Additional notes to my "Operator Theory" course (facts on Banach algebras)

1 Preliminaria

DEFINITIONS: A **Banach algebra** is a pair $(A, \|\cdot\|)$, where A is an algebra (in this course we restrict ourselves to algebras over $\mathbb{C}$) with a norm $\|\cdot\|$ making A a complete space and satisfying $\|x\cdot y\| \leq \|x\|\|y\|$ for any $x, y \in A$. In most cases we assume that A is a **unital** algebra, i.e. it possesses a unit $\mathbf{1}$. Then by G_A we denote the **group of invertible elements** of A. The **spectrum** of an element $x \in A$ is defined as $\sigma(x) := \{\lambda \in \mathbb{C} : (\lambda\mathbf{1} - x) \notin G_A\}$. We also consider
$\mathrm{Sp}(A) := \{\omega : A \to \mathbb{C} : \omega$ is a non-zero homomorphism of algebras $\}$, called the **spectrum of** A.

We say that the mapping $* : A \to A$ is an **involution**, if it is conjugate-linear (i.e. additive and satisfying $(\lambda x)^* = \bar{\lambda}x^*$ and such that $(x^*)^* = x, (xy)^* = y^*x^*$ for all $x, y \in A$. Note that if $1 \in A$, then $1^* = 1$. Indeed, $1 = 1^{**} = (11^*)^* = 1^{**}1^* = 1^*$. Hence $\sigma(x^*) = \{\bar{\lambda} : \lambda \in \sigma(x)\}$. If additionally

$$\forall_{x\in A}\ \|x\|^2 = \|x^*x\| \qquad (C*),$$

we say that A is a **C*-algebra**[1].

EXAMPLES: Two most important examples of unital C*-algebras are $C(K)$ and $\mathcal{B}(H)$.

Here $C(K)$ is the commutative algebra of continuous functions on a compact Hausdorff space K with sup-norm, point-wise algebraic operations and involution given by complex conjugation.

A non-commutative C*-algebra is the space $\mathcal{B}(H)$ of bounded linear operators on a Hilbert space H.

The easiest example of a commutative, non-unital C-* algebra is the following one: If Ω is locally compact, but non-compact, then $C_0(K) = \{f \in C(\Omega) : \forall_{\epsilon>0}\exists_{K\subset\subset\Omega}\forall_{t\in\Omega\setminus K}|f(t)| < \epsilon\}$ is the algebra of continuous functions vanishing at infinity. Here $K \subset\subset \Omega$ means that K is a compact subset of Ω. By taking a 1-point (Alexandroff) compactification $\hat{\Omega} := \Omega \cup \{\infty\}$, where the set of all open neighbourhoods of ∞ is the family $\{\hat{\Omega} \setminus K : K \subset\subset \Omega\}$ and Ω is open in $\hat{\Omega}$, we identify $C_0(\Omega)$ with the maximal ideal $\{f \in C(\hat{\Omega}) : f(\infty) = 0\}$. Here we have a concrete example of the a general method of adjoining the unit (here denoted e, a process called unitisation) of a non-unital Banach algebra A.

Namely, define A_1 as the Cartesian product $A \times \mathbb{C}$ with coordinate-wise vector operations and involution, while the product is $(x, \lambda)\cdot(y, \mu) := (xy + \lambda y + \mu x, \lambda\mu)$. If $x \in A, \lambda \in \mathbb{C}$, we write $(x, \lambda) = x + \lambda e$. Its norm is defined by $\|(x, \lambda)\|_1 = \|\lambda e + x\|_1 := \sup_{\|y\|\leq 1} \|xy + \lambda y\|$. It is sub-multiplicative and satisfies the C*-condition if so does $(A, \|\cdot\|)$. The unit in A_1 is $e := (0, 1)$, as easy computations[2] show.

We can find also partial substitutes to a unit in any C*-algebra called approximate units (which are sequences (k_n) such that $\|x - k_nx\| \to 0$ as $n \to \infty$ -for left approx. units).

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The left- multiplication operators $L_a : A \ni x \mapsto ax \in A$ are linear, bounded with operator norm $\|L_a\| = \|a\|$ (here we consider unital algebras), therefore if in A a series $\sum_{n=0}^{\infty} x_n$ norm-converges, then so does $\sum L_a(x_n)$ and its sum equals $a \cdot \sum x_n$. The same holds true for right multiplications : $x \mapsto xa$. Now if $q := \|\mathbf{1} - x\| < 1$, applying the above with $a := \mathbf{1} - x$, $x_n := a^n$ we obtain that for $S := \sum_{n=0}^{\infty}(\mathbf{1} - x)^n$ we have $(\mathbf{1} - x)S = S - \mathbf{1}$, implying $xS = \mathbf{1}$. Similarly we deduce that $Sx = \mathbf{1}$.

COROLLARIES, Lemmas:

(1) For $x \in A$ if $\|\mathbf{1} - x\| < 1$, then the C.Neumann series $\sum_{n=0}^{\infty}(\mathbf{1} - x)^n$ converges to x^{-1}, so $x \in G_A$.
(2) The set G_A of invertible elements is open.
(3) Spectra $\sigma(a)$ are compact, $r(a) := \sup\{|\lambda| : \lambda \in \sigma(a)\} \leq \|a\|$ for all $a \in A$. For $|\lambda| > \|a\|$

$$(\lambda\mathbf{1} - a)^{-1} = \frac{1}{\lambda}(\mathbf{1} - \frac{1}{\lambda}a)^{-1} = \sum_{n=0}^{\infty} \frac{1}{\lambda^{n+1}}a^n \qquad (*)$$

(4) The inversion operation: $G_A \ni z \mapsto z^{-1}$ is continuous.
(5) Homomorphisms $\omega \in \mathrm{Sp}(A)$ are continuous, $\omega(a) \in \sigma(a)$, $|\omega(a)| \leq \|a\|$ for $a \in A$. Even $\|\omega\| = 1$.
(6) The resolvent mapping $\mathbb{C} \setminus \sigma(a) \ni \lambda \mapsto R_\lambda := (\lambda\mathbf{1} - a)^{-1} \in A$ satisfies the Hilbert equation

$$R_\lambda - R_\mu = (\mu - \lambda)R_\lambda R_\mu, \quad \text{if} \quad \lambda, \mu \in \mathbb{C} \setminus \sigma(\mathrm{a})$$

and are analytic, which means $\forall\psi \in A^*$ the mappings $\lambda \mapsto f(\lambda) := \psi(R_\lambda) \in \mathbb{C}$ are analytic on $\mathbb{C}\setminus\sigma(a)$.
(7) Spectra of elements of A are non-empty.
(7a) If $ab = ba$ and $a \notin G_A$ then $ab \notin G_A$.

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[1] Combining (C*) with $\|x^*x\| \leq \|x^*\|\|x\|$, then dividing both sides of the resulting inequality by $\|x\|$ we obtain $\|x\| \leq \|x^*\|$ (and "$\geq$", by replacing x with x^*). If $\forall_x\ \|x^*\| \leq \|x\|$, then inequality "$\leq$" in (C*) implies "$=$" in (C*).

[2] For any $0 < t < 1$ we find y with $\|y\| = 1, t\|(x, \lambda)\|_1 \leq \|xy + \lambda y\|$. Hence $t^2\|(x, \lambda)\|_1 \leq \|xy+\lambda y\|^2 = \|(xy+\lambda y)^*(xy+\lambda y)\| = \|y^*(\lambda e+x)^*(\lambda e+x)y\|_1 \leq \|(\lambda e+x)^*(\lambda e+x)\|_1$. As $t \to 1^-$, we obtain $\|(\lambda e+x)\|_1{}^2 \leq \|(\lambda e+x)^*\|_1\|(\lambda e+x)\|_1$. Dividing both sides by $\|(\lambda e + x)\|_1$ and next using the previous footnote's conclusion we obtain (C*) for A_1.

