ESTIMATES FOR DOMAINS OF LOCAL INVERTIBILITY OF DIFFEOMORPHISMS

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ABSTRACT. Using a novel Wintner-type formulation of the classical Peano's existence theorem [*Math. Ann.* **37** (1890), 182-228], we enhance Ważewski's result on invertibility of maps defined on closed balls [*Ann. Soc. Pol. Math.* **20** (1947), 81-125] securing the size of the domain of invertibility that agrees with the bounds derived by John [*Comm. Pure Appl. Math.* **21** (1968), 77-110] and Sotomayor [*Z. Angew. Math. Phys.* **41** (1990), 306-310].

1. INTRODUCTION

In 1906, J. Hadamard [7] established a global version of the inverse function theorem and provided sufficient conditions for a given map to be a diffeomorphism (a differentiable map between manifolds which is one-to-one, onto and whose inverse is differentiable); see also Gordon [6].

Theorem 1. *Let f be a map from $\mathbb{R}^n$ onto $\mathbb{R}^n$ and the derivative $f'(x)$ be invertible for every $x \in \mathbb{R}^n$. If*

$$(1.1) \qquad \int_0^{+\infty} \inf_{\|y\| \le u} \frac{1}{\|f'(y)^{-1}\|} du = +\infty,$$

then f is a diffeomorphism of $\mathbb{R}^n$ to $\mathbb{R}^n$.

Since then, the problem of finding the inverse map, estimating the domain of local invertibility for a smooth function, as well as obtaining conditions for global invertibility of smooth and even nonsmooth maps both in finite-dimensional and infinite-dimensional cases attracted significant interest of the researchers. Elevated interest to global invertibility results reflects their importance for electrical network theory, theory of economic equilibria, stability of nonlinear differential equations, nonlinear elasticity, partial differential equations, etc. A variant of Theorem 1 was discovered by Hadamard [7], extended by Lévy [12], and generalized later by

Plastock [20] to include, in addition to the version of Theorem 1 referring to diffeo-morphism between Banach spaces, the Banach-Mazur theorem [1] and Browder's theorem [2]. Furthermore, Plastock noticed that for finite-dimensional spaces, con-dition (1.1) can be replaced by

$$\int_0^{+\infty} \inf_{\|y\|=u} \frac{1}{\|f'(y)^{-1}\|} du = +\infty.$$

For other related results we refer the reader to the papers by John [10], Li et al. [13], Porciau [21], Sotomayor [23], Zampieri [27].

A direct approach to the problem of global invertibility requires a delicate anal-ysis of topological and integrability properties. On the other hand, employment of differential equations for establishing invertibility, and vice versa, proved to be very beneficial; see De Marco et al. [14]. In fact, Sotomayor [23], Ważewski [25], and Zampieri [27] successfully exploited differential equations for obtaining the es-timates for the domain of invertibility of a given map. Conversely, Gasull et al. [5], Olech [17], and Zampieri and Gorni [28] utilized invertibility to prove global stability.

Our research has been stimulated by an ingenious idea of T. Ważewski [25] to examine the properties of solutions of the differential equation

$$(1.2) \qquad\qquad x' = \left(f'(x)^{-1}\right) v,$$

where f is a continuously differentiable map from $\mathbb{R}^n$ into itself and $v \in \mathbb{R}^n$ for ob-taining an estimate for the domain of invertibility. Equation (1.2), known nowadays as the *Ważewski equation*, is well defined in any domain where $f'(x)$ is nonsingular, and its solution $x(t)$ is the function whose image is linear,

$$f(x(t)) = f(x(0)) + vt.$$

As emphasized by Olech [18], (1.2) helps to attack the Markus-Yamabe Fundamen-tal Problem on Global Asymptotic Stability of the autonomous system in the plane [15] which was solved by several authors using lengthy and complicated arguments.

