldsymbol{z}(k) + \boldsymbol{B}_d P_{MR}(k)$$

where $\boldsymbol{z}(k) = [z_1(k) \quad z_2(k) \quad z_3(k) \quad z_4(k)]^T$ is a state vector, and a (multiobjective) *quadratic cost functional*:

$$J = \frac{1}{2} \sum_{k=k_0}^{\infty} [z^T(k) \boldsymbol{Q}_d z(k) + u^T(k) R_d u(k)]$$

The *optimal control* law is given by:

$$u^*(k) = -\boldsymbol{K}_d \boldsymbol{z}^*(k)$$

where:

$$\boldsymbol{K}_d = \left(\boldsymbol{B}_d{}^T \boldsymbol{S}_d \boldsymbol{B}_d + R_d\right)^{-1} \boldsymbol{B}_d{}^T \boldsymbol{S}_d \boldsymbol{A}_d$$

while $\boldsymbol{S}_d$ is a constant, positive definite solution of the discrete-time *Algebraic Riccati Equation*:

$$\boldsymbol{S}_d = \boldsymbol{A}_d{}^T \left[\boldsymbol{S}_d - \boldsymbol{S}_d \boldsymbol{B}_d \left(\boldsymbol{B}_d{}^T \boldsymbol{S}_d \boldsymbol{B}_d + R_d\right)^{-1} \boldsymbol{B}_d{}^T \boldsymbol{S}_d\right] \boldsymbol{A}_d + \boldsymbol{Q}_d$$

The continuous-time system may be discretised using zero-order hold with T_0 being sampling period. Thus (also, *MATLAB* command *c2d* may be used):

$\boldsymbol{A}_d = e^{\boldsymbol{A} \cdot T_0}$

$\boldsymbol{B}_d = \int_0^{T_0} e^{\boldsymbol{A}\tau} \boldsymbol{B} d\tau$ or: $\boldsymbol{B}_d = \boldsymbol{A}^{-1}(\boldsymbol{A}_d - I)\boldsymbol{B}$

$\boldsymbol{C}_d = \boldsymbol{C}$

$\boldsymbol{D}_d = \boldsymbol{D}$

The DLQR optimisation problem may be solved using *dlqr* or *lqrd* functions from *MATLAB/Simulink Optimization toolbox*. These functions calculate the optimal state-feedback gain that minimises the assumed cost functional.