(8) Let $a \in A$ and let $F(z) = \sum_{n=0}^{\infty} c_n z^n$ be holomorphic in $\mathbb{D}_\rho := \{z \in \mathbb{C} : |z| < \rho\}$ for some $\rho > r = \|a\|$. Then the series $F(a) := \sum_{n=0}^{\infty} c_n a^n$ converges and $F(\lambda) \in \sigma(F(a))$ for any $\lambda \in \sigma(a)$.
(9) Spectral radius $r(a)$ defined in (3) above satisfies the relations

$$r(a) = \lim_{n\to\infty} (\|a^n\|)^{\frac{1}{n}} .$$

(10) This formula implies that normal elements in C*-algebras satisfy $\|x\| = r(x)$.
(11) In commutative C*-algebras if $\omega \in \mathrm{Sp}(A)$ and $x \in A$, then we have $\omega(x^*) = \overline{\omega(x)}$.
Theorem 1 (Gelfand- Mazur) *If $A \setminus \{0\} = G_A$ (i.e. if A is a field), then $A = \{\lambda \mathbf{1} : \lambda \in \mathbb{C}\}$.*
Corollary 1.1 (Gelfand) *For commutative Banach algebras the assignment: $Sp(A) \ni \omega \mapsto \ker(\omega)$ is a bijection*[3] *onto the set of all maximal ideals of A. Moreover $\sigma(a) = \{\omega(a) : \omega \in \mathrm{Sp}(A)\}$.*

For a commutative Banach algebra A we have the Gelfand transform $\Gamma : A \ni a \mapsto \hat{a} \in C(\mathrm{Sp}(A))$, where $\hat{a}(\omega) := \omega(a)$. Here the topology on $\mathrm{Sp}(A) \subset A^*$ is the weak-star topology inherited from A^*. A net (= generalised sequence) $(\phi_\iota)_{\iota \in I}$ in A^* converges w* to ψ if it converges point-wise on A.
Theorem 2 (Gelfand- Naimark) *For commutative C^*-algebras Γ is an isometric, bijective isomorphism between A and $C(Sp(A))$, preserving multiplication and involution.*

Proofs of the above results (outline)
`ad (1)` Since $\|a^n\| \le \|a\|^n$, the series is absolutely convergent ($\Rightarrow$ convergent, by completeness of A). Preceding (1) five lines of remarks yield the result.
`ad (2)` If $x \in G_A$, then $y \in G_A \Leftrightarrow x^{-1}y \in G_A$ But $\|\mathbf{1} - x^{-1}y\| = \|x^{-1}(x-y)\| \le \|x^{-1}\|\|x-y\| < 1$ if $\|x-y\| < \frac{1}{\|x^{-1}\|}$. Hence the ball $\{y \in A : \ \|x-y\| < \frac{1}{\|x^{-1}\|}\}$ is contained in G_A.
`ad (3)` $\sigma(a)$ is closed as a pre-image of closed set $A \setminus G_A$ under isometry $\mathbb{C} \ni \lambda \to \lambda\mathbf{1} - a \in A$. If $|\lambda| > \|a\|$, then $x := \mathbf{1} - \frac{1}{|\lambda|}a$ satisfies $\|\mathbf{1} - x\| < 1$, hence x and λx belong to G_A, so $\lambda \notin \sigma(a)$. Formula (*) follows directly from Carl Neumann's series.
`ad (4)` Continuity at $z_0 = \mathbf{1}$: by (1) we estimate that $\|z^{-1} - \mathbf{1}\| = \|\sum_{n=1}^{\infty}(\mathbf{1}-z)^n\| \le \sum_{n=1}^{\infty} \|(\mathbf{1}-z)\|^n = \frac{\|(\mathbf{1}-z)\|}{1-\|(\mathbf{1}-z)\|}$, converging to 0 as $z \to \mathbf{1}$.
Continuity at $x \in G_A$: if $x_n \to x$, then eventually $x_n \in G_A$ and $x_n x^{-1} \to x x^{-1} = \mathbf{1}$. By continuity at $\mathbf{1}$, $(x_n x^{-1})^{-1} = x x_n^{-1} \to 1$ and applying continuity of $L_{x^{-1}}$, we have $x^{-1} x x_n^{-1} \to x^{-1}\mathbf{1}$.
`ad (5)` By (3) if $|\lambda| > \|x\|$ then $\exists(\lambda\mathbf{1} - x)^{-1}$, hence $0 \ne \omega(\lambda\mathbf{1} - x) = \lambda - \omega(x)$, so $|\omega(x)| \le \|x\|$ and $\|\omega\| \le 1$. Since $|\omega(\mathbf{1})| = 1$, we have $\|\omega\| = 1$.
`ad (6)` Since R_λ commutes with R_μ, replacing $(\mu - \lambda)$ by $((\mu\mathbf{1} - a) - (\lambda\mathbf{1} - a))$ in the right -hand term of the equation, we verify it easily. Now (4) implies the continuity of $\lambda \mapsto f(\lambda)$. By linearity of ψ, Hilbert equation gives $\frac{f(\lambda)-f(\mu)}{\lambda-\mu} = -f(\lambda)f(\mu)$, which tends to $-f(\lambda)^2$, if $\mu \to \lambda$. Hence complex derivatives exist at all points of $\mathbb{C} \setminus \sigma(a)$.
`ad (7)` Assume towards a contradiction that $\sigma(a) = \emptyset$ for some $a \in A$. Then $\forall_{\psi \in A^*} \lambda \mapsto f(\lambda) := \psi(R_\lambda) \in \mathbb{C}$ are entire functions (clearly, non-zero). By (*) in (3), $|f(\lambda)| \le \|\psi\| \frac{1}{|\lambda|} \|(\mathbf{1} - \frac{1}{\lambda}a)^{-1}\|$, which goes to 0 as $\lambda \to \infty$, since the last term is bounded. Indeed, by (4), $(\mathbf{1} - \frac{1}{\lambda}a)^{-1}$ goes to $\mathbf{1}$ as $\lambda \to \infty$.
`ad (7a)` Suppose the contrary: that $\exists c := (ab)^{-1}$. Then $a(bc) = 1$ Also $1 = c(ba) = (cb)a$, so $a \in G_A$.
`ad (8)` By power series theory, $\sum_n c_n z^n$ converges absolutely in $\mathbb{D}_r$, so $\sum |c_n| r^n < \infty$. Consequently $\sum c_n a^n$ converges absolutely to $F(a) \in A$, since $\|a^n\| \le r^n$. Let us choose arbitrary $\lambda \in \sigma(a)$, so $|\lambda| \le r$. Algebraic formula for $x^n - y^n$ implies the following factorisation of $F(\lambda)\mathbf{1} - F(a) = \sum_{n=1}^{\infty} c_n(\lambda^n\mathbf{1} - a^n)$ as equal to $(\lambda\mathbf{1} - a)\sum_{n=1}^{\infty} c_n P_{n-1}(\lambda, a)$, where $P_{n-1}(\lambda, a) = \sum_{k=0}^{n-1} \lambda^k a^{n-k-1}$. Since $\|P_{n-1}(\lambda, a)\| \le n r^{n-1}$, the series $b := \sum_{n=1}^{\infty} c_n P_{n-1}(\lambda, a)$ converges in A and its sum, b commutes with a (and with $(\lambda\mathbf{1} - a)$).That $F(\lambda)\mathbf{1} - F(a)$ is not invertible now follows from (7a).
`ad (9)` Since $\sigma(a) \subset \{z \in \mathbb{C} : |z| \le \|a\|\}$, we have $r(a) \le \|a\|$. For $\lambda \in \sigma(a)$ we have $\lambda^n \in \sigma(a^n)$ by (8), so $|\lambda|^n \le \|a^n\|$, i.e. $|\lambda| \le \sqrt[n]{\|a^n\|}$ and $r(a) \le \inf_n \sqrt[n]{\|a^n\|}$.

