L^p Spaces for $0 < p < 1$

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1 L^p Spaces for $0 < p < 1$

1.1 Complete Quasi-Normed Space

Lemma 1. *If $p \in (0,1)$ and $a, b \geq 0$, then*

$$(a+b)^p \leq a^p + b^p$$

with equality if and only if either a or b is zero.

Proof. Define a function $f(t) := (1+t)^p - 1 - t^p$ for $t \geq 0$. Then $f'(t) = p(1+t)^{p-1} - pt^{p-1} < 0$ for all $t \in (0,\infty)$. Since $f(0) = 0$, it follows that $f(t) < 0$ on $(0, \infty$. If $a, b \neq 0$, then substituting $t = \frac{a}{b}$,

$$\left(1 + \frac{a}{b}\right)^p - 1 - \left(\frac{a}{b}\right)^p < 0 \Longleftrightarrow \left(\frac{a+b}{b}\right)^p - 1 - \left(\frac{a}{b}\right)^p < 0 \Longleftrightarrow (a+b)^p - (a^p + b^p) < 0$$

The equality criterion is obvious from the fact that f is strictly decreasing on $(0, \infty)$. □

Recall that a pair $(X, \|\cdot\|)$, consiting of a (real or complex) vector space X and a function $\|\cdot\| : X \to \mathbb{R}^{\geq 0}$ satisfying $\|\lambda x\| = |\lambda| \|x\|$, is a quasinormed space, if there exists $K \geq 1$ such that

$$\|x + y\| \leq K (\|x\| + \|y\|) \qquad \forall x, y \in X$$

Proposition 2. *For $0 < p < \infty$, $(L^p(X,\mu), \|\cdot\|_{L^p})$ is a complete quasinormed space.*

Proof. We can define a distance function on $L^p(X,\mu)$ by

$$d(f,g) := \|f - g\|_{L^p}^p = \int_X |f - g|^p \, d\mu$$

The only metric axiom which isn't obvious is the triangle inequality. Applying the preceding lemma, for all $f, g, h \in L^p(X,\mu)$,

$$d(f,g) + d(g,h) = \int_X (|f-g|^p + |g-h|^p) \, d\mu \geq \int_X (|f-g| + |g-h|)^p \, d\mu \geq \int_X |f-h|^p \, d\mu = d(f,h)$$

Since $\|f_n - f_m\|_{L^p} \to 0, n, m \to \infty \Longleftrightarrow d(f_n, f_m) \to 0, n, m \to \infty$ by the continuity of the maps $x \mapsto x^p$ and $x \mapsto x^{\frac{1}{p}}$, to show that d is a complete metric, it suffices to show that given a sequence $(f_n)_{n=1}^\infty$,

$$\|f_n - f_m\|_{L^p}^p \to 0, n, m \to \infty \Rightarrow \exists f \in L^p, \|f_n - f\|_{L^p}^p \to 0, n \to \infty$$

Let $(f_n)_{n=1}^\infty$ be such a sequence. Then we can construct a subsequence $(f_{n_k})_{k \in \mathbb{N}}$ such that $\left\|f_{n_k} - f_{n_{k+1}}\right\|_{L^p}^p \leq \frac{1}{2^k}$. Define

$$f = f_{n_1} + \sum_{k=1}^\infty \left(f_{n_{k+1}} - f_{n_k}\right)$$

Since

$$\left\|\sum_{k=1}^N \left(f_{n_{k+1}} - f_{n_k}\right)\right\|_{L^p}^p \leq \sum_{k=1}^N \left\|f_{n_{k+1}} - f_{n_k}\right\|_{L^p}^p \leq \sum_{k=1}^N \frac{1}{2^k} \leq 1 \; \forall N \in \mathbb{N}$$

it follows from the monotone convergence theorem, $|f_{n_1}|+\sum_{k=1}^{\infty}\left|f_{n_{k+1}}-f_{n_k}\right| \in L^p(X,\mu)$. Hence, by the Lebesgue dominated convergence theorem, $f \in L^p(X,\mu)$.

$$f_1+\sum_{k=1}^{N}\left(f_{n_{k+1}}-f_{n_k}\right)=f_{n_{N+1}} \Rightarrow \lim_{k\to\infty} f_{n_k}=f$$

Hence, $(f_n)_{n=1}^{\infty}$ is Cauchy with a convergent subsequence and therefore $\|f_n-f\|_{L^p}^p \to 0$, as $n\to\infty$. □

1.2 Inequalities

Proposition 3. *(Reverse Hölder's) Let $q\in(0,1)$. For $r<0$ and $g>0$ $\mu-a.e.$, define $\|g\|_{L^r} := \left\|g^{-1}\right\|_{L^{|r|}}^{-1}$. Then for $f\geq 0$ and $g>0$ $\mu-a.e.$, we have that*

$$\|fg\|_{L^1} \geq \|f\|_{L^q}\|g\|_{L^{q'}}$$

where $\frac{1}{q}+\frac{1}{q'}=1$.