In 1990, studying maximal star-shaped domain $S(0)$ of right-invertibility for the mapping f on $\mathbb{R}^n$, Sotomayor [23] improved Ważewski's theorem, obtaining results closely related to John's work [10] where maximal ball instead of maximal star-shaped domain was used. In particular, employing a modified Ważewski equation,

$$(1.3) \qquad\qquad x' = \delta(x) \left(f'(x)^{-1}\right) v,$$

where δ is a smooth function such that, given a compact set $M \subset \mathbb{R}^n$ with $0 \in M$ and $f'(x) \neq 0$ in $\overline{M}$, one has $\delta(x) > 0$ if $x \in M$, $\delta(x) = 0$ if $x \in \partial M$, and $\delta(x) < 0$ otherwise, Sotomayor obtained integral representation for the radial function $\rho(v)$ naturally associated with the star-shaped domain of invertibility. Namely, if $x(t, v)$ is a solution of (1.3) that satisfies the initial condition $x(0, v) = 0$, then

$$(1.4) \qquad\qquad \rho(v) = \int_0^{+\infty} \delta(x(t, v)) dt, \qquad \|v\| = 1, \ v \in \mathbb{R}^n.$$

Stability properties of the radial function $\rho(v)$ with respect to smooth deformations, called $S(0)$-stability of f, were studied by Sotomayor [24, Section 2] using the techniques from the singularity theory and a modified Ważewski equation,

$$(1.5) \qquad\qquad x' = \left(f'(x)^{-1}\right) [f(x) - f(x_0)].$$

Replacing M with a closed ball D of the radius $b > 0$ centered at 0 and assuming that

$$\theta(r) = \sup_{\|x\| \le r} \left(\limsup_{u \to 0} \frac{\|u - f'(x)u\|}{\|u\|} \right) < 1, \qquad 0 \le r \le b,$$

in D, one obtains an interesting estimate, discovered by Nevanlinna [16], for the radius of a closed ball on which the function f is invertible, namely

$$\rho(v) = b - \int_0^b \theta(r) dr.$$

In this paper, we use the technique developed by Hartman [9, pp. 11-12], in particular, an illuminating hint to the proof of Ważewski's theorem [9, p. 557], and a Wintner-type existence theorem to enhance Ważewski's result [25] magnifying the constants bM^{-1} and $b(MM_1)^{-1}$ up to the values provided by the bounds for the domain of invertibility derived by John [10] and Sotomayor [23].

2. Peano's existence theorem revisited

Peano's existence theorem [19] is one of the key results in the theory of differential equations which, owing to the weakness of assumptions and transparency of the idea underlying its proof, is included in many modern texts on differential equations. An important problem related to Peano's existence theorem considers the possibility for improving the estimate for the interval of existence of a solution; see the fundamental paper by Wintner [26].

In 1988, Lee and O'Regan [11] studied the initial value problem

$$(2.1) \qquad \begin{cases} x' = f(t, x), & t \ge 0, \ x \in \mathbb{R}^n, \\ x(0) = x_0, \end{cases}$$

under the assumption that $f : [0, +\infty) \times \mathbb{R}^n \to \mathbb{R}^n$ is a continuous function satisfying $\|f(t, x)\| \le \psi(\|x\|)$, where $\psi : \mathbb{R}_+ \to (0, +\infty)$ is a certain continuous function. They have established the existence of a solution $x(t)$ of the problem (2.1) defined on the interval $[0, T]$, where T is given by

$$(2.2) \qquad T < T_\infty = \int_{\|x_0\|}^{+\infty} \frac{du}{\psi(u)} \le +\infty.$$

Our approach to the problem slightly differs from that of Lee and O'Regan and relies on the Leray-Schauder alternative [4, Theorem 5.3, pp. 61-62], also known as Schaefer's theorem [3, Theorem 4.4, pp. 133-134]. We recall that the operator $T : E \to \mathcal{N}$ defined on a subset $E \subset \mathcal{N}$ is completely continuous (compact) if T is continuous and for a bounded subset $M \subset E$ the set $T(M)$ is relatively compact; see [3, p. 131].

Theorem 2. *Let $T : \mathcal{N} \to \mathcal{N}$ be a completely continuous operator acting on a normed linear space $\mathcal{N}$ and $\lambda_0 \in [0, 1]$. Then either there is an $x \in \mathcal{N}$ such that*

$$x = \lambda_0 T(x)$$

or, for any $\lambda \in (0, 1)$, the set

$$E(T) = \{x \in X : x = \lambda T(x)\}$$

is not bounded.