Hence it suffices to show that $\limsup_n \sqrt[n]{\|a^n\|} \le r(a)$. Any functional $\varphi \in A^*$ applied to R_λ represented by (*) in (3) gives a Laurant series $\varphi((\lambda\mathbf{1} - a)^{-1}) = \sum_{n=0}^{\infty} \frac{1}{\lambda^{n+1}} \varphi(a^n)$ convergent for any $|\lambda| > r(a)$. Hence its terms are bounded for such λ, so there exists a constant K_φ such that $\forall_n |\varphi(a^n)\lambda^{-n-1}| \le K_\varphi$. By the (Banach-Steinhaus) *Uniform Boundedness Principle* $\forall_{|\lambda| > r(a)} \exists_C \forall_n \|a^n\| |\lambda^{-n-1}| \le C$. Here $C = C_\lambda$ does not depend on n, but may depend on λ and since

$$\|a^n\|^{\frac{1}{n}} \le C_\lambda^{\frac{1}{n}} |\lambda|^{\frac{n+1}{n}}, \quad \textit{we have} \quad \limsup_n \|a^n\|^{\frac{1}{n}} \le |\lambda|$$

for any $|\lambda| > r(a)$, implying $\limsup_n \|a^n\|^{\frac{1}{n}} \le r(a)$.
`ad (10)` Indeed, by the C*-condition $\|x^* x x^* x\| = \|x^* x\|^2 = \|x\|^4$ and since $xx^* = x^* x$, the left-hand

[3] The inverse bijection: if M is a max.ideal, then $\omega_M(a) = \lambda \Leftrightarrow (\lambda\mathbf{1} - a) \in M$ satisfies $\omega_M \in \mathrm{Sp}(A)$, $M = \ker(\omega_M)$.

term equals $\|(x^*)^2x^2\| = \|(x^2)^*(x^2)\| = \|x^2\|^2$. Hence $\|x^2\| = \|x^2\|$ and by induction, for any k of the form $k = 2^m$, where $m \in \mathbb{N}$, $\|x^k\| = \|x\|^k$ Applying (9) to this subsequence we have $r(x) = \|x\|$.
`ad (11)` First we consider unitary elements $u \in A$, so that $u^* = u^{-1}$. It follows that $r(u) = \|u\| = \|u^{-1}\| = 1$ (Indeed, $\|u\|^2 = \|u^*u\| = \|\mathbf{1}\| = 1$.) Hence for any $\lambda \in \sigma(x)$ we have $|\lambda| \leq 1$. Since u^* is also unitary and $\omega(u^{-1}) = (\omega(u))^{-1}$, we obtain even $|\omega(u)| = 1$ for any $\omega \in \mathrm{Sp}(A)$.

In the next step we use the fact that if $a, b \in A$ commute, then $\exp(a)\exp(b) = \exp(a + b)$ - here $\exp(a) := \sum_{n=0}^{\infty} \frac{1}{n!}a^n$. Its proof is a verbatim translation from the scalar case (when A=$\mathbb{C}$). If $a = a^*$ and $t \in \mathbb{R}$, we easily check that $(\exp(ita))^* = \exp(-ita)$ and consequently $\exp(ita)$ is unitary. Then for any $\omega \in \mathrm{Sp}(A), t \in \mathbb{R}$ we have $\omega(\exp(ita) \in \sigma(\exp(ita) \subset \{\lambda : |\lambda| \leq 1\}$. Hence $1 \geq |\omega(\exp(ita))| = |\exp(it\omega(a))|$, which is possible $\forall_{t\in\mathbb{R}}$ only if $\omega(a) \in \mathbb{R}$. Now for any $x \in A$ $x = a + ib$, where a, b are self-adjoint (eg. $a = \frac{x+x^*}{2}$) and then $x^* = a - ib$, so using the fact that $\omega(a), \omega(b)$ are real, we obtain that $\omega(x^*) = \overline{\omega(x)}$.

`Proof of Theorem 1.` If $\lambda \in \sigma(x)$, (such λ exists by (7)) then $\lambda\mathbf{1} - x \in A \setminus G_A = \{0\}$, so $x = \lambda\mathbf{1}$.

`Proof of Corollary 1.1` Note first that by ideal we mean here a proper ($\neq A$) subspace $I \subset A$ such that $a \in A \Rightarrow aI \subset I$. Clearly $\mathbf{1} \notin I$. Hence $G_A \cap I = \emptyset$ and the norm-closure of I is also an ideal disjoint with G_A. Maximal ideals are therefore closed. The quotient space $A/_I$ is also a unital algebra consisting of equivalence classes $[a] = a + I$ with $[a] \cdot [b] := [ab]$. The quotient map $\pi : A \ni a \mapsto [a] \in A/_I$ is a homomorphism of unital algebras. Here $\|[a]\| = \inf\{\|a - x\| : x \in I\}$ is the distance from a to I and it is a sub-multiplicative norm, $\|[\mathbf{1}]\| = 1$. Easy verifications show that pre-image of an ideal $K \subset A/_I$ is an ideal in A containing I (and proper). In case when I is maximal, we must have $\pi^{-1}(K) = I$, i.e. $K = \{[0]\}$. Hence $[0]$ is the only non-invertible element in $A/_I$. Hence

$$\forall_{a\in A}\exists!_{\lambda=\lambda_a\in\mathbb{C}} : a - \lambda\mathbf{1} \in I.$$

The assignment $\omega_I : A \ni a \mapsto \lambda_a \in \mathbb{C}$ is linear and multiplicative, $\omega_I(\mathbf{1}) = 1$, hence $\omega_I \in \mathrm{Sp}(A)$. Also $I = \ker(\omega_I)$. Conversely, if $\omega \in \mathrm{Sp}(A)$, then $\ker(\omega)$ is an ideal J having (as a linear subspace of A) co-dimension 1, which implies its maximality. Clearly, $\omega = \omega_J$ (as a pair of linear functionals having the same kernels and taking the same non-zero value at some vector v (here $v = \mathbf{1}$).

In the proof of (5) we have seen that $\omega(a) \in \sigma(a)$. Conversely, if $\lambda \in \sigma(a)$, then $b := \lambda\mathbf{1} - a \notin G_A$. Then $bA := \{bx : x \in A\}$ is a proper ideal. By Kuratowski-Zorn's Lemma, such ideals are contained in some maximal ideal I. Then $\omega_I(b) = 0$, i.e. $\omega_I(a) = \lambda$.

`Proof of Theorem 2.` By property (11), the image $\Gamma(A)$ of A under the Gelfand transform is a symmetric subalgebra of $C(\mathrm{Sp}(A))$, since $\omega(a^*) = \overline{\omega(a)}$, hence along with $\hat{a}$, this image contains the complex conjugate function $\widehat{a^*}$. Since by previous corollary

$$\sup\{|\hat{a}(\omega)| : \omega \in \mathrm{Sp}(A)\} = \sup\{|\lambda| : \lambda \in \sigma(a)\} = r(a),$$

the Gelfand transform Γ is an isometry. Hence $\Gamma(A)$, as an isometric image of a complete space, is complete, hence closed. It also contains all constant functions and separates the points of $\mathrm{Sp}(A)$. Hence by Stone-Weierstrass Theorem it equals $C(\mathrm{Sp}(A))$ and Γ is surjective. Also Γ is a homomorphism of unital C*-algebras. □

There is a version of Stone-Weierstrass Theorem for locally compact spaces Ω and symmetric subalgebras A of $C_0(\Omega)$. A is then non-unital, instead we assume that for any $\omega \in \Omega$ there exists $f \in A$ with $f(\omega) \neq 0$. Then theorem asserts that A is dense in $C_0(\Omega)$.

