Notes to "Operator Theory" tutorials 9.XI'21

Remaining from the previous tutorials:

- Orthonormal sequences are linearly independent
- The diagonal operator defined by a scalar sequence $\{\alpha_n\}$ in an orhonormal basis $\{e_n\}_{n\in\mathbb{N}}$, denoted $diag(\alpha_n)$ is defined as the only bounded linear operator T such that $Te_n = \alpha_n e_n$ for any $n \in \mathbb{N}$. Show that this operator has the norm equal to $\|\{\alpha_n\}\| := \sup_n |\alpha_n|$ and find when it is compact (resp. when it is a Hilbert-Schmidt operator). Are diagonal operators noormal?

Recall that $N \in \mathcal{B}(H)$ is a normal operator if $N^*N = NN^*$, which is in turn equivalent to $\|T^*x\| = \|Tx\|, x \in H$. Also this implies the normality of $N - \lambda I$ (for all $\lambda \in \mathbb{C}$).

(1) Let $M \subset H$ be a closed subspace and denote by $P = P_M$ the orthoprojection onto M. As we know, $P^2 = P = P^*$ and $\|Px\| = \|x\| \Leftrightarrow x \in M$. We say, that M *is invariant for* $T \in \mathcal{B}(H)$ if $Tx \in M$ for all $x \in H$. If M is invariant both for T and for T^*, M is called *a reducing subspace for* T. Show that M is invariant $\Leftrightarrow TP = PTP$, while M reduces T $\Leftrightarrow TP = PT$

(2) Let M be an invariant subspace for a normal operator $T \in \mathcal{B}(H)$. Show that M is reducing for T iff the restriction $T|_M$ is a normal operator on M. (Use the metric condition for normality)

(3) Show that M is a reducing subspace for $T \Leftrightarrow$ both M and $M^\perp$ are invariant for T.

(4) Are the eigenspaces $\mathcal{M}(T - \lambda I)$ always invariant (resp. reducing) for T ? What is the answer for normal operators?

(5) The numerical range $W(T) := \{\langle Tx, x\rangle : \ \|x\| = 1\}$ is a set of complex numbers of modulus $\leq \|T\|$. Show that if $T = T^*$ then $W(T) \subset \mathbb{R}$ (in the sense that the imaginary part of $\langle Tx, x\rangle$ must be zero).

(6) If $\lambda \in \sigma_a(T)$, so that $\exists_{\|x_n\|=1} \|Tx_n - \lambda x_n\| \to 0$, then $\lambda \in \overline{W(T)}$. Conclude that the spectrum of self-adjoint T must be real.

(7) If $S = S^* \geq 0$, which means that $W(S) \subset [0, +\infty)$, show that the sesquilinear form $q_S(x, y) := \langle Sx, y\rangle$ obeys the Schwarz inequality: $|q_S(x, y)|^2 \leq q_S(x, x)q_S(y, y)$. (Copy the proof from the case when $S = I$, using non-positivity of certain discriminant Δ.)

(8) Assume from now on that $T = T^* \in \mathcal{B}(H)$. If $M = \sup W(T)$, applying the above inequality for $Sx := Mx - Tx$ estimate $\sup\{q_S(x, y) : \|y\| = 1\} = \|Sx\|$ by some constant times $\sqrt{q_S(x, x)}$. Then taking as x the values from (x_n) show, that $M \in \sigma_a(T)$ (The same holds for $m := \inf W(T)$).

(9) By considering $Te_n = \frac{1}{n}e_n$ on the canonical orthonormal basis in $H = \ell^2$ show, that $W(T)$ need not be closed (despite of the separate continuity of the inner product and of Q_T in the weak topology of H and weak compactness of the closed ball in H)

(10) Let $y_x := \|Tx\|^{-1}Tx$ when $Tx \neq 0$. Then $\|T\| = \sup_{\|x\|=1}\langle Tx, y_x\rangle$ and the inner products appearing here are ≥ 0, real-valued. Hence (bearing in mind that $\langle Tx, y_x\rangle \in \mathbb{R}_+$) from the "real part polarisation formula" we have

$$\langle Tx, y_x\rangle = \frac{1}{4}\big(q_T(x + y_x, x + y_x) - q_T(x - y_x, x - y_x)\big) \leq \frac{1}{4}w(T)(\|x + y_x\|^2 + \|x - y_x\|^2).$$

Here $w(T) := \max(|m|, |M|)$ is called the **numerical radius** of $T = T^*$. Deduce, that for selfadjoint $T \in \mathcal{B}(H)$ we have

$$\|T\| = w(T) = \sup\{|\lambda| : \lambda \in \sigma(T)\}.$$

The last quantity, denoted $r(T)$ is called the **spectral radius of** T.

(11) If A is any bounded linear operator on H. Then there are two selfadjoint operators: T and T_1 such that $A = T + iT_1$ Moreover $W(T)$ (respectively $W(T_1)$)consists of the real (resp. imaginary) parts of the points from W(A). Here $T := \frac{1}{2}(A + A^*)$ is called the "real part of A" Deduce that $\|A\| \leq 2w(A)$.

Note that there is no constant C such that $\|A\| \leq Cr(A)$, since for the Volterra integral operator $r(V) = 0$. Here $(Vf)(t) := \int_0^t f(s)\,ds$ and $V \neq 0$.