Material of Functional Analysis classes in April 2024

5 maja 2024

(applications of Separation Theorem to approximation:) For 3 types of hulls: convex, absolutely convex and "span" - criteria, when a point belongs to closures of the corresponding hull of a set (in terms of functionals).

A proof of the Riesz Representation Theorem for a bounded linear functional on $C[a, b]$ based on its norm-preserving extension to the space of bounded functions. (The representation is in terms of Stieltjes integral. proof was only briefly outlined)

Definition and properties of L^p -spaces:

Inequalities of Hölder and Minkowski. Proof of the completeness of $L^p(X, \mathcal{B}, \mu)$.
(We have an important corollary of that proof: *convergent(in* $\|\cdot\|_p$ *) sequences contain subsequences convergent almost everywhere* $[\mu]$).

A condition equivalent to separability of $L^p(\mu)$ for $p < \infty$.

Regularity of Borel σ-finite measures on metric spaces, density of continuous functions.

Inner products, Hilbert spaces:

Sesquilinear forms, hermitian forms, Polarisation Formula, Parallelogram Law, Schwarz's Inequality for non-negative forms. Norm defined by an inner product.

Orthogonal projections onto closed, convex sets in Hilbert spaces H. Properties of orthoprojections. Equivalent conditions on a linear map $P : H \to H$ such that $P \circ P = P$ in order to be an ortho-projection. Orthogonal decomposition w.r. to a subspace. Example of projections in $L^2(\mu)$: -multiplication operators by characteristic functions ("indicator functions") of a measurable set.

Riesz-Fréchet Theorem representing bounded linear functionals on H in terms of the inner product.

Definition of the adjoint T^* of a linear operator (-for a bounded one and for a possibly unbounded, but densely-defined operator). Definition of symmetric and self-adjoint operators. Closed operators (T^* is always closed for any densely defined operator T).

Orthogonal and orthonormal sequences in Hilbert spaces. Complete orthonormal sequences (= Hilbertian bases).

Fourier coefficients w.r.to an orthonormal sequence. Bessel Inequality. Convergence of orthogonal series, conditions equivalent to completeness of an orthonormal sequence $(e_n)_{n\in\mathbb{N}}$:
$[\forall_{x\in H} x = \sum_{n=1}^{\infty} \langle x, e_n\rangle e_n] \Leftrightarrow$ [Parseval Equality holds $\forall_{x\in H}$] $\Leftrightarrow [span\{e_n : n \in \mathbb{N}\} = H]$.

Application of orthoprojections: Gramm-Schmidt orthogonalisation process. Gramm determinant of a set of n vectors. The existence of Hilbertian bases in all Hilbert spaces.

In May classes I will start with showing that all separable Hilbert spaces are (isometrically) isomorphic (+ Riesz-Fischer Theorem). This will end the basic information on spaces. Next we will study bounded linear operators in more detail. Beginning with basic consequences of Baire Category Theorem: *Banach-Stinhaus and Uniform Boundedness theorems*, 3 fundamental Banach's theorems *(Open Mapping, Inverse Mapping and Closed Graph theorems)*. Then we restrict our attention to bounded operators on Hilbert spaces, especially we shall focus on spectrla theory of normal and self-adjoint operators. Our course will end with *Spectral Theorem for self-adjoint operators.*

`The list of topics for oral exam will contain no more than 23 items` (its Polish version is in the file zakraf2023.pdf on my web-page) We may discuss- which parts of it will concern you, the Erasmus students. Feel invited for my official consultations (Tuesdays, 16:35-17:25 B7 B-7 room 2.10, but if these hours/or day are not suitable for you, we may arrange some other dates of meetings). It is important not to postpone them for the "last minute".