Now a non-unital commutative C*-algebra A is isometrically identified with $\{(a, 0) = a + 0 \cdot e : a \in A\} \subset A_1$ -a maximal ideal in A_1 and $\mathrm{Sp}(A_1)$ is the one-point compactification of $\mathrm{Sp}(A)$. One verifies that $\mathrm{Sp}(A) \cup \{0\}$ is a w*-closed subset of the (w*-compact) unit ball in A_1^*. Since $\{0\}$ is closed, $\mathrm{Sp}(A)$ is locally compact. For any $a \in A \subset A_1$ its Gelfand transform $\hat{a}$ is continuous on $\mathrm{Sp}A_1$ and vanishes on the ideal A. The corresponding to this ideal element $\varphi \in \mathrm{Sp}(A_1)$ is given by $\varphi(a + \lambda e) = \lambda$. The spectrum of $C_0(\Omega)$ is homeomorphic to Ω via evaluation functionals. The details are omitted, but simple.

Functional Calculus

Let $a \in A$ be a normal element in a unital C*- algebra. The (norm)-closure of polynomials in 3 variables: $a, a^*, \mathbf{1}$ is a commutative subalgebra of A, denoted[4] $C^*(a, 1)$. For $x \in C^*(a, 1)$ denote by $\sigma_*(x)$ its spectrum in this subalgebra. We shall prove that $\sigma_*(a) = \sigma(a)$ (the spectrum w.r. to A.

Theorem 3. *If $a \in A$ is normal then $C^*(a, 1)$ is isometrically *-isomorphic to $C(\sigma(a))$. We also have unital isomorphism of C*-algebras $\Phi : C(\sigma(a) \ni f \mapsto f(a) \in C^*(a, 1) \subset A$ that extends polynomial functional calculus in a, i.e.$\Phi(id) = a, \Phi(1) = \mathbf{1}$. Moreover*

$$\sigma(f(a)) = f(\sigma(a)),\ \|f\| = \|f(a)\| \ \text{ and } \ \widehat{f(a)} = f \circ \hat{a} \quad \text{for } f \in C(\sigma(a)).$$

Here $id(\lambda) = \lambda, 1(\lambda) = 1$ and $\|f\| := \sup\{|f(\lambda)| : \lambda \in \sigma(a)\}$.

[4]If $E \subset A$, one denotes by $C^*(E)$ the smallest C*-subalgebra of A containing E (i.e. generated by E).

Proof First note that $\hat{a} : \mathrm{Sp}(C^*(a,1)) \ni \omega \mapsto \omega(a) \in \sigma_*(a)$ is surjective by Corollary 1.1, continuous and injective (hence a homeomorphism between these compact spaces). Its injectivity follows from the fact that if $\omega_1(a) = \omega_2(a)$ then since $\omega_j(a^*) = \overline{\omega_j(a)}, j = 1,2$, these continuous, linear functionals agree on generators, whose products of powers together with 1 span a dense subset of $C^*(a,1)$. Now the composition operator $C(\mathrm{Sp}(C^*(a,1))) \ni f \mapsto f \circ \hat{a}$ composed with the Gelfand transform $\Gamma : \mathrm{Sp}(C^*(a,1)) \to C(\mathrm{Sp}(C^*(a,1)))$ establishes our isomorphism between $C^*(a,1)$ and $C(\sigma_*(a))$. Once we show that $\sigma_*(a) = \sigma(a)$, the remaining claims will be consequence of this isomorphism. The inclusion "$\supset$" is trivial, as $C^*(a,1) \subset A$, so inverse of $\lambda 1 - a$ in $C^*(a,1)$ will also be its inverse in A.

Suppose towards a contradiction that there exists $\alpha \in \sigma_*(a) \setminus \sigma(a)$. Then $b := (a - \alpha 1)^{-1}$ exists in A. The function $f : \sigma_*(a) \ni \lambda \mapsto \lambda - \alpha$ is continuous and for positive $\epsilon < \|b\|^{-1}$ the set $K := \{\lambda \in \sigma_*(a) : |f(\lambda)| \geq \epsilon\}$ is compact. Since $\alpha \notin K$, by Urysohn's Lemma, we find $g \in C(\sigma_*(a))$ vanishing on K, equal 1 at α and such that $0 \leq g \leq 1$. Clearly $\|fg\|$ - the sup-norm over $\sigma_*(a)$ is $\leq \epsilon$. By the above isomorphism, we have $g = \hat{\gamma}$ for sone $\gamma \in C^*(a,1)$ and $b(a-\alpha 1)\gamma = \gamma$ has norm 1, since the spectrum of γ is the range of g. But this contradicts the estimate $\|b(a-\alpha 1)\gamma\| \leq \|\hat{b}\|\|(\hat{a}-\alpha)g\| < 1$.

The functional calculus in a is the inverse mapping to the isomorphism: $C^*(a,1) \to C(\sigma(a))$. $\square$

In particular, unital subalgebras generated by a self-adjoint element a in C* algebra can be identified with $C(\sigma(a)) \subset \mathbb{R}$ by Corollary (11) and $\sigma(x^*x)$, as the range of $\bar{\hat{x}}\hat{x}$ is a subset of $[0,+\infty) = \mathbb{R}_+$ if x is normal. We say that $a \in A$ is a **positive element**, if $a = a^*$ and $\sigma(a) \subset \mathbb{R}_+$. Note that 0 is also positive. Functional calculus using functions $t \mapsto t_+ := \max(t,0)$ (resp.$t_- = (-t)_-, t \in \mathbb{R}$ allows us to write any $x = x^* \in A$ as a difference $x = x_+ - x_-$ of two positive elements. By applying the above isomorphic description of $C^*(a,1)$ we have the following useful characterisation:

Lemma 4. *For a self-adjoint element $a \in A \setminus \{0\}$ the following are equivalent:*

(i) a is positive,

(ii) $\|\frac{1}{\|a\|}a - \mathbf{1}\| \leq 1$, i.e. $\|\|a\|\mathbf{1} - a\| \leq \|a\|$,

(iii) $a = x^*x$ for some $x \in A$ (namely, for $x = \sqrt{a}$)[5]

The second condition can be applied to show that **sum of two positive elements is positive**, [6] a fact used in proof that $(iii) \Rightarrow (i)$ (here their commutativity is not assumed!). Also note that

$$(\dagger) \qquad \forall_x \, \|x^*x\|\mathbf{1} - x^*x \quad \text{is positive.}$$

If a is positive, ten so is x^*ax (for all $x \in A$), since $x^*ax = (a^{\frac{1}{2}}x)^*(a^{\frac{1}{2}}x)$.

Definition A linear functional $\varphi : A \to \mathbb{C}$ on a C*-algebra A is **positive**, in symbols, $\varphi \geq 0$, if its values on positive elements are non-negative: $\forall_{x \in A}\, \varphi(x^*x) \geq 0$. **States** of A are positive functionals having value 1 at **1**. **Pure states** are extreme points in the set of all states on A.

By ($\dagger$) we have $\varphi(x^*x) \leq \|x^*x\|\varphi(1)$, if $\varphi \geq 0$.

