Pathological Real-Valued Continuous Functions

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Abstract

We look at continuous functions with pathological properties, in particular, two examples of continuous functions that are nowhere differentiable. The first example was discovered by K. W. T. Weierstrass in 1872 and the second by B. L. Van der Waerden in 1930. We also present an example of a continuous strictly monotonic function with a vanishing derivative almost everywhere, discovered by Zaanen and Luxemburg in 1963.

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CHAPTER 1

Introduction

The aim of this project is to investigate pathological counterexamples of continuous functions, that is, functions whose behaviour is atypical and counterintuitive. Theorems and concepts from real analysis are used in the construction of these pathological functions.

Continuity and differentiability are two related concepts in real analysis. A well-known theorem states that if a function is differentiable at a point, then it must be continuous there as well. The converse of this statement, however, is false, as shown by the absolute value function at zero. This notion of a function being continuous yet not differentiable can be extended to the entire real line. However, these functions are not as simple as the absolute value function. In fact, up to the early nineteenth century, most mathematicians believed that every continuous function is differentiable at almost all points. The first counterexample, the Weierstrass function, proved this conjecture to be false. First presented in July 1872, this prototypical example shocked the mathematical world. After the Weierstrass function was published, many other continuous non-differentiable functions were discovered. Another more geometrically intuitive example presented in this paper is a function first given by Van der Waerden in 1930.

Monotonicity and differentiability are related as well, and there is a theorem that associates these concepts: if a function has a positive derivative at every point in an interval, then the function must be strictly increasing on the interval. Again, we can use a simple example such as the cubic function to illustrate that the converse is false. To improve on this notion, we introduce the Cantor function, affectionately known as the Devil's Staircase. We use the Cantor function to construct an example of a continuous strictly monotonic function with a derivative that vanishes almost everywhere. This highly atypical function was first given by Zaanen and Luxemburg in 1963.

The project is structured as follows. In Chapter 2, we review continuity and differentiability. Since all the pathological examples in this paper are constructed with sequences of functions, sequences and series are reviewed here as well. Chapter 3 then presents the two examples of continuous yet nowhere differentiable functions, given by K. W. T. Weierstrass and B. L. Van der Waerden. In Chapter 4, monotonicity and some important theorems are presented. Finally, in Chapter 5, we investigate the second type of pathological example of a continuous strictly increasing function with a vanishing derivative almost everywhere, given by Zaanen and Luxemburg. The main examples are taken from *Counterexamples in Analysis* [**1**], by Gelbaum and Olmsted.

CHAPTER 2

Continuity and Differentiability

1. Preliminaries

In this chapter we review the necessary definitions and theorems regarding continuity and differentiability of real-valued functions. Our aim is to lay the foundation for later discussion of the main examples.

Definition 2.1 ([**2**]). Let $I \subseteq \mathbb{R}$ be an interval, let $f : I \to \mathbb{R}$, and let $c \in I$. We say that f is **continuous at** c if, given any number $\epsilon > 0$, there exists $\delta(\epsilon) > 0$ such that if x is any point of I satisfying $|x - c| < \delta(\epsilon)$, then $|f(x) - f(c)| < \epsilon$.

Equivalently, f is continuous at c when

$$\lim_{x \to c} f(x) = f(c).$$

Definition 2.2 ([**2**]). Let $I \subseteq \mathbb{R}$ be an interval, let $f : I \to \mathbb{R}$, and let $c \in I$. We say that a real number L is the **derivative of f at c** if given any $\epsilon > 0$ there exists $\delta(\epsilon) > 0$ such that if $x \in I$ satisfies $0 < |x - c| < \delta(\epsilon)$, then

$$\left| \frac{f(x) - f(c)}{x - c} - L \right| < \epsilon.$$

In other words, the derivative of f at c is given by the limit

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}$$

provided this limit exists.

Next we state one of the most significant results in real analysis, the Mean Value Theorem, which relates the values of a function to the values of its derivative.

Theorem 2.3 (Mean Value Theorem [**3**]). *Suppose that f is continuous on a closed interval $I = [a, b]$, and that f has a derivative in the open interval (a, b). Then there exists at least one point c in (a, b) such that*

$$f(b) - f(a) = f'(c)(b - a).$$

Proof. The proof of this fundamental result can be found on page 154 of [**3**]. $\square$

We relate differentiability with continuity of a function with the following theorem. The pathological property of interest concerns its converse.

THEOREM 2.4 ([**3**]). *If $f : I \to \mathbb{R}$ has a derivative at $c \in I$, then f is continuous at c.*

PROOF. For all $x \in I, x \neq c$, we have

$$f(x) - f(c) = \left(\frac{f(x) - f(c)}{x - c}\right)(x - c).$$

Since $f'(c)$ exists, we have

$$\lim_{x\to c}(f(x) - f(c)) = \lim_{x\to c}\left(\frac{f(x) - f(c)}{x - c}\right)\left(\lim_{x\to c}(x - c)\right) = f'(c) \cdot 0 = 0.$$

Therefore, $\lim_{x\to c} f(x) = f(c)$ so that f is continuous at c. $\square$

However the converse of the theorem is not necessarily true, which can be easily seen with the classic example of the absolute value function.

EXAMPLE 2.5. Let $f(x) = |x|$. Then f is continuous but not differentiable at $x = 0$.

For any $\epsilon > 0$, there exists $\delta = \epsilon > 0$ such that for all $x \in \mathbb{R}$, $|x - 0| < \delta$ we have

$$|f(x) - f(0)| = \big||x| - |0|\big| = |x| < \delta = \epsilon.$$

Hence f is continuous at $x = 0$.

For $x \neq 0$ we have

$$\frac{f(x) - f(0)}{x - 0} = \frac{|x|}{x} = \begin{cases} 1 & \text{if } x > 0 \\ -1 & \text{if } x < 0. \end{cases}$$

Hence the left and right hand limits are not equal:

$$\lim_{x\to 0^-} \frac{f(x) - f(0)}{x - 0} \neq \lim_{x\to 0^+} \frac{f(x) - f(0)}{x - 0}$$

and we conclude that f is not differentiable at $x = 0$.

Graphically, we can see that the function has a sharp, non-differentiable point at $x = 0$.

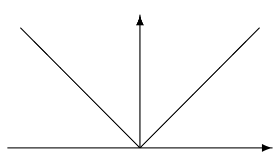

Figure 1 Graph of $f(x) = |x|$.

With this simple first example, we see that continuity at a point does not ensure the existence of the derivative at that point. Indeed, this property of being continuous yet not differentiable can be seen on the entire domain of certain functions, not just at a single point; that is, there exist continuous functions that are *nowhere* differentiable. However, these functions are not as simple as the example of the absolute value function, as will be seen in the construction of the examples in Chapter 3.

2. Sequences of functions

The examples of continuous but nowhere differentiable functions discussed in this paper are constructed with sequences and infinite series of continuous functions. We now review some important properties of sequences and series using definitions and theorems from [**2**], beginning with uniform convergence of a sequence of functions.