Proof. If $fg\notin L^1(X,\mu)$ (i.e. $\|fg\|_{L^1}=\infty$) or $g^{-1}\notin L^{q'}(X,\mu)$, then the inequality is trivial. So assume otherwise. Since $q\in(0,1)$ and $1=\frac{1}{q}+\frac{1}{q'}$, we have that $q'<0$ and

$$\frac{1}{q}=\frac{1}{1}+\frac{1}{|q'|}$$

By Hölder's inequality applied to fg and $g^{-1}\in L^{|q'|}$,

$$\|f\|_{L^q}=\left\|fgg^{-1}\right\|_{L^q}\leq \|fg\|_{L^1}\left\|g^{-1}\right\|_{L^{|q'|}} \Rightarrow \|f\|_{L^q}\|g\|_{L^{q'}}=\|f\|_{L^q}\left\|g^{-1}\right\|_{L^{|q'|}}\leq\|fg\|_{L^1}$$

□

Proposition 4. *(Reverse Minkowski's) Let $f_1,\cdots,f_N\in L^p(X,\mu)$, where $0<p<1$ Then*

$$\sum_{j=1}^{N}\|f_j\|_{L^p}\leq\left\|\sum_{j=1}^{N}|f_j|\right\|_{L^p}$$

Proof. By induction it suffices to consider the case $N=2$. If $\||f_1|+|f_2|\|_{L^p}=\infty$, then the stated inequality is trivially true, so assume otherwise. Furthermore, if either f_1 or f_2 are zero $\mu-.a.e$, then the inequality is also trivial, so assume otherwise. By the reverse Hölder's inequality,

$$\begin{aligned}\||f_1|+|f_2|\|_{L^p}^p=\int_X ||f_1|+|f_2||^p\,dx &= \int_X |f_1|\,||f_1|+|f_2||^{p-1}\,dx+\int_X |f_2|\,||f_1|+|f_2||^{p-1}\,dx\\ &\geq \|f_1\|_{L^p}\left\|(|f_1|+|f_2|)^{p-1}\right\|_{L^{\frac{p}{p-1}}}+\|f_2\|_{L^p}\left\|(|f_1|+|f_2|)^{p-1}\right\|_{L^{\frac{p}{p-1}}}\\ &= (\|f_1\|_{L^p}+\|f_2\|_{L^p})\,\||f_1|+|f_2|\|_{L^p}^{p-1}\end{aligned}$$

Dividing both sides by $\|f_1+f_2\|_{L^p}^{p-1}$ yields the stated inequality. □

The preceding proposition shows that $(L^p(X,\mu),\|\cdot\|_{L^p})$ is not a normed space when $0<p<\infty$.

Lemma 5. *Suppose $1\leq\theta<\infty$. Then for $a_1,\cdots,a_N\in\mathbb{R}^{\geq 0}$,*

$$\left(\sum_{j=1}^{N}a_j\right)^{\theta}\leq N^{\theta-1}\sum_{j=1}^{N}a_j^{\theta}$$

Proof. Since $\theta\geq 1$, the function $f(x)=x^{\theta}$ is convex. Hence,

$$\left(\sum_{j=1}^{N}a_j\right)^{\theta}=f\left(\frac{\sum_{j=1}^{N}Na_j}{N}\right)\leq\frac{1}{N}\sum_{j=1}^{N}f(Na_j)=N^{\theta-1}\sum_{j=1}^{N}a_j^{\theta}$$

□

Proposition 6. *For $0 < p < 1$,*

$$\left\| \sum_{j=1}^{N} f_j \right\|_{L^p} \leq N^{\frac{1-p}{p}} \sum_{j=1}^{N} \|f_j\|_{L^p}$$

Furthermore, $N^{\frac{1-p}{p}}$ is the best possible constant.