Properties of positive functionals For any $x, y \in A$ and $\varphi : A \to \mathbb{C}$-positive we have

1. $\varphi(x^*) = \overline{\varphi(x)}$ (we say that functionals satisfying 1. are hermitian)
2. $|\varphi(y^*x)|^2 \leq \varphi(x^*x)\varphi(y^*y)$,
3. $|\varphi(x)|^2 \leq \varphi(1)\varphi(x^*x)$
4. $\|\varphi\| = \varphi(1)$.
5. *Conversely, if $f \in A^*$ and $f(a) = \|f\|\|a\|$ for some $a = x^*x \in A \setminus \{0\}$, then $f \geq 0$.*

To prove 5., without loss of generality, we may assume that $\|f\| = 1 = \|a\|, f(a) = 1$. To show first that $f(1) \in \mathbb{R}$ assume that $f(1) = \alpha + i\beta, \alpha, \beta \in \mathbb{R}$. If $\beta \neq 0$, then $\forall_{t \in \mathbb{R}} |f(1 + tia)| = |\alpha + i(t+\beta)| \geq |t + \beta|$ But $|f(1+tia)| \leq \|1 + tia\| \leq \sqrt{t + t^2}$. Hence $|t + \beta| \leq \sqrt{t+t^2}$, i.e. $t^2 + 2t\beta + \beta^2 \leq 1 + t^2$ for all $t \in \mathbb{R}$ - a contradiction. Hence $f(1)$ is real. Suppose $f(1) < 1$. Then $|f(1-2a)| = |f(1) - 2| > 1$ and $|f(1-2a)| \leq \|1 - ea\| = \|(1-a) - a\| \leq 1$ -a contradiction. Hence $f(1) = 1$.

Now if $b = b^*$ and suppose that $f(b) = \alpha + i\beta$, where $\alpha, \beta \in \mathbb{R}$. If $\beta \neq 0$, then $\forall_{t \in \mathbb{R}} |f(b + it1)| \geq |\beta + t|$ and $|f(b + it1)| \leq \|b + it1\| = \sqrt{\|b\|^2 + t^2}$, a contradiction. Hence $f(b) \in \mathbb{R}$ if $b = b^*$. Now take any $h = h^* \geq 0$ and suppose that $f(h) < 0$. Then $f(1-h) = 1 - f(h) > 1$, but this contradicts $|f(1-h)| \leq \|1 - h\| \leq 1$. $\square$.

[5] The proof that (iii) implies (i) is a bit tricky, but for algebras of operators it follows easily from the fact that the closure of numerical range contains the spectrum.

[6] If $a, b \in A^+$, then for $\alpha := \|a\|, \beta := \|b\|, \gamma := \alpha + \beta, c = a + b$, we have by (ii) that $\|\gamma\mathbf{1} - c\| \leq \|\alpha\mathbf{1} - a\| + \|\beta\mathbf{1} - b\| \leq \gamma$. Since $c = c^*$, this gives $\sigma(\gamma\mathbf{1} - c) \subset [-\gamma, \gamma]$, which implies $\sigma(c) \subset [0, 2\gamma]$.

In particular, if K is a compact Hausdorff space, then the set of states of $C(K)$ is the set of probabilistic regular Borel measures on K and pure states are delta-measures.

If $\omega : A \to \mathbb{C}$ is a state, then $A \times A \ni (a,b) \mapsto \langle a|b\rangle := \omega(b^*a)$ is a sesquilinear form, obeying the **Schwarz inequality:** $|\langle a|b\rangle|^2 \le \langle a|a\rangle\langle b|b\rangle$ for $a,b \in A$. Hence $\mathcal{N}_\omega := \{x \in A : \omega(x^*x) = 0\}$ is a closed left ideal. The quotient space H°_ω is a pre-Hilbert space with inner product $\langle [a]|[b]\rangle_\omega := \omega(b^*a)$ and it is a dense subspace of its completion denoted by H_ω. The "left multiplication" operators $\pi_\omega(a) : H^\circ_\omega \ni [b] \mapsto [ab]$ extend by continuity on H_ω and $A \ni a \mapsto \pi_\omega(a) \in \mathcal{B}(H_\omega)$ is a representation (i.e. a unital *-homomorphism of C*-algebras). One shows that the direct (orthogonal) sum

$$\bigoplus_\omega \pi_\omega : A \to \mathcal{B}(H_\omega),$$

where ω runs through the set of all states on H –is an isometric *-homomorphism called the **universal representation of** A. This (so called) **GNS construction** allows us to treat any C*-algebra as an algebra of operators on some Hilbert space.

Here we concentrate on commutative (subalgebras of) C*-algebras in order to obtain spectral theorem for normal operators. We begin by an important fact on **commutants**[7] $\mathcal{R}'$ of sets $\mathcal{R} \subset \mathcal{B}(H)$.

Definition $\mathcal{R}' := \{S \in \mathcal{B}(H) : \forall_{T\in\mathcal{R}}\, TS = ST\}$. We say that S,T doubly commute if $ST = TS$ and $S^*T = TS^*$. (Clearly $\mathcal{R}'$ is a unital subalgebra of $\mathcal{B}(H)$, closed in WOT-topology :=the topology defined on $\mathcal{B}(H)$ by the family of semi-norms $p_{x,y}(T) := \langle Tx|y\rangle, x\, y \in H$.)

It is not obvious, but for a normal operator $T \in \mathcal{B}(H)$ its commutant, $\{T\}'$ is conjugate -closed: Operators commuting with T also doubly commute with T, as follows from the following more general

Theorem 5. (Fuglede-Putnam) *If $M, N \in \mathbf{B}(H)$ are normal and "intertwined by $T \in \mathcal{B}(H)$" i.e. such that $MT = TN$, then $M^*T = TN^*$.*

Proof. Note that T also intertwines M^k with N^k for $k \in \mathbb{N}$ and T intertwines $\exp(M) := \sum_{k=0}^\infty \frac{1}{k!}M^k$ with $\exp(N)$. Hence $T = \exp(-M)T\exp(N)$. Multiplying this equality by $U := \exp(M - M^*) = \exp(M)(\exp(M^*)^{-1})$ from the left and by $W := (\exp N)^{-1}\exp(N^*)$ from the right, we have $UTW = \exp(M^*)T\exp(-N^*)$. Since $V^* = -V$ for $V := M - M^*$, we have $U^* = (\exp(V))^* = \exp(-V) = \exp(V)^{-1}$, operators U and W are unitary, hence having norm 1. Consequently, $\|\exp(M^*)T\exp(-N^*)\| = \|UTW\| \le \|T\|$. Replacing N and M by zN , zM with arbitrary $z \in \mathbb{C}$ we deduce the boundedness of (analytic) function $z \mapsto F(z) := \exp(zM^*)T\exp(-zN^*)$. As in the proof of (7), by Liouville's theorem, F must be constant equal $F(0) = T$. Hence $\exp(zM^*)T = T\exp(zN^*)$. Comparing the coefficients at z^1 we obtain the result. $\square$

Corollary 5.1 Similar normal operators are unitarily equivalent. This means that if $N, M \in \mathcal{B}(H)$ are normal and for some bijective $S \in B(H)$ we have $SM = NS$, then there exist a unitary U with $UM = NU$.