Let $a, b > 0$, $x_0 \in \mathbb{R}^n$, and $f : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ be a continuous function. Consider the initial value problem

$$(2.3) \qquad\qquad x' = f(t, x), \qquad x(t_0) = x_0$$

on the "box" B defined by

$$B = \{|t - t_0| \le a, \qquad \|x - x_0\| \le b\}.$$

Introduce a majorizing function $\psi : [0, b] \to [0, +\infty)$ by the formula

$$(2.4) \qquad \psi(r) = \sup\{\|f(t, x)\| : |t - t_0| \le a, \ \|x - x_0\| \le r\}, \qquad 0 \le r \le b.$$

Note that $\psi(0) = 0$ if and only if (2.3) has the constant solution $x(t) = x_0$ on the interval $[t_0 - a, t_0 + a]$. Assuming now that $\psi(0) > 0$, we introduce the function $G : [0, b] \to [0, +\infty)$ by the formula

$$G(v) = \int_0^v \frac{du}{\psi(u)}, \qquad 0 \le v \le b.$$

Theorem 3 (Peano's existence theorem)**.** *Consider the initial value problem* (2.3) *on the box B. Then either*

(i) $\psi(0) = 0$ *and* $x(t) = x_0$ *is a solution of the problem* (2.3) *defined on the interval* $[t_0 - a, t_0 + a]$*, or*

(ii) $\psi(0) > 0$*, and there exists a nonconstant solution of the problem* (2.3) *defined on the interval* $[t_0 - \tau, t_0 + \tau]$*, where* $\tau = \min [a, G(b)]$*.*

Proof. Without loss of generality, assume that $x_0 = 0$ and $\psi(0) > 0$. Let $\mathcal{N} = C([t_0 - \tau, t_0 + \tau], \mathbb{R}^n)$ be a Banach space of continuous functions endowed with the standard sup-norm and let T be defined by

$$(2.5) \qquad T(x)(t) = \int_{t_0}^t f(s, x(s))ds, \qquad x \in \mathcal{N}, \ t \in [t_0 - \tau, t_0 + \tau].$$

Following Hale [8, Theorem 1.1, p. 15], one can deduce that the operator T is completely continuous. We claim that *the set $E(T)$ is bounded*. Indeed, for $u \in E(T)$, it follows from the equation

$$u(t) = \lambda \int_{t_0}^t f(s, u(s))ds, \qquad |t - t_0| \le \tau,$$

that

$$(2.6) \qquad\qquad \|u(t)\| \le \int_{t_0}^t \psi(\|u(s)\|)ds,$$

for all $t \in [t_0, t_0 + \tau]$ and for any $\lambda \in (0, 1)$. Denote the right-hand member of (2.6) by $z(t)$. Obviously,

$$z'(t) \le \psi(z(t)).$$

Integrating the latter inequality over $[t_0, t]$, we conclude that

$$(2.7) \qquad\qquad \int_0^{z(t)} \frac{du}{\psi(u)} \le t - t_0 \le \tau,$$

for all t in $[t_0, t_0 + \tau]$. Note that (2.7) also holds for all $t \in [t_0 - \tau, t_0]$ if one chooses

$$z(t) = \int_t^{t_0} \psi(\|u(s)\|)ds.$$

Taking into account that (2.7) can be written as

$$(2.8) \qquad \|u(t)\| \leq z(t) \leq G^{-1}(\tau) \leq b, \qquad |t - t_0| \leq \tau,$$

the conclusion follows by application of the Leray-Schauder alternative. $\qquad \square$

3. WAŻEWSKI'S RESULT AND BOUNDS FOR THE DOMAIN OF INVERTIBILITY

Noticing that

$$(3.1) \qquad \inf_{\|y\| \leq u} \frac{1}{\|f'(y)^{-1}\|} = \frac{1}{\sup_{\|y\| \leq u} \|f'(y)^{-1}\|},$$

we write the Hadamard-Lévy-Plastock condition (1.1) in the equivalent form

$$\int_0^{+\infty} \frac{du}{\sup_{\|y\| \leq u} \|f'(y)^{-1}\|} = +\infty.$$

This representation helps to reveal the link between invertibility of smooth functions and a Wintner-type result (Theorem 3) established by using the comparison function ψ defined by (2.4).