Definition 2.6. A sequence $\{f_n\}$ of functions on an interval $I \subseteq \mathbb{R}$ to $\mathbb{R}$ is said to **converge uniformly on** $I_0 \subseteq I$ to a function $f : I_0 \to \mathbb{R}$ if for each $\epsilon > 0$ there is a natural number $K(\epsilon)$ (depending on ϵ but not on $x \in I_0$) such that if $n \geq K(\epsilon)$, then

$$|f_n(x) - f(x)| < \epsilon \quad \text{for all } x \in I_0.$$

Theorem 2.7. *Let $\{f_n\}$ be a sequence of continuous functions on a set $D \subseteq \mathbb{R}$ and suppose that $\{f_n\}$ converges uniformly on D to a function $f : D \to \mathbb{R}$. Then f is continuous on D.*

Proof. Let $x_0 \in D$. To show that f is continuous at x_0, let $\epsilon > 0$ be given. By the uniform convergence of f_n, there exists an $N \in \mathbb{N}$ such that $|f_n(x) - f(x)| < \frac{\epsilon}{3}$ for all $n > N$ and all $x \in D$. So for $n = N + 1$, we have $|f_{N+1}(x) - f(x)| < \frac{\epsilon}{3}$ for all $x \in D$. Since f_{N+1} is continuous at x_0, there exists a $\delta > 0$ such that $|f_{N+1}(x) - f_{N+1}(x_0)| < \frac{\epsilon}{3}$ for all $|x - x_0| < \delta, x \in D$. Hence

$$\begin{aligned} |f(x) - f(x_0)| &\leq |f(x) - f_{N+1}(x)| + |f_{N+1}(x) - f_{N+1}(x_0)| + |f_{N+1}(x_0) - f(x_0)| \\ &< \tfrac{\epsilon}{3} + \tfrac{\epsilon}{3} + \tfrac{\epsilon}{3} = \epsilon. \end{aligned}$$

Thus f is continuous at the arbitrary point $x_0 \in D$. □

This theorem is used in the proof of the analogous Theorem 2.9 for infinite series, which is a necessary tool for proving continuity in the main examples.

Definition 2.8. If $\{f_n\}$ is a sequence of functions defined on $D \subseteq \mathbb{R}$ with values in $\mathbb{R}$, the sequence of **partial sums** $\{s_n\}$ of the infinite series $\sum f_n$ is defined for $x \in D$ by

$$\begin{aligned} s_1(x) &:= f_1(x), \\ s_2(x) &:= s_1(x) + f_2(x), \\ &\vdots \\ s_{n+1}(x) &:= s_n(x) + f_{n+1}(x), \\ &\vdots \end{aligned}$$

In case the sequence $\{s_n\}$ of functions converges on D to a function f, we say that the infinite series of functions $\sum f_n$ **converges** to f on D and write

$$\sum f_n \quad \text{or} \quad \sum_{n=1}^{\infty} f_n$$

to denote either the series or the limit function, when it exists.

The following is a direct translation of Theorem 2.7 for series.

Theorem 2.9. *Let D be a subset of $\mathbb{R}$. If f_n is a continuous function on D for each $n \in \mathbb{N}$ and if $\sum f_n$ converges to f uniformly on D, then f is continuous on D.*

Proof. We simply view $\sum f_n$ as a sequence (of partial sums) and apply Theorem 2.7. We have that $s_n(x)$ is continuous on D for each $n \in \mathbb{N}$ and the sequence of partial sums $s_n(x) = f_1(x) + f_2(x) + \cdots + f_n(x)$ converges to $f(x)$ uniformly on D. By Theorem 2.7, we conclude that f is continuous on D. $\square$

Theorem 2.10 (Cauchy's Criterion). *Let $\{f_n\}$ be a sequence of functions on $D \subseteq \mathbb{R}$ to $\mathbb{R}$. The series $\sum f_n$ is uniformly convergent on D if and only if for every $\epsilon > 0$ there exists an $M(\epsilon)$ such that if $m > n \geq M(\epsilon)$, then*

$$|f_{n+1}(x) + \cdots + f_m(x)| < \epsilon \text{ for all } x \in D.$$

The following is a well-known theorem that we also use to prove continuity of our main examples.

Theorem 2.11 (Weierstrass M-Test). *Let $\{M_n\}$ be a sequence of positive real numbers such that $|f_n(x)| \leq M_n$ for $x \in D, n \in \mathbb{N}$. If the series $\sum M_n$ is convergent, then $\sum f_n$ is uniformly convergent on D.*

Proof. Since $\sum M_n$ is convergent, we have that for any $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that if $t > s \geq N$, then

$$\left| \sum_{n=1}^{t} M_n - \sum_{n=1}^{s} M_n \right| = M_{s+1} + M_{s+2} + \cdots + M_t < \epsilon.$$

Then for all $x \in D$ we have

$$\begin{aligned} \left| \sum_{n=1}^{t} f_n(x) - \sum_{n=1}^{s} f_n(x) \right| &= |f_{s+1}(x) + f_{s+2}(x) + \cdots + f_t(x)| \\ &\leq |f_{s+1}| + |f_{s+2}| + \cdots + |f_t(x)| \\ &\leq M_{s+1} + M_{s+2} + \cdots + M_t \\ &< \epsilon. \end{aligned}$$

Thus $\sum f_n$ is uniformly convergent on D. $\square$

Having laid the proper foundation, we are now ready to examine two pathological examples of functions that are continuous yet nowhere differentiable. In both cases, the Weierstrass M-test and Theorem 2.9 are used to show continuity.

CHAPTER 3

Two Continuous Functions that are Nowhere Differentiable

We look at two examples of continuous functions that are nowhere differentiable. K. Weierstrass gave the first example of such a function in a paper presented to the Königliche Akademie der Wissenschaften in July 1872. Another more geometrically intuitive example of a function demonstrating this intriguing behaviour was first given by Van der Waerden in 1930.

1. Example given by Weierstrass (1872)

THEOREM 3.1 ([**4**]). *Let a be a positive odd integer and $0 < b < 1$ such that $ab > 1+\frac{3}{2}\pi$. Then*

$$f(x) = \sum_{n=0}^{\infty} b^n cos(a^n \pi x)$$

is a continuous function that is nowhere differentiable on $\mathbb{R}$.

We first define two functions that will be used in the main proof.

LEMMA 3.2 ([**5**]). *Define functions $\alpha : \mathbb{R} \to \mathbb{R}$ and $\xi : \mathbb{R} \to \mathbb{R}$ by*

$$\alpha(x) = \lfloor x + \tfrac{1}{2} \rfloor \quad \text{and} \quad \xi(x) = x - \alpha(x) = x - \lfloor x + \tfrac{1}{2} \rfloor.$$

Then $x = \alpha(x) + \xi(x)$ and also the following properties hold:

(i) $\alpha(x) \in \mathbb{Z}$,
(ii) $\xi(x+1) = \xi(x)$ *for all* $x \in \mathbb{R}$,
(iii) $\xi(x) = x$ *for all* $x \in [-\frac{1}{2}, \frac{1}{2})$, *and*
(iv) $|\xi(x)| \leq \frac{1}{2}$ *for all* $x \in \mathbb{R}$.

PROOF. (i) This is true since the floor function maps to integers.
(ii) We have

$$\begin{aligned}\xi(x+1) &= (x+1) - \lfloor (x+1) + \tfrac{1}{2} \rfloor = (x+1) - \left(\lfloor x + \tfrac{1}{2} \rfloor + 1\right) \\ &= x - \lfloor x + \tfrac{1}{2} \rfloor = \xi(x).\end{aligned}$$

(iii) Since $-\frac{1}{2} \leq x < \frac{1}{2}$, we have $0 \leq x + \frac{1}{2} < 1$ and hence $\lfloor x + \frac{1}{2} \rfloor = 0$. Then for $x \in [-\frac{1}{2}, \frac{1}{2})$, we have

$$\xi(x) = x - \lfloor x + \tfrac{1}{2} \rfloor = x - 0 = x.$$

(iv) If $x \geq \frac{1}{2}$, then

$$\begin{aligned}\xi(x) &= \xi\left((x-1)+1\right) \overset{\text{(ii)}}{=} \xi(x-1) \\ &= \xi(x-2) = \cdots = \xi(x-m), \quad \text{where} \quad x-m \in [-\tfrac{1}{2}, \tfrac{1}{2}).\end{aligned}$$

So $|\xi(x)| = |\xi(x-m)| \overset{\text{(iii)}}{=} |x-m| \leq \frac{1}{2}$.