Proof: From the above theorem, $SM^* = N^*S$ and by taking adjoints, $MS^* = S^*N$. Combined with our assumption this yields $S^*SM = S^*NS = MS^*S$, so M commutes with $|S| = (S^*S)^{\frac{1}{2}}$. Since S is bijective, U in its polar decomposition $S = U|S|$ –is unitary. Now $UMU^* = U|S|M|S|^{-1}U^{-1} = SMS^{-1} = N$. $\square$

We say that a sequence of self-adjoint operators $T_n \in \mathcal{B}(H)$ is *monotone increasing* if $T_n \le T_{n+1}$ for all n and *bounded* if for some $C \in \mathbb{R}_+$ we have $T_n \le C\mathbf{1}$, where $\mathbf{1}((x) = x, x \in H$.We have the following result:

Lemma 6. *For all bounded monotone sequences (T_n) of self-adjoint operators there exists $T \in \mathcal{B}(H)$ such that $\|Tx - T_nx\| \to 0$ as $n \to \infty$. (Analogous result holds for monotone nets).*

Indeed, as bounded and monotone, the sequence $\langle T_nx|x\rangle$ converges in $\mathbb{R}$ for any $x \in H$. By polarisation, so do the sequences $\langle T_nx|y\rangle \in \mathbb{C}$ if $x, y \in H$, their limits define a bounded sesquilinear form corresponding to a bounded operator $T = T^*$. Hence $T_n \to T$ in WOT topology. The SOT convergence follows since

$$\|(T - T_n)x\|^2 = \langle (T - T_n)^2x|x\rangle \le \|T - T_n\|\langle (T - T_n)x|x\rangle \to 0 \quad (\text{as } n \to \infty).$$

Here we are using the obvious inequality $S^2 \le \|S\|S$ for positive elements S in commutative C*-algebras identified with $C(\Omega)$.

Definition. By **spectral measure** in a Hilbert space H defined on a sigma field $\mathfrak{M}$ of subsets of Ω we understand the mapping $E : \mathfrak{M} \to \mathcal{B}(H)$ having the following properties: $E(\emptyset) = 0$, $E(\Omega) = \infty$, $\forall_{\Delta\in\mathfrak{M}} E(\Delta)^* = E(\Delta)^2 = E(\Delta)$ and $\mu_x(\Delta) := \langle E(\Delta)x|x\rangle$ is σ-additive for all $x \in H$.

Note that by polarisation formula, also $\mu_{x,y} : \Delta \mapsto \langle E(\Delta)x|x\rangle$ are countably additive (i.e. $\mu_{x,y}$ are complex measures). From Lemma 6. we obtain countable additivity of $\Delta \mapsto E(\Delta)$ not only in WOT, but in strong operator topology as well.

[7] called also centralizers, denoted in W.Rudin's book by $\Gamma(\mathcal{R})$ instead of $\mathcal{R}'$. The reason is that "prime" is often used there to denote the dual space X' for a Banach space X, but if only operators on Hilbert spaces are considered, most authors use "prime " for commutators.

Let $L^\infty(\mathfrak{M})$ be the Banach algebra of bounded, $\mathfrak{M}$ –measurable functions $f : \Omega \to \mathbb{C}$ (with sup-norm). For $f \in L^\infty(\mathfrak{M})$ the (bounded, sesquilinear) form $H \times H \ni (x, y) \mapsto \int_\Omega f d\mu_{x,y}$ defines a bounded linear operator denoted $\int f(\omega)\, E(d\omega)$, or $\int_\Omega f\, dE$ and called the **spectral integral of** f **w.r. to** E. There is an alternative way of defining such integrals:
(a1) For characteristic functions $\int \chi_\Delta\, dE := E(\Delta)$.
(a2) This together with linearity requirement forces one to define $\int \sum_{k=1}^n c_k \chi_{\Delta_k}\, dE := \sum_{k=1}^n c_k E(\Delta_k)$.
(a3) Then using uniform approximation of f by simple functions f_n one defines $\int f\, dE$ for $f \in L^\infty(\mathfrak{M})$ as the norm-limit of the corresponding integrals of f_n.

Verification that two approaches are equivalent is simple (using Lebesgue's Dominated Convergence Theorem for $\int f_n d\mu_{x,y}$.) The correspondence $f \mapsto \int_\Omega f\, dE \in \mathcal{B}(H)$ is **not only linear, but multiplicative** as well. This key fact relies on the following property of orthogonal projections:

Lemma. If P, Q are orthogonal projections in H, then the following are equivalent:

(i) $(P + Q)^2 = P + Q$ (i.e. $P + Q$ is a projection),

(ii) $PQ = 0$,

(iii) $M := P(H) \perp Q(H) =: N$.

Proof. Since $\langle Pf|f\rangle + \langle Qf|f\rangle = \langle (P + Q)f|f\rangle \leq \|f\|^2$, (by (i), applying this for $f = Ph$, we have $\|PPh\|^2 + \|QPh\|^2 \leq \|Ph\|^2$ and since $PPh = Ph$, (i) implies that $QPh = 0$. Next, (ii) implies that for $f \in N$ we have $0 = PQf = Pf$, implying that $f = Qf \perp M$, since $\ker P = M^\perp$. Implication (iii)⇒(i) is verified by an easy computation. □

Now if $\Delta_1, \Delta_2 \in \mathfrak{M}$ are disjoint, then $E(\Delta_1) + E(\Delta_2)$ is a projection, hence $E(\Delta_1)E(\Delta_2) = 0$. Now easy computation for arbitrary $\Delta_1, \Delta_2 \in \mathfrak{M}$ with $\Delta := \Delta_1 \cap \Delta_2$ after cancelling products of values of E at disjoint pairs of sets (namely $(\Delta_1 \setminus \Delta, \Delta_2 \setminus \Delta); (\Delta_1 \setminus \Delta, \Delta); (\Delta, \Delta_2 \setminus \Delta)$) yields

$$(\ddagger) \qquad E(\Delta_1)E(\Delta_2) = \{E(\Delta_1 \setminus \Delta) + E(\Delta)\}\{E(\Delta_2 \setminus \Delta) + E(\Delta)\} = E(\Delta)^2 = E(\Delta).$$

Therefore (because $\chi_{\Delta_1} \cdot \chi_{\Delta_2} = \chi_{\Delta_1 \cap \Delta_2}$), we have multiplicativity at stage (a1) of our construction of the spectral integral. By taking intersections, we may represent a pair (g, h) of simple functions as a pair of linear combinations of characteristic functions from a common partition $(\Delta_k)_{k \leq n}$ of Ω into pairwise disjoint measurable sets: $g = \sum_{k=1}^n a_k \chi_{\Delta_k}, h = \sum_{k=1}^n b_k \chi_{\Delta_k}$ and then (taking into account only non-empty intersections) we have $fg = \sum_{k=1}^n a_k b_k \chi_{\Delta_k}$. Computing product $(\int g\, dE)(\int h\, dE)$ and again taking into account only non-empty intersections, due to $(\ddagger)$, we see that it equals $\int gh\, dE$. Passing to stage (a3) we only use continuity of the product with respect to operator norm.

Since $\int f\, dE$ depends linearly on the function f and $\int \bar{f}\, dE = (\int f\, dE)^*$ (by similar reasons - starting with simple functions), we have the following

Proposition 7. *The mapping* $\Phi : L^\infty(\mathfrak{M}) \ni f \mapsto \int f\, dE \in \mathcal{B}(H)$ *is a *-homomorphism between C*-algebras. In particular,* $\| \int f\, dE\| \leq \|f\|_\Omega := \sup\{|f(\omega)| : \omega \in \Omega\}$.