Proof. If $\|f_j\|_{L^p} = \infty$ for some j, then the inequality trivially holds, so assume otherwise. Since $\frac{1}{p} > 1$, by the preceding lemma,

$$\left\| \sum_{j=1}^{N} f_j \right\|_{L^p} = \left(\int_X \left| \sum_{j=1}^{N} f_j \right|^p dx \right)^{\frac{1}{p}} \leq \left(\sum_{j=1}^{N} \int_X |f_j|^p \, dx \right)^{\frac{1}{p}} \leq N^{\frac{1}{p}-1} \sum_{j=1}^{N} \left(\int_X |f_j|^p \, dx \right)^{\frac{1}{p}} = N^{\frac{1-p}{p}} \sum_{j=1}^{N} \|f_j\|_{L^p}$$

To see that $N^{\frac{1-p}{p}}$ is the best possible constant, let E be a measurable set such that $\mu(E) = \alpha < \infty$, and set $E_j := E$ and $f_j := \mathbf{1}_E$ for $1 \leq j \leq N$. Then

$$\left\| \sum_{j=1}^{N} f_j \right\|_{L^p} = \left(\sum_{j=1}^{N} \mu(E_j) \right)^{\frac{1}{p}} = (N\alpha)^{\frac{1}{p}} = N^{\frac{1-p}{p}} \left(N\alpha^{\frac{1}{p}} \right) = N^{\frac{1-p}{p}} \sum_{j=1}^{N} \mu(E_j)^{\frac{1}{p}} = N^{\frac{1-p}{p}} \sum_{j=1}^{N} \|f_j\|_{L^p}$$

□

1.3 Day's theorem

Lemma 7. *Let $(X, \mathcal{A}, \mu)$ be a measure space with the property that given any $f \in L^p(X, \mu)$ for $p \in (0, 1)$, the functional*

$$\mathcal{A} \to \mathbb{R}, E \mapsto \int_E |f|^p \, d\mu$$

assumes all values between 0 and $\|f\|_{L^p}^p$. Then $L^p(X, \mu)$, with $0 < p < 1$, contains no convex open sets, other than $\emptyset$ and $L^p(X, \mu)$.

Proof. Let Ω be a nonempty convex open neighborhood of the origin in $L^p(X)$ and $f \in L^p(X)$ be arbitrary. Since Ω is open, there exists a ball B_δ about the origin contained in Ω. Choose $n \in \mathbb{Z}^{\geq 1}$ such that $\frac{\|f\|_{L^p}^p}{n^{1-p}} \leq \delta$ (i.e. $nf \in B_{n\delta}$). Note that we can choose such a n precisely because $p \in (0, 1)$. Using the intermediate value hypothesis for the measure space, there exists a measurable set E_1 such that

$$\int_{E_1} |f|^p \, d\mu = \frac{1}{n} \int_X |f|^p \, d\mu = \frac{\|f\|_{L^p}^p}{n}$$

Repeating the argument for $f_1 = f\mathbf{1}_{E_1^c}$ and apply induction, we obtain a partition $\{E_1, \cdots, E_n\}$ of X into disjoint measurable subsets such that $\int_{E_j} |f|^p = \frac{\|f\|_{L^p}^p}{n}$ $\forall j = 1, \cdots, n$. Define $h_j := nf\mathbf{1}_{E_j}$. Then by our choice of n,

$$\int_X |h_j|^p \, d\mu = \int_{E_j} n^p |f|^p \, d\mu = \frac{1}{n^{1-p}} \int_X |f|^p \, d\mu \leq \delta$$

Hence, $h_j \in B_\delta \subset \Omega$ $\forall j = 1, \cdots, n$. By convexity,

$$f = \frac{h_1 + \cdots + h_n}{n} \in \Omega$$

Since $f \in L^p(X, \mu)$ was arbitrary, we obtain that $\Omega = L^p(X, \mu)$. □

Corollary 8. *With $(X, \mathcal{A}, \mu)$ as above, the natural topology for $L^p(X, \mu)$, with $0 < p < 1$, is not locally convex.*

The following result, originally proven by M.M. Day, shows that the Hahn-Banach theorem fails for $L^p(X, \mu)$, when $0 < p < 1$. Specifically, the Hahn-Banach theorem may fail when we only assume the underlying space is quasi-normed.