Similarly, if $x < -\frac{1}{2}$, we have

$$|\xi(x)| = |\xi(x+1)| = |\xi(x+2)| = \cdots = |\xi(x+m)| = |x+m| \leq \tfrac{1}{2}, \quad \text{where} \quad x+m \in [-\tfrac{1}{2}, \tfrac{1}{2}).$$

Hence $|\xi(x)| \leq \frac{1}{2}$ for all $x \in \mathbb{R}$.

$\square$

Proof of Theorem 3.1 ([**5**]). Returning to the main theorem, we first show that f is everywhere continuous. It is clear that the partial sum $s_n(x) = \sum_{i=0}^{n} b^i \cos(a^i \pi x)$ is continuous for each n. Since $0 < b < 1$, the geometric series $\sum_{i=0}^{\infty} b^i$ converges, and for all $x \in \mathbb{R}$ and $n \in \mathbb{N}$ we have

$$|b^n \cos(a^n \pi x)| \leq b^n.$$

By the Weierstrass M-Test (Theorem 2.11) with $M_n = b^n$, the series $\sum_{n=0}^{\infty} b^n \cos(a^n \pi x)$ converges uniformly on $\mathbb{R}$. Now by Theorem 2.9, since each partial sum is continuous and the convergence of the series to f is uniform, we conclude that f is continuous.

Next we show that f is nowhere differentiable. Let $x \in \mathbb{R}$ be arbitrary. To show that

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \quad \text{does not exist,}$$

we find a sequence $\{h_n\}_{n \geq 0}$ such that $h_n \to 0$ as $n \to \infty$ but

$$\lim_{n \to \infty} \frac{f(x+h_n) - f(x)}{h_n} \quad \text{does not exist.}$$

First fix an $n \in \mathbb{N}$. Then we have

$$\frac{f(x+h_n) - f(x)}{h_n} = \frac{1}{h_n} \left(\sum_{k=0}^{\infty} b^k \cos(a^k \pi (x+h_n)) - \sum_{k=0}^{\infty} b^k \cos(a^k \pi x) \right) = T_n + R_n,$$

$$\begin{aligned}\text{where } T_n &:= \frac{1}{h_n} \sum_{k=0}^{n-1} b^k \left(\cos a^k \pi (x+h_n) - \cos(a^k \pi x) \right) \\ \text{and } R_n &:= \frac{1}{h_n} \sum_{k=n}^{\infty} b^k \left(\cos a^k \pi (x+h_n) - \cos(a^k \pi x) \right).\end{aligned}$$

By the Triangle Inequality, $|R_n + T_n| \geq |R_n| - |T_n|$ so

$$\left| \frac{f(x+h_n) - f(x)}{h_n} \right| \geq |R_n| - |T_n| \quad \text{for every } n \geq 0. \tag{1}$$

Using the trigonometric identity

$$\cos u - \cos v = -2\sin\left(\frac{u-v}{2}\right)\sin\left(\frac{u+v}{2}\right)$$

where $u = a^k\pi(x+h_n)$ and $v = a^k\pi x$, we have

$$\begin{aligned} |\cos a^k\pi(x+h_n) - \cos a^k\pi x| &= \left| 2\ \sin\left(\frac{a^k\pi h_n}{2}\right)\sin\left(\frac{2a^k\pi x + a^k\pi h_n}{2}\right)\right| \\ &\leq \left| 2\ \sin\left(\frac{a^k\pi h_n}{2}\right)\right| \ \leq \ 2\left|\frac{a^k\pi h_n}{2}\right| \\ &= a^k\pi|h_n|. \end{aligned}$$

Next, with the above relation, we have

$$\begin{aligned} |T_n| &= \left|\frac{1}{h_n}\sum_{k=0}^{n-1} b^k\left(\cos a^k\pi(x+h_n) - \cos(a^k\pi x)\right)\right| \\ &\leq \frac{1}{|h_n|}\sum_{k=0}^{n-1} b^k|\cos a^k\pi(x+h_n) - \cos(a^k\pi x)| \\ &\leq \frac{1}{\cancel{|h_n|}}\sum_{k=0}^{n-1} b^k a^k\pi\cancel{|h_n|} \ = \ \pi\sum_{k=0}^{n-1}(ab)^k \ = \ \pi\,\frac{(ab)^n-1}{ab-1} \\ &< \pi\,\frac{(ab)^n}{ab-1}. \end{aligned} \tag{2}$$

To find the desired $\{h_n\}$, we use the two functions as shown in Lemma 3.2:

$$\alpha(x) = \lfloor x + \tfrac{1}{2}\rfloor \quad \text{and} \quad \xi(x) = x - \lfloor x + \tfrac{1}{2}\rfloor.$$

Denote $\alpha(a^n x) := \alpha_n$ and $\xi(a^n x) := \xi_n$. Then

$$a^n x = \alpha(a^n x) + \xi(a^n x) = \alpha_n + \xi_n,$$

and $\alpha_n \in \mathbb{Z}$ and $|\xi_n| \leq \frac{1}{2}$.

Now for any $k \geq n$,

$$\begin{aligned} \cos a^k\pi x &= \cos a^{k-n}a^n\pi x = \cos a^{k-n}\pi(\alpha_n + \xi_n) \\ &= \cos a^{k-n}\pi\alpha_n \cos a^{k-n}\pi\xi_n - \sin a^{k-n}\pi\alpha_n \sin a^{k-n}\pi\xi_n \\ &= \cos a^{k-n}\pi\alpha_n \cos a^{k-n}\pi\xi_n - 0 && \left(\sin(a^{k-n}\alpha_n)\pi = 0\right) \\ &= (-1)^{a^{k-n}\alpha_n}\cos a^{k-n}\pi\xi_n && \left(\cos k\pi = (-1)^k \text{ for } k \in \mathbb{Z}\right) \\ &= (-1)^{\alpha_n}\cos a^{k-n}\pi\xi_n \end{aligned} \tag{3}$$

and

$$\begin{aligned}\cos a^k\pi(x+h_n) &= \cos a^{k-n}a^n\pi(x+h_n)\\ &= \cos a^{k-n}\pi[a^nx+a^nh_n]\\ &= \cos a^{k-n}\pi(\alpha_n+\xi_n+a^nh_n).\end{aligned} \tag{4}$$

Define

$$h_n := \frac{1-\xi_n}{a^n}.$$

Then $\lim_{n\to\infty} h_n = 0$ since

$$h_n = |h_n| \le \frac{1+|\xi_n|}{a^n} \overset{(3.2)(\text{iii})}{\le} \frac{3}{2a^n}. \tag{5}$$

By (4), we have

$$\begin{aligned}\cos a^k\pi(x+h_n) &= \cos a^{k-n}\pi\big(\alpha_n+\cancel{\xi_n}+(1-\cancel{\xi_n})\big)\\ &= \cos a^{k-n}\pi(\alpha_n+1)\\ &= (-1)^{a^{k-n}(\alpha_n+1)}\\ &= (-1)^{\alpha_n+1}.\end{aligned} \tag{6}$$