However to obtain an isomorphism we have to consider "equality almost everywhere $[dE]$". There exists a countable family $\{D_n : n \in \mathbb{N}\}$ of open disks forming a base of topology in $\mathbb{C}$. For a measurable (w.r. to $\mathfrak{M}$) function $f : \Omega \to \mathbb{C}$ we define its **essential range** to be the intersection:

$$E - \text{ess rg}(f) := \bigcap \{\mathbb{C} \setminus D_n : E(f^{-1}(D_n)) = 0\}.$$

We say that a property $P(\omega)$ holds E-a.e. (almost everywhere), if the set $N_P := \{\omega \in \Omega : P(\omega) \text{ does not hold }\}$ is contained in some $\Delta \in \mathfrak{M}$ such that $E(\Delta) = 0$. Then $E-$ess rg(f) is the smallest closed set in $\mathbb{C}$ that contains $f(\omega)$ for almost every $\omega \in \Omega$. Let $\|f\|_\infty := \sup\{|\lambda| : \lambda \in E-\text{ess rg}(f)\}$. This is a semi-norm (also sub-multiplicative) and $N_E := \{f \in L^\infty(\mathfrak{M}) : \|f\|_\infty = 0\} = \{f : f = 0 \text{ E}-a.e.\}$ is a closed ideal in $L^\infty(\mathfrak{M})$. we define $L^\infty(E)$ as the quotient algebra $L^\infty(\mathfrak{M})/_{N_E}$ normed by $\|[f]\|_\infty := \|f\|_\infty$, where $[f] = f + N_E$ is the equivalence class of f. It is easy to show that neither $\|f\|_\infty$, nor $\int f\, dE$ depend on the choice of representative f of the equivalence class $[f]$, allowing us to use simplified notation f instead of $[f]$ for elements of $L^\infty(E)$. We have the following refinement of Proposition 7 (which we leave now without proof):

Theorem 8. *Let* $E : \mathfrak{M} \to \mathcal{B}(H)$ *be a spectral measure. The mapping* $L^\infty(E) \ni f \to \Psi(f) := \int f\, dE$ *is an isometric *-isomorphism from* $L^\infty(E)$ *onto a commutative, unital C* -subalgebra of* $\mathcal{B}(H)$. *Moreover* $\|\Psi(f)x\|^2 = \int |f|^2\, d\mu_x$ *and* $\sigma(\Psi(f)) = E - \textit{ess rg}(f)$. □

From now on, $\mathfrak{M}$ will be the sigma-field of Borel subsets of a compact Hausdorff space Ω. In the case when $\Omega = \sigma(T)$, we say that E is a spectral measure for $T \in \mathcal{B}(H)$ if for $f_1(\lambda) = \lambda$ we have $T = \int_{\sigma(T)} \lambda\, E(d\lambda)$.

Our main goal is the construction of spectral measure for a normal operator T. We shall consider even more general construction representing a commutative, unital C*-subalgebra A of $B(H)$ applicable for representing commuting families of normal operators.

One of the key techniques of the proof will be the following comparison principle for two measures based on the by uniqueness part of Riesz-Markof-Kakutani theorem (RMK-for short):

Lemma C. *Given $\rho \in L^\infty(\mathfrak{M})$ and a Borel complex measure μ on Ω define $\rho \cdot \mu$ by equality $\rho \cdot \mu(\Delta) = \int_\Delta \rho \, d\mu$. Then for a Borel complex measure μ on Ω we have*

$$\nu = \rho \cdot \mu \Leftrightarrow (\forall_{f \in C(\Omega)} \int f \, d\nu = \int f \rho \, d\mu).$$

If $\nu = \rho \cdot \mu$, the last equalities hold also for all $f \in L^\infty(\mathfrak{M})$.

Theorem 9. *Let $\Omega = Sp(A)$ be the spectrum of a commutative, unital C^*-subalgebra A of $B(H)$.*
(i) Then there exists a unique Borel spectral measure E on Ω such that

$$T = \int_\Omega \hat{T} \, dE \quad \text{for any } T \in A, \tag{e9}$$

where $\hat{T} = \Gamma(T) \in C(\Omega)$ is the Gelfand transform of T.
(ii) The C^*-algebra homomorphism $\Phi : L^\infty(\mathfrak{M}) \ni f \mapsto \int f \, dE$ acting onto a closed C^*-subalgebra B of $\mathcal{B}(H)$ extends the mapping $\Gamma^{-1} : C(\Omega) \to A$.
(iii) If $U \neq \emptyset, U \subset \Omega$ is w^*- open, then $E(U) \neq 0$.
(iv) An operator $S \in \mathcal{B}(H)$ commutes with A iff[8] it commutes with all $E(\Delta)$, $\Delta \in \mathfrak{M}$.

Proof. ad (i): By Gelfand-Naimark Theorem, $\Gamma^{-1} : C(\Omega) \ni \hat{T} \mapsto T \in A$ is a *-isomorphism. Hence (using (RMK) theorem) given $x, y \in H$ we find complex Borel measures μ_{xy} on Ω such that $\int \hat{T} \, d\mu_{x,y} = \langle Tx | y \rangle$ for any $T \in A$. The correspondence: $(x, y) \mapsto \mu_{x,y}$ is linear in x and anti-linear in y, i.e. for any Borel set $\Delta \subset \Omega$ the mapping: $H \times H \ni (x, y) \mapsto \mu_{x,y}(\Delta)$ is a sesqui-linear form. This form is bounded, since $|\langle Tx|y\rangle| \leq \|T\|\|x\|\|y\|$ and $\|T\| = r(T) = \|\hat{T}\|_\Omega$. The norm of functional $C(\Omega) \ni f \mapsto \int f \, d\mu_{x,y}$ is the total variation norm of $\mu_{x,y}$. By the 1-1 correspondence between bounded sesqui-linear forms and operators, there is a unique operator $\Phi(f)$ such that $\langle \Phi(f)x|y\rangle = \int f \, d\mu_{x,y}$ for all $x, y \in H$. In particular, for any $\Delta \in \mathfrak{M}$ there exists a unique operator $E(\Delta) \in \mathcal{B}(H)$ such that

$$\langle E(\Delta)x|y\rangle = \mu_{x,y}(\Delta) \quad \text{for any } x, y \in H.$$

Since Γ and its inverse are *-morphisms, we have $\Gamma^{-1}(\bar{f}) = (\Gamma^{-1}(f))^*$ for any $f \in C(\Omega)$ and consequently $T \in A$ is self-adjoint iff $\hat{T}(\Omega) \subset \mathbb{R}$ and for such T we have $\int \hat{T} \, d\mu_{x,y} = \langle Tx|y\rangle = \langle x|Ty\rangle = \overline{\langle Ty|x`\rangle} = \overline{\int \hat{T} \, d\mu_{y,x}}$. Hence $\mu_{y,x} = \overline{\mu_{x,y}}$ (by uniqueness claim of (RMK) theorem). Hence $E(\Delta)$ are self-adjoint. Since $\mu_{x,y}$ are sigma-additive, so is $\Delta \mapsto E(\Delta)$ (in WOT-topology). Once we show that they are projections, the remaining claims concerning Φ will be consequences of Lemma 6. and Proposition 7. (To show that $B = \Phi(L^\infty(\mathfrak{M})$ is closed one needs to verify that B is also the range of Ψ and to apply isometry of Ψ asserted in Theorem 8 and completeness of $L^\infty(E)$.)

To show (iii) we argue by contradiction: suppose that $E(U) = 0$ and take a non-zero $f \in C(\Omega)$ with support contained in U. For $T_0 := \Gamma^{-1}(f)$ we have $T_0 = \int_\Omega f \, dE = \int_U f \, dE = 0$, a contradiction, since $\|T_0\| = r(T_0) \neq 0$.

ad (iv): For $S, T \in A$, we have

$$\int_\Delta \hat{S}\hat{T} d\mu_{x,y} = \langle STx|y\rangle = \int_\Omega \hat{S} d\mu_{Tx,y}.$$

Applying Lemma C and the fact that any $f \in C(\Omega)$ is of the form $\hat{S}$ for some $S \in A$, we obtain for any $f \in L^\infty(\mathfrak{M})$ that

$$\int f\hat{T} d\mu_{x,y} = \langle \Phi(f)Tx|y\rangle = \langle Tx|(\Phi(f))^*y\rangle = \int \hat{T} d\mu_{x,(\Phi(f))^*y}.$$

By analogous reason we may replace here $\hat{T}$ by $g \in L^\infty(\mathfrak{M})$ obtaining for such g that

$$\langle \Phi(fg)x|y\rangle = \int fg \, d\mu_{x,y} = \int g \, d\mu_{x,(\Phi(f))^*y} = \langle \Phi(g)x|(\Phi(f))^*y\rangle = \langle \Phi(f)\Phi(g)x|y\rangle,$$

which implies that $\Phi(fg) = \Phi(f)\Phi(g)$ for all $f, g \in L^\infty(\mathfrak{M})$. Applying this for $f = g = \chi_\Delta$, we have shown that $E(\Delta)$ (which is $\Phi(\chi_\Delta)$) is a projection.