Theorem 9. *(M.M. Day) Let $p \in (0,1)$ and let $T : L^p(X,\mu) \to Y$ be a continuous linear mapping of $L^p(X,\mu)$ into a locally convex T_0 space Y (i.e. singletons are closed). Then T is the zero map. In particular, $L^p(X,\mu)^* = \{0\}$.*

Proof. Let T be such a map, and let $\mathcal{B}$ be a convex local base for Y at the origin. Let $W \in \mathcal{B}$. Then $T^{-1}(W)$ is a nonempty open convex subset of $L^p(X,\mu)$, hence by the preceding lemma, $T^{-1}(W) = L^p(X,\mu)$. Hence, $T(L^p(X,\mu)) \subset W$ for all $W \in \mathcal{B}$. I claim that $\bigcap_{W\in\mathcal{B}} W = \{0\}$. Assume the contrary, and let $x \neq 0$ be in the intersection. Since singletons are closed in Y, $Y \setminus \{x\}$ is an open neighborhood of 0. Hence, $\bigcap_{W\in\mathcal{B}} W \subset (Y \setminus \{x\})$, which is a contradiction. We conclude that $T(L^p(X,\mu)) = \{0\} \iff T = 0$. □

1.4 Non-Normability

One might ask if $L^p(X,\mu), 0 < p < 1$, is normable for an arbitrary measure space $(X, \mathcal{A}, \mu)$. The following example shows that it is not, even for a nice measure space.

Proposition 10. *Let $(f_n)_{n=1}^{\infty}$ be a sequence in $L^p([0,1], \mathcal{L}, \lambda)$, where $\mathcal{L}$ is the Lebesgue σ-algebra and λ is the Lebesgue measure on $[0,1]$. Then there does not exist a norm $\|\cdot\|$ on $L^p([0,1])$ such that for any sequence $(f_n)_{n\in\mathbb{N}} \subset L^p([0,1]), f_n \to 0$ in $L^p \Rightarrow \|f_n\| \to 0, n \to \infty$.*

Proof. Suppose such a norm $\|\cdot\|$ exists. I claim that there exists a positive constant $C < \infty$ such that $\|f\| \leq C\|f\|_{L^p}$ $\forall f \in L^p([0,1])$. Indeed, the map $L^p([0,1]) \to \mathbb{R}, f \mapsto \|f\|$ is evidently continuous. Hence, there exists $\delta > 0$ such that $\|f\|_{L^p} < \delta \Rightarrow \|f\| \leq 1$. Then $\forall f \in L^p([0,1]), \frac{\alpha\delta f}{\|f\|_{L^p}} \in B_\delta$, where $0 < |\alpha| < 1$. Hence,

$$\left\| \frac{\alpha\delta f}{\|f\|_{L^p}} \right\| \leq 1 \Rightarrow \|f\| \leq \frac{1}{\alpha\delta} \|f\|_{L^p}$$

Letting $\alpha \to 1$, we see that the inequality holds for $C = \frac{1}{\delta}$. Choose $C = \inf\{K : \|f\| \leq K\|f\|_{L^p} \ \forall f \in L^p([0,1])\}$ (Note that we do not exclude the possibility that $C = 0$). By the intermediate value theorem, there exists $c \in (0,1)$ such that

$$\int_0^c |f|^p \, d\lambda = \int_c^1 |f|^p \, d\lambda = \frac{1}{2}\int_0^1 |f|^p \, d\lambda$$

Set $g = f\chi_{[0,c]}$ and $h = f\chi_{(c,1]}$. Then $f = g + h$ and $\|g\|_{L^p} = \|h\|_{L^p} = 2^{-\frac{1}{p}}\|f\|_{L^p}$. By the triangle inequality,

$$\|f\| \leq \|g\| + \|h\| \leq C\left(\|g\|_{L^p} + \|h\|_{L^p}\right) = \frac{C}{2^{\frac{1}{p}-1}}\|f\|_{L^p}$$

Since $p \in (0,1)$, $\frac{C}{2^{\frac{1}{p}-1}} \leq C \Rightarrow C = 0 \Rightarrow \|f\| = 0 \ \forall f \in L^p(X,\mu)$, which contradicts that $\|\cdot\|$ is a norm. □

Remark 11. In fact, the non-normability of $L^p([0,1])$, when $0 < p < 1$, follows from M.M. Day's thoerem. If $L^p([0,1])$ were normable, then the Hahn-Banach theorem would hold, contradicting that $L^p([0,1])^* = \{0\}$. So we have the more general assertion that given any measure space $(X, \mathcal{A}, \mu)$ which satisfies the hypotheses of Day's theorem, $L^p(X,\mu)$ is non-normable.