Now by (3), (4), and (6),

$$\begin{aligned}|R_n| &= \left|\frac{1}{h_n}\sum_{k=n}^{\infty} b^k[\cos a^k\pi(x+h_n)-\cos a^k\pi x]\right|\\ &= \frac{1}{|h_n|}\left|\sum_{k=n}^{\infty} b^k\Big[(-1)^{\alpha_n+1}-(-1)^{\alpha_n}\cos a^{k-n}\pi\xi_n\Big]\right|\\ &= \frac{1}{h_n}\left|(-1)^{\alpha_n+1}\right|\left|\sum_{k=n}^{\infty} b^k(1+\cos a^{k-n}\pi\xi_n)\right|\\ &= \frac{1}{h_n}\sum_{k=n}^{\infty} b^k(1+\cos a^{k-n}\pi\xi_n)\\ &\ge \frac{1}{h_n}b^n(1+\cos\pi\xi_n)\\ &\overset{(3.2)(\text{iii})}{\ge} \frac{1}{h_n}b^n \qquad\qquad \text{since}\quad \pi\xi_n\in[-\tfrac{\pi}{2},\tfrac{\pi}{2}]\\ &\overset{(5)}{\ge} \frac{2}{3}(ab)^n.\end{aligned} \tag{7}$$

So by (1), we see that

$$\begin{aligned}\left|\frac{f(x+h_n)-f(x)}{h_n}\right| &\geq |R_n| - |T_n| \\ &\underset{(7)}{\overset{(2)}{\geq}} \frac{2}{3}(ab)^n - \pi\frac{(ab)^n}{ab-1} \\ &= \frac{2}{3}(ab)^n\left(1-\pi\frac{3}{2}\frac{1}{ab-1}\right) \\ &= \frac{2}{3}(ab)^n\frac{ab-(1+\frac{3}{2}\pi)}{ab-1}. \end{aligned} \tag{8}$$

As $n \to \infty$, we have that

$$\frac{2}{3}(ab)^n\frac{ab-(1+\frac{3}{2}\pi)}{ab-1} \to \infty$$

since $ab > 1+\frac{3}{2}\pi$. Finally, from the inequality in (8), we have that

$$\lim_{n\to\infty}\frac{f(x+h_n)-f(x)}{h_n} \quad \text{does not exist.}$$

Therefore, since x was arbitrary, we conclude that f is not differentiable anywhere. □

2. Example given by Van der Waerden (1930)

The second example of a continuous nowhere differentiable function is constructed through the infinite series of a sequence of continuous "sawtooth" functions. Intuitively, from Figure 2, it can be seen that as the sequence progresses, the functions remain continuous and have smaller periods that alternate between positive and negative slopes more frequently. As well, the sharp points where the functions are non-differentiable cover the entire real line. This behaviour leads to the eventual non-differentiability of the infinite series.

EXAMPLE 3.3 ([**6**]). Let $f_0 : \mathbb{R} \to \mathbb{R}$ be defined by

$$f_0(x) := \operatorname{dist}(x, \mathbb{Z}) = \inf\{|x-k| : k \in \mathbb{Z}\},$$

so that f_0 is a continuous "sawtooth" function whose graph consists of lines with slope ± 1 on the intervals $[\frac{k}{2}, \frac{(k+1)}{2}], k \in \mathbb{Z}$. For each $m \in \mathbb{N}$, let $f_m(x) := (\frac{1}{4^m})f_0(4^m x)$, so that f_m is also a continuous sawtooth function whose graph consists of lines with slope ± 1 and with $0 \leq f_m(x) \leq \frac{1}{2\cdot 4^m}$.

The following shows the graphs of the first three functions in the sequence f_0, f_1 and f_2 defined as above. Underneath each triangle lies four triangles of the next function in the sequence, all of which are a fourth of the size of the bigger triangle. Note that Figure 2 displays a single period of f_0; the graphs extend across the entire real line.

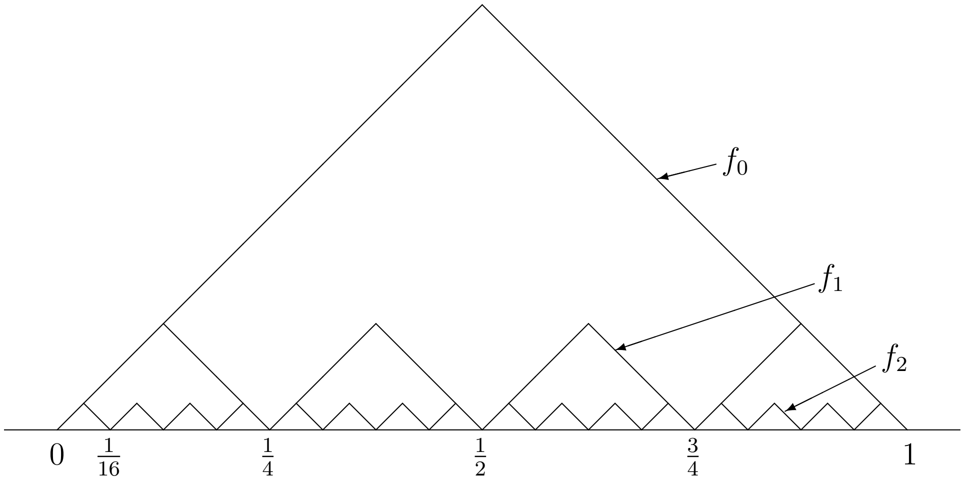

Figure 2 Graphs of f_0, f_1, and f_2.

Theorem 3.4 ([**2**]). *Let f_n be given as in Example 3.3. Then*

$$f(x) = \sum_{n=1}^{\infty} f_m(x) = \sum_{m=1}^{\infty} \frac{f_0(4^m x)}{4^m}.$$

is everywhere continuous and nowhere differentiable.

Proof ([**2**]). First we show that f is everywhere continuous. Since $\left|f_m(x)\right| \leq \frac{1}{4^m}$ for each $m \in \mathbb{N}$ and all $x \in \mathbb{R}$, by the Weierstrass M-Test (2.11), we have that $f(x) = \sum f_m(x)$ is uniformly convergent. Also since each partial sum is continuous, by Theorem 2.9, we conclude that f is continuous.

Next we show f is not differentiable at any point of $\mathbb{R}$. Fix an $x \in \mathbb{R}$. Since the intervals $\left[\frac{k}{2}, \frac{(k+1)}{2}\right]$ with $k \in \mathbb{Z}$ divide the entire real line, we have that for each $n \in \mathbb{N}$, there exists a k_0 such that the number $4^n x$ belongs to the interval $\left[\frac{k_0}{2}, \frac{(k_0+1)}{2}\right]$. Let $h_n := \pm\frac{1}{4^{n+1}}$, with the sign chosen so that both $4^n x$ and $4^n(x+h_n)$ lie in the same interval $\left[\frac{k_0}{2}, \frac{(k_0+1)}{2}\right]$. This is possible since the interval has length $\frac{1}{2}$ and $\left|4^n x - 4^n(x+h_n)\right| = \left|4^n\ h_n\right| = \left|4^n\ \frac{1}{4^{n+1}}\right| = \frac{1}{4}$. Since f_0 has slope ± 1 on the interval $\left[\frac{k_0}{2}, \frac{(k_0+1)}{2}\right]$, then

$$\begin{aligned}
\epsilon_n := \frac{f_n(x+h_n) - f_n(x)}{h_n} &= \frac{(\frac{1}{4^n})f_0\big(4^n(x+h_n)\big) - (\frac{1}{4^n})f_0(4^n x)}{h_n} \\
&= \frac{f_0\big(4^n(x+h_n)\big) - f_0(4^n x)}{4^n h_n} \\
&= \frac{f_0(4^n x + 4^n h_n) - f_0(4^n x)}{(4_n x + 4^n h_n) - 4^n x} \\
&= \pm 1.
\end{aligned}$$

