List of topics for exam[1] (2026) for "Operator Theory" course

Lemmas:
(1) For $x \in A$ if $\|\mathbf{1} - x\| < 1$, then the C.Neumann series $\sum_{n=0}^{\infty}(\mathbf{1}-x)^n$ converges to x^{-1}, so $x \in G_A$.
(2) The set G_A of invertible elements is open.
(3) Spectra $\sigma(a)$ are compact, $r(a) := \sup\{|\lambda| : \lambda \in \sigma(a)\} \le \|a\|$ for all $a \in A$. For $|\lambda| > \|a\|$

$$(\lambda\mathbf{1} - a)^{-1} = \frac{1}{\lambda}(\mathbf{1} - \frac{1}{\lambda}a)^{-1} = \sum_{n=0}^{\infty} \frac{1}{\lambda^{n+1}} a^n \qquad (*)$$

(5) Homomorphisms $\omega \in \mathrm{Sp}(A)$ are continuous, $\omega(a) \in \sigma(a)$, $|\omega(a)| \le \|a\|$ for $a \in A$. Even $\|\omega\| = 1$.
(6) The resolvent mapping $\mathbb{C} \setminus \sigma(a) \ni \lambda \mapsto R_\lambda := (\lambda\mathbf{1} - a)^{-1} \in A$ satisfies the Hilbert equation

$$R_\lambda - R_\mu = (\mu - \lambda)R_\lambda R_\mu, \quad \text{if} \quad \lambda, \mu \in \mathbb{C} \setminus \sigma(\mathrm{a})$$

and are analytic, which means $\forall \psi \in A^*$ the mappings $\lambda \mapsto f(\lambda) := \psi(R_\lambda) \in \mathbb{C}$ are analytic on $\mathbb{C}\setminus\sigma(a)$.
(7) Spectra of elements of A are non-empty.
(10) The Spectral radius formula implies that normal elements in C*-algebras satisfy $\|x\| = r(x)$.
Theorem 1 (Gelfand- Mazur) *If $A \setminus \{0\} = G_A$ (i.e. if A is a field), then $A = \{\lambda\mathbf{1} : \lambda \in \mathbb{C}\}$.*
Corollary 1.1 (Gelfand) *For commutative Banach algebras the assignment: $Sp(A) \ni \omega \mapsto \ker(\omega)$ is a bijection onto the set of all maximal ideals of A. Moreover $\sigma(a) = \{\omega(a) : \omega \in \mathrm{Sp}(A)\}$.*

Theorem 2 (Gelfand- Naimark) *For commutative C*-algebras Γ is an isometric, bijective isomorphism between A and $C(Sp(A))$, preserving multiplication and involution.* (here we may quote the earlier result that for self-adjoint elements $x = x^* \Rightarrow \omega(x) \in \mathbb{R}$ for $\omega \in \mathrm{Sp}(A)$.)
Theorem 3. *If $a \in A$ is a normal element in a unital C*- algebra then $C^*(a, 1)$ (C*- unital subalgebra generated by a) is isometrically *-isomorphic to $C(\sigma(a))$. We also have unital isomorphism of C*-algebras $\Phi : C(\sigma(a) \ni f \mapsto f(a) \in C^*(a, 1) \subset A$ that extends polynomial functional calculus in a, i.e.$\Phi(id) = a, \Phi(1) = \mathbf{1}$. Moreover*

$$\sigma(f(a)) = f(\sigma(a)),\ \|f\| = \|f(a)\| \ \text{ and } \ \widehat{f(a)} = f \circ \hat{a} \quad \text{for } f \in C(\sigma(a)).$$

Here $id(\lambda) = \lambda, 1(\lambda) = 1$ and $\|f\| := \sup\{|f(\lambda)| : \lambda \in \sigma(a)\}$.
Lemma 4. *If $a \in A \subset \mathcal{B}(H), a = a^*$ **and** $\|a\| = 1$, then the following are equivalent:*

(i) a is positive, i.e. $\sigma(a) \subset [0, +\infty)$,

(ii) $\|a - \mathbf{1}\| \le 1$,

(iii) $a = x^*x$ for some $x \in A$

Definition A linear functional $\varphi : A \to \mathbb{C}$ on a C*-algebra A is **positive**, in symbols, $\varphi \ge 0$, if its values on positive elements are non-negative: $\forall_{x\in A}\, \varphi(x^*x) \ge 0$. **States** of A are positive functionals having value 1 at $\mathbf{1}$. **Pure states** are extreme points in the set of all states on A.
Definition of spectral measure in a Hilbert space H acting on a sigma field $\mathfrak{M}$ of subsets of Ω and the related measures $\mu_{x,y}$. Definition of **spectral integral of f w.r. to E.**
Proposition 7. *The mapping $\Phi : L^\infty(\mathfrak{M}) \ni f \mapsto \int f\, dE \in \mathcal{B}(H)$ is a *-homomorphism between C*-algebras. In particular, $\|\int f\, dE\| \le \|f\|_\Omega := \sup\{|f(\omega)| : \omega \in \Omega\}$.*
Theorem 9.(ONLY BRIEF OUTLINE OF ITS PROOF). *Let $\Omega = Sp(A)$ be the spectrum of a commutative, unital C*-subalgebra A of $B(H)$.*
(i) Then there exists a unique Borel spectral measure E on Ω such that $T = \int_\Omega \hat{T}\, dE$ for any $T \in A$, where $\hat{T} = \Gamma(T) \in C(\Omega)$ is the Gelfand transform of T.
(ii) The C-algebra homomorphism $\Phi : L^\infty(\mathfrak{M}) \ni f \mapsto \int f\, dE$ acting onto a closed C*-subalgebra B of $\mathcal{B}(H)$ extends the mapping $\Gamma^{-1} : C(\Omega) \to A$.*
(iii) If $U \ne \emptyset, U \subset \Omega$ is w- open, then $E(U) \ne 0$.*
(iv) An operator $S \in \mathcal{B}(H)$ commutes with $A \Leftrightarrow$ it commutes with all $E(\Delta)$, $\Delta \in \mathfrak{M}$.
Theorem 10. =spectral theorem for normal operators.
Corollary 11. *If T is a normal operator than T is self-adjoint iff $\sigma(T) \subset \mathbb{R}$ and T is unitary iff $\sigma(T)$ is a subset of the unit circle $\partial D = \{\lambda \in \mathbb{C} : |\lambda| = 1\}$. If for $f \in C(\sigma(T))$ we define $\Delta_0 := f^{-1}(\{0\})$, then for $x \in H$ we have $\|f(T)x\|^2 = \langle (f(T))^* f(T)x|x\rangle = \int |f|^2\, d\mu_x$ and*

$$f(T)x = 0 \Leftrightarrow x \in \mathcal{R}(E(\Delta_0)) := E(\Delta_0)(H).$$

Proposition 12 If E is the spectral measure for a normal operator $N = \int zE(dz)$, then N is compact iff for any $\delta > 0$ the projection $P_\delta := E(\{z \in \mathbb{C} : |z| > \delta\}$ has finite rank.

[1]This procedure applies when a student wants to raise his proposed mark (based on his long-term performance during tutorials in this semester)