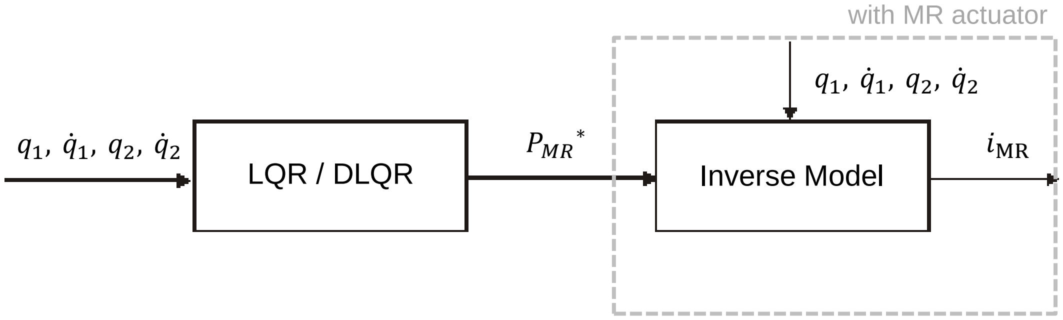

Fig. 2. LQR with MR damper inverse model

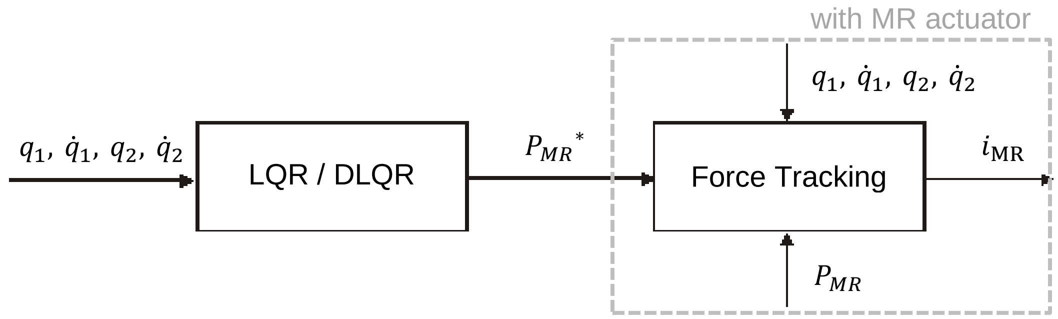

Fig. 3. LQR / DLQR with MR damper force tracking algorithm

(e) Optimal control for *nonlinear* systems

Pontryagin maximum principle based nonlinear optimal control solutions, i.e. optimal control / *model predictive control*, or *quasi-optimal control* / optimal based *modified ground-hook* law, are directly applicable for on-line and real-time implementation. An actuator control signal (e.g. MR damper control current) is directly determined as their output rather than the required *force* (as for two-stage cascade approaches), thus a force tracking algorithm (MR damper inverse model) that always results in control inaccuracy is entirely omitted. Moreover, all of the system / actuator (e.g. MR damper) *constraints*: the inability to generate active forces, lower and upper force value limitations, nonlinear operational characteristics with hysteresis, are embedded as an *intrinsic* part of this nonlinear optimal control technique, thus the solution is *optimal* (or quasi optimal) *for the assumed actuator*, respecting its constraints.

Consider the equation of motion of a vibrating system with an MR TVA (Fig. 1) in a form of (1), where $\boldsymbol{z}(t)$ is a state vector assumed as in *section 3*(*c*). Thus:

$$\boldsymbol{f}(\boldsymbol{z}(t),u(t),t)=\begin{bmatrix} z_2(t) \\ \frac{1}{m_1}\left(-(k_1+k_2)z_1(t)-c_1z_2(t)+k_2z_3(t)-P_{MR}(\boldsymbol{z}(t),u(t),t)+P(t)\right) \\ z_4(t) \\ \frac{1}{m_2}\left(k_2z_1(t)-k_2z_3(t)+P_{MR}(\boldsymbol{z}(t),u(t),t)\right) \end{bmatrix}$$

where:

$$P_{MR}(\boldsymbol{z}(t),u(t),t)= \\ =\left(C_1i_{MR}(u(t),t)+C_2\right)tanh\{v[\left(z_2(t)-z_4(t)\right)+\left(z_1(t)-z_3(t)\right)]\}+ \\ +\left(C_3i_{MR}(u(t),t)+C_4\right)[\left(z_2(t)-z_4(t)\right)+\left(z_1(t)-z_3(t)\right)]$$

is the MR damper force according to a hyperbolic tangent model, while $i_{MR}(u(t),t)$ is the MR damper coil current. To account for the control constraints, i.e. the MR damper current limitation to $[0,i_{max}]$ range ($i_{max}>0$), it may be assumed:

$$i_{MR}(u(t),t)=i_{max}\,sin^2\left(u(t)\right).$$

The considered quality function in (2) is:

$$g(\boldsymbol{z}(t),u(t),t)= \\ g_{11}{z_1}^2(t)+g_{12}{z_2}^2(t)+g_{13}\left(z_1(t)-z_3(t)\right)^2+g_{21}{i_{MR}}^2(u(t),t)+g_{22}{P_{MR}}^2(\boldsymbol{z}(t),u(t),t)$$

to account for the primary system/structure displacement z_1 and velocity z_2 minimisation, the MR damper stroke $(\mathrm{z}_1-\mathrm{z}_3)$ minimisation, and the MR damper control current i_{MR} and force P_{MR} minimisation.