If $m < n$, then on the interval between x and $x + h_n$, the graph of f_n lies completely under one half of the graph of f_m, so that f_m also has slope ± 1. So

$$\epsilon_m := \frac{f_m(x+h_n) - f_m(x)}{h_n} = \frac{f_m(x+h_n) - f_m(x)}{(x+h_n) - x} = \pm 1 \quad \text{for } m < n.$$

On the other hand, if $m > n$, then $4^m(x+h_n) - 4^m x = 4^m h_n = \pm 4^{m-n-1}$ is an integer. Since f_0 is a periodic function with an integer as its period,

$$f_m(x+h_n) - f_m(x) = (\tfrac{1}{4^m})[f_0(4^m(x+h_n)) - f_0(4^m x)] = 0.$$

Hence we have

$$\frac{f(x+h_n) - f(x)}{h_n} = \sum_{m=0}^{n} \frac{f_m(x+h_n) - f_m(x)}{h_n} + \sum_{m=n+1}^{\infty} \frac{f_m(x+h_n) - f_m(x)}{h_n} = \sum_{m=0}^{n} \epsilon_m,$$

where the difference quotient

$$\frac{f(x+h_n) - f(x)}{h_n}$$

is an odd integer if n is even, and an even integer if n is odd.

Therefore the limit

$$\lim_{h \to 0} \frac{f(x+h_n) - f(x)}{h_n}$$

does not exist, so f is not differentiable at the arbitrary point $x \in \mathbb{R}$. $\square$

CHAPTER 4

Monotonic Functions and their Derivatives

Our final pathological example is a monotonic function. In this chapter, we define the necessary terms and state some important theorems required for the construction of this special function.

1. Preliminaries

Definition 4.1 ([**7**]). A property holds **almost everywhere** if the set of elements for which the property does not hold is a set of measure zero. The term almost everywhere is abbreviated a.e.

Definition 4.2 ([**2**]). If $A \subseteq \mathbb{R}$, then a function $f : A \to \mathbb{R}$ is said to be **increasing on** A if whenever $x_1, x_2 \in A$ and $x_1 \leq x_2$, then $f(x_1) \leq f(x_2)$. We say that f is **strictly increasing on** A if we have $f(x_1) < f(x_2)$ whenever $x_1 < x_2$ for $x_1, x_2 \in A$. We can similarly define when a function is **decreasing** and **strictly decreasing on** A but we will be concerned with strictly increasing functions.

We say that f is **strictly monotone** on A if a function is either strictly increasing or strictly decreasing on A.

If a function is differentiable, we can determine the intervals on which f increases or decreases by examining the sign of the first derivative. A function having a positive derivative must be strictly increasing as stated by the following theorem. The second pathological property of interest concerns the converse of this theorem.

Theorem 4.3 ([**3**]). *If a function f is defined on an interval I and f has a positive derivative at every point on I, then f is strictly increasing.*

Proof. Let $x_1, x_2 \in I$ and $x_1 < x_2$. It follows from the Mean Value Theorem (Theorem 2.3) that there exists $c \in I$ such that

$$f(x_2) - f(x_1) = f'(c)(x_2 - x_1).$$

Since $f'(c) > 0$ and $x_2 - x_1 > 0$, we have that $f(x_2) - f(x_1) > 0$. Hence $f(x_2) > f(x_1)$ so we conclude that f is strictly increasing. $\square$

Again, as in Chapter 2, we can disprove the converse of this theorem using a simple example.

EXAMPLE 4.4. The function $f(x) = x^3$ is a strictly increasing function but $f'(x) = 0$ at $x = 0$.

Note that f is strictly increasing because if $x_1 < x_2$, then $f(x_1) = x_1^3 < x_2^3 = f(x_2)$. Also $f'(x) = x^2$ so $f'(0) = 0$, that is, $f'(0)$ is not positive.

From the graph, it can be seen that the tangent line at $x = 0$ has a slope of 0, that is, a **vanishing derivative** exists at $x = 0$.

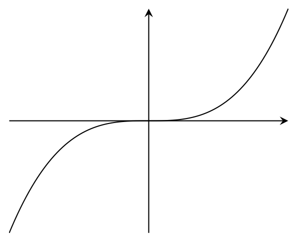

Figure 3 Graph of $f(x) = x^3$.

Again, we wish to extend this property of being strictly increasing but having a vanishing derivative to the entire domain. To construct a function of this type, we prepare by looking at a few necessary theorems.

2. Some notable theorems

Note that monotone functions are not necessarily continuous. For example, if $f(x) := 0$ for $x \in (-\infty, 0]$ and $f(x) := 1$ for $x \in (0, \infty)$, then f is increasing on $\mathbb{R}$ but is not continuous at $x = 0$. The next result shows that the only discontinuities that a strictly increasing function can have are jump discontinuities.

THEOREM 4.5. *If f is a strictly increasing function and discontinuous at c, then f must have a jump discontinuity at c.*

PROOF. Suppose that f is discontinuous at the point $x = c$. If f is increasing, then for any $y < c$ and $z > c$, we have that $f(y) \leq f(c) \leq f(z)$. Then for the one-sided limits of f at c, we have

$$\lim_{x \to c^-} f(x) \leq f(c) \leq \lim_{x \to c^+} f(x).$$

But f is discontinuous at c, so $\lim\limits_{x \to c} f(x) \neq f(c)$ which implies that

$$\lim_{x \to c^-} f(x) < \lim_{x \to c^+} f(x).$$

Thus f has a jump discontinuity at the point $x = c$. $\square$

Now we look at some fundamental theorems that will be of use later.

LEMMA 4.6. *Let A be a dense subset of $(0,1)$ and let $f : A \to \mathbb{R}$ be an increasing function. Then f can be extended to an increasing function on $(0,1)$.*

PROOF. Let $x_0 \in (0,1) \sim A$. Since A is dense in $(0,1)$, x_0 is a limit point of A. It follows from the fact that f is increasing on A that $\lim\limits_{x\to x_0} f(x)$ exists. Define $f(x_0) = \lim\limits_{x\to x_0} f(x)$. Then f is a function on $(0,1)$ and it is clear that f is increasing. □

Next we present a well-known theorem on the differentiability of monotone functions.

THEOREM 4.7 (Lebesgue's Theorem [**8**]). *Let $f : [a,b] \to \mathbb{R}$ be a monotone increasing function. Then $f'(x)$ exists for almost all $x \in [a,b]$ and*

$$\int_a^b f'(x)\,dx \le f(b) - f(a).$$

PROOF. The proof requires the use of results in measure theory that are beyond the scope of this paper. See Theorem 5.2 on page 96 of [**8**]. □

Using Lebesgue's Theorem, we obtain the Theorem of Fubini on the termwise differentiability of series with monotone terms, to be used in constructing our last pathological example.