Theorem 4. *Let $n \geq 1$ and let $G \subseteq \mathbb{R}^n$ be an open set containing the closed ball D of radius $b > 0$ centered at 0. Suppose further that $f : G \to \mathbb{R}^n$ is a continuously differentiable function such that $f(0) = 0$ and $\det[f'(y)] \neq 0$, for all $y \in D$. Consider the sets*

$$D_0 = \{x \in \mathbb{R}^n : \|x\| \leq \Omega(b)\} \qquad D_2 = \left\{y \in \mathbb{R}^n : \|y\| \leq M_1^{-1}\Omega(b)\right\},$$

where

$$(3.2) \qquad \Omega(b) = \int_0^b \inf_{\|y\| \leq u} \frac{1}{\|f'(y)^{-1}\|} du$$

and

$$M_1 = \max_{y \in D} \|f'(y)\|.$$

Then there exists a set D_1, $D_2 \subseteq D_1 \subseteq D$, such that $f : D_1 \to D_0$ is a diffeomorphism.

Proof. Consider the initial value problem

$$(3.3) \qquad \begin{cases} \frac{dy}{dt} = \left(f'(y)^{-1}\right)\xi, \\ \|y\| \leq b, \qquad |t| \leq 1, \\ y(0) = 0, \end{cases}$$

where $\xi \in \mathbb{R}^n \setminus \{0\}$ is a fixed vector such that $\|\xi\| \leq \Omega(b)$. Introduce the function $\psi : [0, b] \to [0, +\infty)$ by

$$\psi(r) = \|\xi\| \sup\{\|f'(y)^{-1}\| : \|y\| \leq r\}.$$

It follows from (3.1) that

$$G(b) = \|\xi\|^{-1} \int_0^b \frac{du}{\sup_{\|y\| \leq u} \|f'(y)^{-1}\|} \geq 1,$$

and Theorem 3 yields existence of a solution $y(t; \xi)$, not necessarily unique, of the problem (3.3), which is defined on $[-1, 1]$. Writing (3.3) in the form

$$\begin{cases} f'(y(t; \xi))\frac{dy(t; \xi)}{dt} = \xi, & |t| \leq 1, \\ y(0) = 0, \end{cases}$$

we conclude, upon integration, that

(3.4)
$$f(y(t; \xi)) = t\xi, \qquad |t| \leq 1.$$

Now the set D_0 can be redefined as

$$D_0 = \{t\xi : \ (t, \xi) \in \mathbb{R} \times \mathbb{R}^n, \ |t| \leq 1, \ \|\xi\| \leq \Omega(b)\}.$$

Clearly, $D_0 \subseteq f(D)$. Application of the inverse function theorem (see, for instance, [4, Theorem 4.1, pp. 28-30] or [22, Theorem 9.24, pp. 221-223]) yields the existence of the unique solution $y(t; \xi)$ of (3.4). Letting $g(\xi) = y(1; \xi)$, we conclude that the function $g : D_0 \to D$ is the inverse of f and g is continuously differentiable.

Consider now the initial value problem

(3.5)
$$\begin{cases} \frac{dx}{ds} = \left(g'(x)^{-1}\right)\eta, \\ \|x\| \leq \Omega(b), & |s| \leq 1, \\ x(0) = 0, \end{cases}$$

where $\eta \in \mathbb{R}^n \setminus \{0\}$ is a fixed vector satisfying $\|\eta\| \leq M_1^{-1}\Omega(b)$. Since, for all $z \in D_0$, one has $f'(g(z))g'(z) = I_n$, where I_n stands for an $n \times n$ identity matrix, we can choose $\psi(r) = \|\eta\| M_1$. Taking into account that

$$G(\Omega(b)) = \|\eta\|^{-1} \int_0^{\Omega(b)} \frac{du}{M_1} \geq 1,$$

we deduce from Theorem 3 and the inverse function theorem that there exists a unique solution $x(s; \eta)$ of the problem (3.5) defined on the interval $[-1, 1]$. Finally, it follows from the relation

$$g(x(s; \eta)) = s\eta, \qquad |s| \leq 1, \ \|\eta\| \leq M_1^{-1}\Omega(b),$$

which is equivalent to

$$f(s\eta) = x(s; \eta),$$

that $D_2 \subseteq g(D_0)$. The proof is complete. $\qquad\square$

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