For the regarded system, the co-state vector is $\boldsymbol{\xi}(t)=[\xi_1(t)\quad \xi_2(t)\quad \xi_3(t)\quad \xi_4(t)]^T$, while $\boldsymbol{f}_z^{*T}(\boldsymbol{z}^*(t),u^*(t),t)$ in (4) is:

$$\boldsymbol{f}_z^{*T}(\boldsymbol{z}^*(t),u^*(t),t)= \\ \begin{bmatrix} 0 & -\frac{1}{m_1}\left(k_1+k_2+\tilde{P}_{MR}(\boldsymbol{z}^*(t),u^*(t),t)\right) & 0 & \frac{1}{m_2}\left(k_2+\tilde{P}_{MR}(\boldsymbol{z}^*(t),u^*(t),t)\right) \\ 1 & -\frac{1}{m_1}\left(c_1+\tilde{P}_{MR}(\boldsymbol{z}^*(t),u^*(t),t)\right) & 0 & \frac{1}{m_2}\tilde{P}_{MR}(\boldsymbol{z}^*(t),u^*(t),t) \\ 0 & \frac{1}{m_1}\left(k_2+\tilde{P}_{MR}(\boldsymbol{z}^*(t),u^*(t),t)\right) & 0 & -\frac{1}{m_2}\left(k_2+\tilde{P}_{MR}(\boldsymbol{z}^*(t),u^*(t),t)\right) \\ 0 & \frac{1}{m_1}\tilde{P}_{MR}(\boldsymbol{z}^*(t),u^*(t),t) & 1 & -\frac{1}{m_2}\tilde{P}_{MR}(\boldsymbol{z}^*(t),u^*(t),t) \end{bmatrix}$$

where:

$$\tilde{P}_{MR}(\boldsymbol{z}^*(t), u^*(t), t) = \nu(C_1 i_{MR}(u^*(t), t) + C_2)\{1 - tanh^2[\nu(z_1^*(t) + z_2^*(t) - z_3^*(t) - z_4^*(t))]\} + (C_3 i_{MR}(u^*(t), t) + C_4)$$

thus:

$$\tilde{P}_{MR}(\boldsymbol{z}^*(t), u^*(t), t) = \frac{\partial P_{MR}(\boldsymbol{z}^*(t), u^*(t), t)}{\partial z_1^*} = \frac{\partial P_{MR}(\boldsymbol{z}^*(t), u^*(t), t)}{\partial z_2^*} = -\frac{\partial P_{MR}(\boldsymbol{z}^*(t), u^*(t), t)}{\partial z_3^*} = -\frac{\partial P_{MR}(\boldsymbol{z}^*(t), u^*(t), t)}{\partial z_4^*}$$

and:

$$g_z^T(\boldsymbol{z}^*(t), u^*(t), t) = \begin{bmatrix} 2g_{11} z_1^*(t) + 2g_{13}(z_1^*(t) - z_3^*(t)) + 2g_{22} P'_{MR}(z^*(t), u^*(t), t) \\ 2g_{12} z_2^*(t) + 2g_{22} P'_{MR}(z^*(t), u^*(t), t) \\ -2g_{13}(z_1^*(t) - z_3^*(t)) - 2g_{22} P'_{MR}(z^*(t), u^*(t), t) \\ -2g_{22} P'_{MR}(z^*(t), u^*(t), t) \end{bmatrix}$$

where:

$$P'_{MR}(z^*(t), u^*(t), t) = P_{MR}(\boldsymbol{z}^*(t), u^*(t), t)\tilde{P}_{MR}(\boldsymbol{z}^*(t), u^*(t), t).$$