THEOREM 4.8 (Fubini's Theorem [**9**]). *Let $f_k : [a,b] \to \mathbb{R}$ be a monotone increasing function for each $k = 1, 2, \ldots$ and assume that the series*

$$f(x) = \sum_{k=1}^{\infty} f_k(x)$$

converges pointwise on $[a,b]$. Then

$$f'(x) = \sum_{k=1}^{\infty} f_k'(x)$$

for almost all $x \in [a,b]$.

PROOF. First we split the infinite sum to a finite sum with a remainder by writing

$$f(x) = \sum_{k=1}^{n} f_k(x) + R_n(x)$$

where

$$R_n(x) = \sum_{k=n+1}^{\infty} f_k(x)$$

is the nth remainder for the nth partial sum $\sum_{k=1}^{n} f_k(x)$.

By Lebesgue's Theorem (4.7), since f and R_n are both monotone increasing functions, these functions are differentiable almost everywhere. Let E be the set of points where all the functions f_k and f are differentiable. Since each of these functions is differentiable almost everywhere, by definition, $[a, b] \sim E$ has measure zero. If $x \in E$, then the partial sum $\sum_{k=1}^{n} f_k(x)$ is differentiable at x and so we have the existence of

$$R_n'(x) = f'(x) - \sum_{k=1}^{n} f_k'(x).$$

Also, since $R_n'(x) \geq 0$, we have

$$f'(x) = \sum_{k=1}^{n} f_k'(x) + R_n'(x) \geq \sum_{k=1}^{n} f_k'(x)$$

for all $x \in E$.

Taking the limit as $n \to \infty$ gives $f'(x) \geq \sum_{k=1}^{\infty} f_k'(x)$ for all $x \in E$ and hence

$$\sum_{k=1}^{\infty} f_k'(x) \leq f'(x) \quad a.e. \tag{9}$$

We apply Theorem 4.7 to the monotone increasing functions R_n to get

$$0 \leq \int_a^b R_n'(x)\, dx \leq R_n(b) - R_n(a) = \sum_{k=n+1}^{\infty} \big(f_k(b) - f_k(a)\big). \tag{10}$$

We know that the series $\sum_{k=1}^{\infty} f_k(x)$ converges pointwise on $[a, b]$, so both the series $\sum_{k=1}^{\infty} f_k(a)$ and $\sum_{k=1}^{\infty} f_k(b)$ converge, resulting in the convergence of $\sum_{k=1}^{\infty}\big(f_k(b) - f_k(a)\big)$.

So

$$\begin{aligned}
\lim_{n\to\infty} \sum_{k=n+1}^{\infty} \big(f_k(b) - f_k(a)\big) &= \lim_{n\to\infty} \left(\sum_{k=1}^{\infty}\big(f_k(b) - f_k(a)\big) - \sum_{k=1}^{n}\big(f_k(b) - f_k(a)\big) \right) \\
&= \sum_{k=1}^{\infty}\big(f_k(b) - f_k(a)\big) - \sum_{k=1}^{\infty}\big(f_k(b) - f_k(a)\big) \\
&= 0.
\end{aligned}$$

Combining this result with (10) and applying the Squeeze Theorem gives

$$\lim_{n\to\infty} \int_a^b R_n'(x)\, dx = 0. \tag{11}$$

Also

$$\int_a^b f'(x)\,dx = \int_a^b \Big(\sum_{k=1}^n f_k(x)\Big)' dx \;+ \int_a^b R_n'(x)\,dx$$
$$= \int_a^b \sum_{k=1}^n f_k'(x)\,dx \;+ \int_a^b R_n'(x)\,dx$$
$$\leq \int_a^b \sum_{k=1}^\infty f_k'(x)\,dx \;+ \int_a^b R_n'(x)\,dx.$$

Now letting $n \to \infty$ and using (11) gives

$$\int_a^b f'(x)\,dx \leq \int_a^b \sum_{k=1}^\infty f_k'(x)\,dx. \tag{12}$$

But by (9), we have that $\sum_{k=1}^\infty f_k'(x) \leq f'(x)$ almost everywhere. Therefore the only way for (9) and (12) to hold simultaneously is if

$$f'(x) = \sum_{k=1}^\infty f_k'(x) \qquad a.e.$$

This completes the proof. □

With these important theorems, we are ready to explore the final pathological example.

CHAPTER 5

A Strictly Monotone Function with a Vanishing Derivative Almost Everywhere

In this chapter, we will see an example of a continuous strictly monotonic function with a derivative that vanishes almost everywhere. The function is constructed using the Cantor function, which we first define.

1. Construction of the Cantor function

Let A and B be nonempty subsets of $\mathbb{R}$. We say that A is smaller that B, denoted by $A < B$, if for each $x \in A$ and each $y \in B$, $x \leq y$. We use this ordering when constructing a sequence of open intervals.

Next we simultaenously construct the Cantor set and the Cantor function in a manner similar to the method given in Chapter 8 on page 96 of [**1**]. The Cantor function is increasing but its derivative is zero almost everywhere.

The Cantor set C is obtained from the closed unit interval $[0,1]$ by a sequence of deletions of open intervals known as "middle thirds" as follows.

Step 1: Remove the middle third open interval $\left(\frac{1}{3}, \frac{2}{3}\right)$ from $[0,1]$ to obtain the set

$$F_1 := \left[0, \tfrac{1}{3}\right] \cup \left[\tfrac{2}{3}, 1\right].$$

F_1 is the union of 2 increasing closed intervals of length $\frac{1}{3}$.
The sequence of all removed open interval is $\left\{\left(\frac{1}{3}, \frac{2}{3}\right)\right\}$.
The total lengths of the removed intervals is $\frac{1}{3}$.
Define $\phi(x) = \frac{1}{2}$ for all $x \in \left[\frac{1}{3}, \frac{2}{3}\right]$.

Step 2: Remove the middle third open interval from each of the closed interval in F_1 to obtain the set

$$F_2 := \left[0, \tfrac{1}{9}\right] \cup \left[\tfrac{2}{9}, \tfrac{1}{3}\right] \cup \left[\tfrac{2}{3}, \tfrac{7}{9}\right] \cup \left[\tfrac{8}{9}, 1\right].$$

F_2 is the union of 2^2 increasing closed intervals of length $\frac{1}{3^2}$.
The sequence of all removed open interval is

$$\left\{ \left(\tfrac{1}{9}, \tfrac{2}{9}\right), \left(\tfrac{1}{3}, \tfrac{2}{3}\right), \left(\tfrac{7}{9}, \tfrac{8}{9}\right) \right\}.$$

The total lengths of the removed intervals is $\frac{1}{3} + \frac{2}{3^2}$.

We define

$$\phi(x) = \tfrac{1}{2^2} \text{ for } x \in \left[\tfrac{1}{9}, \tfrac{2}{9}\right], \;\; \phi(x) = \tfrac{3}{2^2} \text{ for } x \in \left[\tfrac{7}{9}, \tfrac{8}{9}\right]$$

and $\phi(x) = \frac{2}{2^2}$ for $x \in \left[\frac{1}{3}, \frac{2}{3}\right]$ as in Step 1.

Step 3: Remove the middle third open interval from each of the closed interval in F_2 to obtain the set F_3 which is the union of 2^3 increasing closed intervals of length $\frac{1}{3^3}$. The open intervals removed from F_2 are $\left(\frac{1}{27}, \frac{2}{27}\right), \left(\frac{7}{27}, \frac{8}{27}\right), \left(\frac{19}{27}, \frac{20}{27}\right), \left(\frac{25}{27}, \frac{26}{27}\right)$. They are in an increasing order and each has length $\frac{1}{3^3}$.