Finally, if $S \in \mathcal{B}(H)$ commutes with any $T \in A$, we have for any $x, y \in H$ and for $z := S^*y$ that $\langle STx|y\rangle = \langle Tx|z\rangle = \int \hat{T}], d\mu_{x,z} = \langle TSx|y\rangle = \int \hat{T} d\mu_{Sx,y}$ and again by uniqueness claim of

[8]"iff" is an abbreviation of "if and only if"

(RMK) theorem, $\mu_{x,z} = \mu_{Sx,y}$. Then for any $\Delta \in \mathfrak{M}$, $\langle SE(\Delta)x|y\rangle = \langle E(\Delta)x|z\rangle = \mu_{x,z}(\Delta)$ and $\langle E(\Delta)Sx|y\rangle = \mu_{Sx,y}(\Delta)$, hence $SE(\Delta) = E(\Delta)S \Leftrightarrow \mu_{x,z} = \mu_{Sx,y}$. By reversing this argument we obtain the opposite implication: $SE(\Delta) = E(\Delta)S \Rightarrow ST = TS$ $(\forall T \in A)$. $\square$

As a corollary, we obtain spectral theorem for normal operators.

Theorem 10. *For a normal operator $T \in \mathcal{B}(H)$ there exists a unique Borel spectral measure on $\sigma(T)$ such that*

$$T = \int_{\sigma(T)} \lambda\, E(d\lambda). \tag{e10}$$

An operator $S \in \mathcal{B}(H)$ commutes with $T \Leftrightarrow SE(\Delta) = E(\Delta)S$ for any Borel set $\Delta \subset \sigma(T)$.

Proof. We apply Theorem 9. for A, the (commutative) C*-algebra generated by T, I, whose maximal ideal space is identified with $\sigma(T)$ by the homeomorphism $\hat{T}$ (the Gelfand transform of T, as in the proof of Theorem 3. Any point $\lambda \in \sigma(T)$ is the image under the bijection $\hat{T}$ of a unique element ω_λ of $\Omega := \mathrm{Sp}(A)$. For any Borel set $\Delta \subset \sigma(T)$ the set $\hat{T}^{-1}(\Delta) = \{\omega_\lambda : \lambda \in \Delta\}$ is a Borel subset of Ω and if E_A is the spectral measure for A constructed in Theorem 9., then $E(\Delta) := E_A(\hat{T}^{-1}(\Delta))$ is the spectral measure on $\sigma(T)$. Formally, E is the transport (or the image) of measure E_A under the mapping $\hat{T}$ and using for integrals of simple functions χ_Δ the obvious equality $\chi_{\hat{T}^{-1}(\Delta)} = \chi_\Delta \circ \hat{T}$ and then passing to continuous functions (which are uniform limits of simple functions) we see that

$$\int_{\sigma(T)} f(\lambda)\, E(d\lambda) = \int \Omega (f \circ \hat{T})(\omega)\, E_A(d\omega).$$

Since $f_1(\lambda) = \lambda$ acts as identity on the range of $\hat{T}$, we have $f \circ \hat{T} = \hat{T}$, the equation (e9) of the last theorem implies that $T = \int \lambda\, E(d\lambda)$.

To show the uniqueness, note first that by *-homomorphism of the spectral integrals (Proposition 7) or by Theorem 8, for $f(z) = \sum_{k=0}^{n}(a_k \bar{z}^k + b_k z^k)$ - a complex polynomial in $(z, \bar{z})$, we have $\int f(z)\, E(dz) = \sum_{k=0}^{n}(a_k (T^*)^k + b_k T^k)$ for any spectral measure E satisfying (e10). By Stone-Weierstrass Theorem, such functions are uniformly dense in $C(\sigma(T))$ implies[9] the uniqueness of such E.

If $ST = TS$, then by Theorem 5., also $ST^* = T^*S$ and consequently, S commutes with any element of $A = C^*(T, I)$. By the last part of Theorem 9, S commutes with each $E(\Delta)$. Conversely, if S commutes with $E(\Delta)$, it commutes with spectral integrals of simple functions and the latter functions approximate $f_1(\lambda) = \lambda$ uniformly on $\sigma(T)$. Hence $ST = TS$. $\square$

If $f \in L^\infty(\mathfrak{M})$, then instead of $\Phi(f)$ we will also write $f(T)$, calling Φ the extended functional calculus in a normal operator T. Since for $f_1 \in C(\sigma(T)$, where $f_1(\lambda) = \lambda$ we have $T^* = \bar{f}_1(T)$, $T^*T = (|f_1|^2)(T)$ and $\chi_\Delta(T) = E(\Delta)$ we have the following

Corollary 11. *If T is a normal operator than T is self-adjoint iff $\sigma(T) \subset \mathbb{R}$ and T is unitary iff $\sigma(T)$ is a subset of the unit circle $\partial D = \{\lambda \in \mathbb{C} : |\lambda| = 1\}$. If for $f \in C(\sigma(T)$ we define $\Delta_0 := f^{-1}(\{0\})$, then for $x \in H$ we have $\|f(T)x\|^2 = \langle (f(T))^* f(T)x|x\rangle = \int |f|^2\, d\mu_x$ and*

$$f(T)x = 0 \Leftrightarrow x \in \mathcal{R}(E(\Delta_0)) := E(\Delta_0)(H).$$

Proof. Only the last equivalence needs now to be explained. If g is the characteristic function of Δ_0, then $fg = 0$, so $0 = f(T)g(T) = f(T)E(\Delta_0)$, showing that "$\Leftarrow$" is true. If $\Delta_n := \{\lambda \in \sigma(T) : |f(\lambda)| \geq \frac{1}{n}\}$, then $\Delta_* := \bigcup_{n\in\mathbb{N}} \Delta_n = \sigma(T) \setminus \Delta_0$ and $f_n(\lambda)$ equal $\frac{1}{f(\lambda)}$ for $\lambda \in \Delta_n$ and equal 0 outside Δ_n is in $L^\infty(\mathfrak{M})$ and such that $f_n f = \chi_{\Delta_n}$, hence $f_n(T)f(T) = E(\Delta_n)$. Now if $f(T)x = 0$, then $E(\Delta_n)x = 0$ and by sigma-additivity, also $E(\Delta_*)x = 0$. Since $E(\Delta_*)x + E(\Delta_0)x = E(\sigma(T))x = x$, we obtain $x = E(\Delta_0)x$, showing "$\Rightarrow$". $\square$

In particular, if for $\lambda_0 \in \sigma(T)$ we take $f(\lambda) = f_1(\lambda) - \lambda_0$, then $E(\{\lambda_0\})$ is the projection onto $\ker(T - \lambda_0 I)$, the eigenspace of T corresponding to eigenvalue λ_0 if $E(\{\lambda_0\}) \neq 0$. Isolated points of $\sigma(T)$ (as relatively open subsets of $\sigma(T)$) are eigenvalues of T (in view of part (iii) of Theorem 9.).

[9] This uniqueness is verified first for the measures $\mu_x(\Delta) := \langle E(\Delta)x|x\rangle$