The Hamiltonian maximisation condition is:

$$\frac{\partial H(\boldsymbol{\xi}(t), \boldsymbol{z}^*(t), u(t), t)}{\partial u(t)} = \left\{\left(-\frac{1}{m_1}\xi_2(t) + \frac{1}{m_2}\xi_4(t) - 2g_{22} P_{MR}(\boldsymbol{z}^*(t), u(t), t)\right)\frac{\partial P_{MR}(\boldsymbol{z}^*(t), u(t), t)}{\partial i_{MR}(u(t), t)} - 2 i_{max} g_{21} \sin^2(u(t))\right\} \sin(2u(t)) i_{max} = 0 \qquad (7)$$

with the appropriate sign change regime, where:

$$\frac{\partial P_{MR}(\boldsymbol{z}^*(t), u(t), t)}{\partial i_{MR}(u(t), t)} = C_1 \tanh[\nu(z_1^*(t) + z_2^*(t) - z_3^*(t) - z_4^*(t))] + C_3(z_1^*(t) + z_2^*(t) - z_3^*(t) - z_4^*(t))$$

From equation (7) we obtain:

$$\sin(2u(t)) = 0$$

or:

$$\sin^2(u(t)) = \frac{1}{2 i_{max} g_{21}}\left(-\frac{1}{m_1}\xi_2(t) + \frac{1}{m_2}\xi_4(t) - 2g_{22} P_{MR}(z^*(t), u(t), t)\right)\frac{\partial P_{MR}(z^*(t), u(t), t)}{\partial i_{MR}(u(t), t)} \qquad (8)$$

Thus, finally:

$$i_{MR}^*(u^*(t), t) = \begin{cases} 0, & if\ RHS(8) < 0 \\ \frac{1}{2g_{21}}\left(-\frac{1}{m_1}\xi_2(t) + \frac{1}{m_2}\xi_4(t) - 2g_{22} P_{MR}(z^*(t), u(t), t)\right)\frac{\partial P_{MR}(z^*(t), u(t), t)}{\partial i_{MR}(u(t), t)}, & if\ RHS(8) \in (0\ \ 1) \\ i_{max}, & if\ RHS(8) > 1 \end{cases} \qquad (9)$$

where $RHS(8)$ is the right-hand side of equation (8); an exemplary range of $[0, \pi)$ was considered here to fix an attention (regarding a period of both: $sin(2u(t))$ and $sin^2(u(t))$).

(f) Implementation of optimal control for *nonlinear* systems – *model predictive control* (*MPC*)

The common approach to the optimal control of nonlinear systems is computation of $u^*(t)$ using the maximum principle by solving the *two point boundary value problem* (*TPBVP*) (1)÷(5) offline. However, so calculated *open loop* control suffers from a lack of robustness to operating uncertainties, perturbations of external forces/disturbances or initial conditions, and to unmodeled dynamics that is always present for strongly nonlinear systems. To improve robustness to various types of uncertainties, perturbation control technique, among others, is used. In this method, the (correcting) feedback control $\partial u^*(t)$ is determined on-line from a linearization of the system about the optimal control pair $\left(\boldsymbol{z}^*(t), u^*(t)\right)$, on the basis of e.g. LQR theory. However, for highly nonlinear systems with implicit relations between state, co-state and control, proper linearization may be an issue.

To cope with that problem, *TPBVP* (1)÷(5) may be solved at every sample step, with the state and co-state dynamics on-line implementation. An optimisation horizon length $(t_1 - t_0)$ may be equal to some finite *N* integration (sample) steps, including *N=1* case (one-step optimality).

A standard *model predictive control* (MPC) approach is to minimise an objective function repeatedly at each sample step over a finite prediction horizon of *N* steps. The solution of such an optimal control problem depends on the current state and leads to an optimal control sequence of length *N,* whose first sample $u^*(t)$ is applied to the system, while the remaining future samples are discarded. The optimisation is repeated at the next sample step over a shifted time horizon, and so on (thus this procedure is also known as *receding horizon control*).

MATLAB/Simulink implementation procedure of *optimal control MPC*:

- in *Simulink*, state (1) and co-state (4) dynamics, as well as Hamiltonian maximisation condition (9) should be implemented, as well as:
- a dedicated level-2 s-function, implementing (1)(4) dynamics and (e.g.) *bvp4c MATLAB* function, should be called at every sample / integration step with actual external disturbances value(s) and actual control value(s) as the *inputs* for (1) and (4), actual state as the *initial condition* for (1), along with zero (5) as the *terminal condition* for (4),
- in this way a *TPBVP* problem (1)÷(5) is solved at every sample / integration step with an optimisation horizon length equal to some finite integral number of *N* (*N≥1*) steps, yielding the (*missing*) *initial condition* for co-state (4); then, all co-state *Integrator* blocks are reset to these initial condition values, and the procedure is repeated for the next sampling step.