The sequence of all removed open intervals is

$$\left\{ \left(\tfrac{1}{27}, \tfrac{2}{27}\right), \left(\tfrac{1}{9}, \tfrac{2}{9}\right), \left(\tfrac{7}{27}, \tfrac{8}{27}\right), \left(\tfrac{1}{3}, \tfrac{2}{3}\right), \left(\tfrac{19}{27}, \tfrac{20}{27}\right), \left(\tfrac{7}{9}, \tfrac{8}{9}\right), \left(\tfrac{25}{27}, \tfrac{26}{27}\right) \right\}.$$

The total lengths of the removed intervals is $\frac{1}{3} + \frac{2}{3^2} + \frac{2^2}{3^3}$.

We define

$$\begin{aligned}
&\phi(x) = \tfrac{1}{2^3} \text{ for } x \in \left[\tfrac{1}{27}, \tfrac{2}{27}\right], \;\; \phi(x) = \tfrac{3}{2^3} \text{ for } x \in \left[\tfrac{7}{27}, \tfrac{8}{27}\right], \\
&\phi(x) = \tfrac{5}{2^3} \text{ for } x \in \left[\tfrac{19}{27}, \tfrac{20}{27}\right], \;\; \phi(x) = \tfrac{7}{2^3} \text{ for } x \in \left[\tfrac{25}{27}, \tfrac{26}{27}\right]
\end{aligned}$$

and we have, as in Step 1 and Step 2,

$$\begin{aligned}
&\phi(x) = \tfrac{2}{2^3} = \tfrac{1}{2^2} \text{ for } x \in [\tfrac{1}{9}, \tfrac{2}{9}], \;\; \phi(x) = \tfrac{4}{2^3} = \tfrac{1}{2} \text{ for } x \in [\tfrac{1}{3}, \tfrac{2}{3}], \\
&\phi(x) = \tfrac{6}{2^3} = \tfrac{3}{2^2} \text{ for } x \in [\tfrac{7}{9}, \tfrac{8}{9}].
\end{aligned}$$

Step n: We continue in this way. In general, when we finish Step n, we have that F_n is a union of 2^n increasing closed intervals of length $\frac{1}{3^n}$.

The sequence of all the removed open intervals from F_{n-1} is

$$\{O_1, O_2, \ldots, O_{k_n}\},$$

where the sequence is in increasing order and $k_n = 1 + 2 + 2^2 + 2^3 + \cdots + 2^{n-1} = 2^n - 1$,

The total lengths of the removed intervals is $\frac{1}{3} + \frac{2}{3^2} + \frac{2^2}{3^3} + \cdots + \frac{2^{n-1}}{3^n}$ and we define $\phi(x) = \frac{k}{2^n}$ when $x \in O_k$ for $k = 1, 2, \ldots, 2^n - 1$.

Step $n+1$: We remove the middle third open interval O_{ni} from each sub-interval in F_n, where $i = 1, 2, \ldots, 2^n$. Then each O_{ni} has length $\frac{1}{3^{n+1}}$. We assume that $O_{n1}, O_{n2}, \ldots, O_{n2^n}$ is in increasing order.

The total lengths of the removed intervals is $\frac{1}{3} + \frac{2}{3^2} + \frac{2^2}{3^3} + \cdots + \frac{2^{n-1}}{3^n} + \frac{2^n}{3^{n+1}}$.

The sequence of all removed open interval in increasing order is

$$\{O_{n,1}, O_1, O_{n,2}, O_2, \ldots, O_{k_n}, O_{n,2^n}\}.$$

We rename them by

$$\{ \tilde{O}_1, \tilde{O}_2, \ldots, \tilde{O}_{k_{n+1}} \},$$

where $k_{n+1} = 2^{n+1} - 1$.

For $k = 1, 3, 5, \ldots, 2^{n+1} - 1$, we define $\phi(x) = \frac{k}{2^{n+1}}$ when x is in the closure of $\tilde{O}_k$. For $k = 2, 4, 6, \ldots, 2^{n+1} - 2$, as in Step n, we have $\phi(x) = \frac{\frac{k}{2}}{2^n} = \frac{k}{2^{n+1}}$ when x is in the closure of $\tilde{O}_k$ since $\tilde{O}_k = O_{\frac{k}{2}}$ as above.

Let O be the union of all the open intervals removed. Then ϕ is an increasing function defined on O.

Since all the F_n are closed subsets and $F_{n+1} \subseteq F_n$ for each n, $\bigcap_{n=1}^{\infty} F_n$ is nonempty. The **Cantor set** is defined as $C = \bigcap_{n=1}^{\infty} F_n$.

The total length of the removed intervals is

$$\frac{1}{3} + \frac{2}{3^2} + \frac{2^2}{3^3} + \cdots + \frac{2^{n-1}}{3^n} + \frac{2^n}{3^{n+1}} + \ldots \;=\; \frac{1}{3}\,\frac{1}{1-\frac{2}{3}} \;=\; 1.$$

Then ϕ is an increasing function defined on O. Since the total length of O is 1, O is dense in $[0, 1]$. Using Lemma 4.6, we can extend ϕ to an increasing function on $[0, 1]$. Because ϕ is increasing and its range is dense in $[0, 1]$, the function ϕ has no jump discontinuities. By Theorem 4.5, a monotonic function can have no discontinuities other than jump discontinuities, so ϕ is continuous.

Since the Cantor set C has measure zero and ϕ is locally constant on the open subset $[0, 1] \sim C$, we have that $\phi'(x) = 0$ almost everywhere in $[0, 1]$. The function ϕ is called the **Cantor function**.

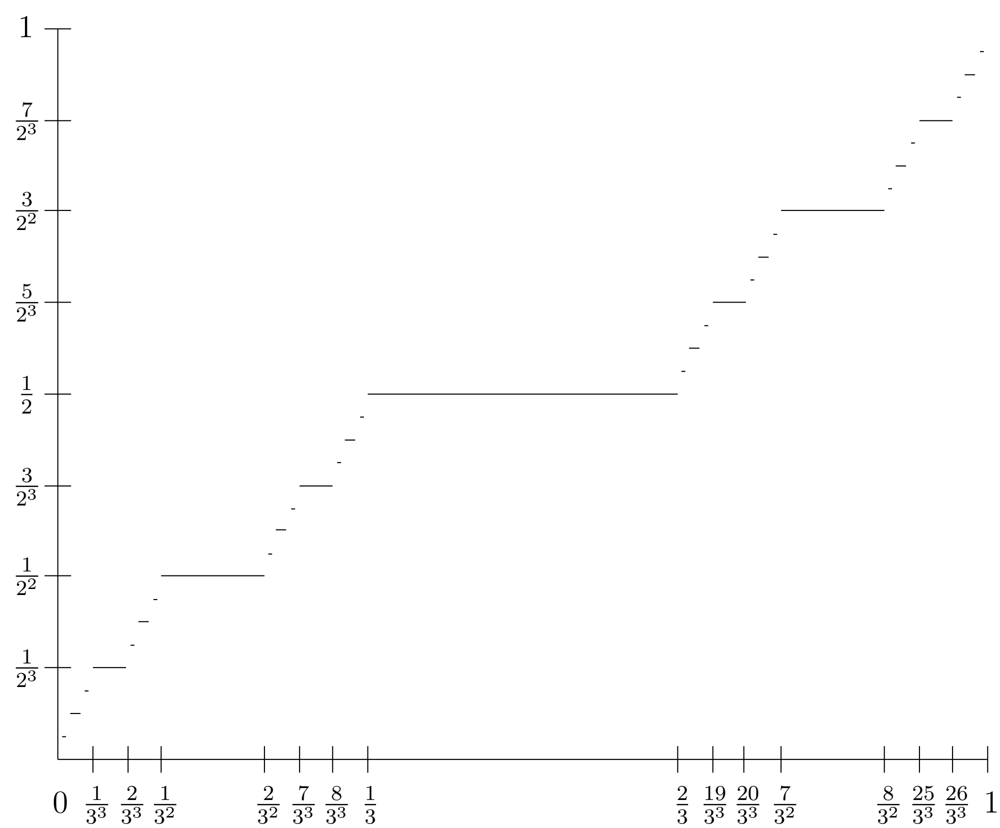

Figure 4 Graph of the Cantor function.