Remark 2: As thorough research along with simulation and experimental analyses proved [5,6], for some nonlinear systems (e.g. tower-nacelle model with MR or hybrid TVA(s)), the *TPBVP* solving at every sample / integration step may be omitted to avoid a large computational load; in this case the co-state *Integrator* blocks are reset to *zero* initial condition values at each sample / integration step. The error of this approach is negligible (except for a numerable number of time instants at which the displacement/deflection of the primary system/structure changes sign) and the quality of vibration control does not noticeably differ from the quality of *optimal control MPC* implementation, assuming the appropriate sampling frequency. This solution is termed *quasi-optimal control*.

Remark 3: Another simplified procedure not requiring the *TPBVP* solving is two-level *modified ground-hook* law; it does not require the implementation of the Hamiltonian maximization condition (9), nor the model of the state (1) and co-state dynamics (4); in its basic approach it minimises the displacement/deflection amplitude of the primary system/structure.

4. Tasks

For the regarded system with parameters *m1*, *c1*, *k1* *m2*, *k2* determined during *Lab. 1*, MR damper RD-1097-1 model and parameters according to *Lab. 2*, and i_{max}=0.5 [A]:

1. build *LQR* / *DLQR* controllers according to *section 3*, points (*c*)(*d*) (assume T_0=1 ms, and T_0=10 ms; use MR damper inverse model and force tracking algorithm from *Lab. 3*), using *MATLAB/Simulink* environment; assume weights $\boldsymbol{Q}$ and R with regard to minimisation of: the primary system/structure displacement z_1, the MR damper stroke $(z_1 - z_3)$, and the MR damper force P_{MR}; assume excitation *P*(*t*) as a sine input of amplitude 61 N and angular frequency vector $\Omega=\omega_{1d}*$[0.50:0.05:1.50] rd/s with ω_{1d} according to relation (5b) from *Lab. 1*,
2. SUPPLEMENTARY: build *optimal control MPC* solution according to *section 3*, points (*e*)(*f*) (assume T_0=1 ms), using *MATLAB/Simulink* environment; assume weights with regard to minimisation of: the primary system/structure displacement z_1, the MR damper stroke $(z_1 - z_3)$, and the MR damper force P_{MR} or current i_{MR}; assume excitation *P*(*t*) as a sine input of amplitude 61 N and angular frequency vector *Ω* – see *Task 1* (maximum of 3 additional points),
3. for each of the control solutions execute 21 simulations applying each of the consecutive excitation angular frequencies from the regarded vector *Ω* (*Task 1*), and determine *DAF* for steady state oscillations; consider simulation time long enough (longer than a transient response);

   Remark 4: for each of the *LQR* / *DLQR* controllers use *both MR-damper-based* configurations (according to Fig. 2 and Fig. 3), as well as *one additional* configuration with an ideal *active cylinder/actuator* of [–100, 100] N output force range, instead of the MR damper (see Fig. 1),
4. print *DAF* frequency response characteristics (*DAF* [-] vs. angular frequency [rd/s] curve) for all of the regarded control solutions in one graph; for each solution print in a legend section its design parameters i.e.: T_0, Q_1, Q_2, Q_3, R (designations in the *Appendix* below; controllers from *section 3*(*c*)(*d*)), or: $g_{11}, g_{12}, g_{13},\ g_{21}, g_{22}$ (controllers from *section 3*(*e*)(*f*)).
5. compare *DAF* frequency responses of *Task 4* with those obtained for constant MR damper current values (from the vector I_{MR}=[0.0, 0.1, 0.2, 0.5] A), using graph/data (or *Simulink* model) from *Lab. 2*.