2. Example given by Zaanen and Luxemburg (1963)

The Cantor function is an increasing function with a derivative equal to zero almost everywhere. Our goal is to use the Cantor function to construct our final example of a *strictly* increasing function with a vanishing derivative almost everywhere.

THEOREM 5.1 ([**1**]). *If* ϕ *is the Cantor function on* $[0,1]$*, let*

$$\psi(x) = \begin{cases} \phi(x) & \text{if } x \in [0,1], \\ 0 & \text{if } x < 0, \\ 1 & \text{if } x > 1. \end{cases}$$

If $\{[a_n, b_n]\}$ *is the sequence of closed intervals* $[0,1], [0,\frac{1}{2}], [\frac{1}{2},1], [0,\frac{1}{4}], [\frac{1}{4},\frac{1}{2}], [\frac{1}{2},\frac{3}{4}], [\frac{3}{4},1],$ $[0,\frac{1}{8}], \ldots,$ *then*

$$f(x) = \sum_{n=1}^{\infty} \frac{1}{2^n}\, \psi\Big(\frac{x-a_n}{b_n-a_n}\Big)$$

is a continuous strictly increasing function with a vanishing derivative almost everywhere.

PROOF ([**10**]). First to prove continuity, we see that for each n, the function

$$\psi\Big(\frac{x-a_n}{b_n-a_n}\Big)$$

is a Cantor function on $[a_n, b_n]$. So for each n,

$$f_n(x) = \frac{1}{2^n}\, \psi\Big(\frac{x-a_n}{b_n-a_n}\Big)$$

is continuous and has a vanishing derivative almost everywhere, that is,

$$f_n'(x) = \left[\frac{1}{2^n}\, \psi\Big(\frac{x-a_n}{b_n-a_n}\Big)\right]' = 0 \quad a.e. \tag{13}$$

Also for each n, we have that

$$\big|f_n(x)\big| = \left|\ \frac{1}{2^n}\, \psi\Big(\frac{x-a_n}{b_n-a_n}\Big)\ \right| \le \left|\frac{1}{2^n}\right|.$$

So by the Weierstrass M-Test (2.11), the convergence of $\sum \frac{1}{2^n}$ implies that

$$f(x) = \sum_{n=1}^{\infty} \frac{1}{2^n}\, \psi\Big(\frac{x-a_n}{b_n-a_n}\Big)$$

converges uniformly on $\mathbb{R}$. Since each partial sum is continuous and the convergence of the series to f is uniform, by Theorem 2.9, we have that f is continuous.

To show that f is strictly increasing, let $x_1 < x_2$ with $x_1, x_2 \in [0,1]$. Then there exists n_0 such that $x_1 < a_{n_0}$ and $x_2 > b_{n_0}$.

Since

$$\frac{x_1 - a_{n_0}}{b_{n_0} - a_{n_0}} < \frac{a_{n_0} - a_{n_0}}{b_{n_0} - a_{n_0}} = 0 \quad \text{and} \quad \frac{x_2 - a_{n_0}}{b_{n_0} - a_{n_0}} > \frac{b_{n_0} - a_{n_0}}{b_{n_0} - a_{n_0}} = 1,$$

we have

$$\psi\Big(\frac{x_1 - a_{n_0}}{b_{n_0} - a_{n_0}}\Big) = 0 \quad \text{and} \quad \psi\Big(\frac{x_2 - a_{n_0}}{b_{n_0} - a_{n_0}}\Big) = 1. \tag{14}$$

Now for each n, since we have the inequality

$$\psi\Big(\frac{x_1 - a_n}{b_n - a_n}\Big) \leq \psi\Big(\frac{x_2 - a_n}{b_n - a_n}\Big)$$

and from (14), we obtain the strict inequality

$$\psi\Big(\frac{x_1 - a_{n_0}}{b_{n_0} - a_{n_0}}\Big) < \psi\Big(\frac{x_2 - a_{n_0}}{b_{n_0} - a_{n_0}}\Big),$$

so by taking the infinite series, we get the strict inequality

$$\sum_{n=1}^{\infty} \frac{1}{2^n}\, \psi\Big(\frac{x_1 - a_n}{b_n - a_n}\Big) < \sum_{n=1}^{\infty} \frac{1}{2^n}\, \psi\Big(\frac{x_2 - a_n}{b_n - a_n}\Big).$$

Hence $f(x_1) < f(x_2)$, so f is strictly increasing.

Finally to show that f has a vanishing derivative almost everywhere, we use Fubini's Theorem (4.8) to get

$$\begin{aligned} f'(x) &= \sum_{n=1}^{\infty} f_n'(x) \\ &= \sum_{n=1}^{\infty} \left[\frac{1}{2^n}\, \psi\Big(\frac{x - a_n}{b_n - a_n}\Big)\right]' \\ &\overset{(13)}{=} \sum_{n=1}^{\infty} 0 \\ &= 0 \quad a.e. \end{aligned}$$

Therefore,

$$f(x) = \sum_{n=1}^{\infty} \frac{1}{2^n}\, \psi\Big(\frac{x - a_n}{b_n - a_n}\Big)$$

is a strictly increasing function with a vanishing derivative almost everywhere. $\square$

CHAPTER 6

Concluding Remarks

In this paper, we examined two types of pathological continuous functions, namely, two functions that are continuous yet nowhere differentiable, and a continuous strictly monotonic function that has derivative zero almost everywhere.

In all three examples, infinite series of sequences of continuous functions were constructed. To show that each function is continuous, the same method was used: the Weierstrass M-test showed uniform convergence, which was then used with Theorem 2.9 to prove continuity.

The first example was the Weierstrass function (3.1),

$$f(x) = \sum_{n=0}^{\infty} b^n \cos(a^n \pi x),$$

which has necessary restrictions on a, b, and ab. Its construction involved using special floor functions and proving its non-differentiability was shown to be quite technical.

A more geometrically intuitive example was the Van der Waerden function (3.4),

$$f(x) = \sum_{n=1}^{\infty} f_m(x) = \sum_{m=1}^{\infty} \frac{f_0(4^m x)}{4^m},$$

where f_m is a sequence of continuous sawtooth functions with rapidly alternating positive and negative slopes which suggests its non-differentiability.

The final example, first given by Zaanen and Luxemburg (5.1) involved revising the Cantor function, which is increasing yet has a vanishing derivative almost everywhere. We extended the Cantor function to the entire real line and modified it to be strictly increasing.

The original discovery of pathological continuous real-valued functions challenged traditional notions and emphasized the need for analytical rigour in mathematical analysis. Understanding these special functions may lead to new theories, more general results, and real-world applications, but ultimately, pathological functions are simply fascinating to study.

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