Appendix: LQR weights selection

LQR cost functional for displacement ($z_1 = q_1$), MR damper stroke ($z_1 - z_3 = q_1 - q_2$), and MR damper force ($u = P_{MR}$) minimisation:

$$J = \frac{1}{2}\int_{t_0}^{\infty} \boldsymbol{z}^T(t)\boldsymbol{Q}\boldsymbol{z}(t) + u^T(t)Ru(t)\,dt$$

where:

$$\boldsymbol{Q} = \boldsymbol{Q}^T = \begin{bmatrix} Q_1 + Q_3 & 0 & -Q_3 & 0 \\ 0 & Q_2 & 0 & 0 \\ -Q_3 & 0 & Q_3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \geq 0, R = R^T > 0$$

are matrices of weights, thus:

$$J = \frac{1}{2}\int_{t_0}^{\infty} [q_1(t) \quad \dot{q}_1(t) \quad q_2(t) \quad \dot{q}_2(t)] \begin{bmatrix} Q_1 + Q_3 & 0 & -Q_3 & 0 \\ 0 & Q_2 & 0 & 0 \\ -Q_3 & 0 & Q_3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} q_1(t) \\ \dot{q}_1(t) \\ q_2(t) \\ \dot{q}_2(t) \end{bmatrix} \boldsymbol{z}(t) + P_{MR}(t) R P_{MR}(t)\, dt$$

$$J = \frac{1}{2}\int_{t_0}^{\infty} Q_1 {q_1}^2(t) + Q_2 {\dot{q}_1}^2(t) + Q_3\big(q_1(t) - q_2(t)\big)^2 + R{P_{MR}}^2(t)\, dt$$

It may be assumed e.g.: $Q_1 = 10^6$ (or greater); $Q_2 = 1$; $Q_3 = 1$ or $Q_3 = 10^4$; $R = 10^{-4}$ (try your own weight value selections).

References:

[1] MATLAB/Simulink documentation: http://www.mathworks.com
[2] J.P. Den Hartog, *Mechanical Vibrations*. Mineola: Dover Publications, 1985.
[3] P. Martynowicz *Vibration control of wind turbine tower-nacelle model with magnetorheological tuned vibration absorber*, Journal of Vibration and Control 23(20), 2017.
[4] P. Martynowicz *Control of a magnetorheological tuned vibration absorber for wind turbine application utilising the refined force tracking algorithm*, Journal of Low Frequency Noise, Vibration & Active Control, 36(4), 2017.
[5] P. Martynowicz *Nonlinear optimal-based vibration control for systems with MR tuned vibration absorbers,* Journal of Low Frequency Noise; Vibration and Active Control, 38(3-4), 2019.
[6] P. Martynowicz *Real-time implementation of nonlinear optimal-based vibration control for a wind turbine model*, Journal of Low Frequency Noise; Vibration and Active Control, 38(3-4), 2019.
[7] Lord Rheonetic, *MR Controllable Friction Damper RD-1097-01 Product Bulletin*, Lord Co. 2002.
[8] M. Maślanka, B. Sapiński and J. Snamina, *Experimental Study of Vibration Control of a Cable With an Attached MR Damper*, Journal of Theoretical and Applied Mechanics 45(4), 2007.
[9] A.D. Ioffe and V.M. Tihomirov *Theory of extremal problems. Studies in mathematics and its applications*, Amsterdam–New York–Oxford: North-Holland Publishing Company, 1979.
[10] D.S. Naidu *Optimal Control Systems*, CRC Press, 2003.
[11] W.S. Oates and R.C. Smith *Nonlinear optimal control techniques for vibration attenuation using magnetostrictive actuators*, J Intell Mater Syst Struct, 19(2), 2008.
[12] K. Deb *Multi-Objective Optimization Using Evolutionary Algorithms: An Introduction*, KanGAL Report No. 2011003, 2011.
[13] A. Bemporada and D. Muñoz de la Peña *Multiobjective model predictive control*, Automatica 45, 2009.