Orthogonal projections, strong operator convergence. Let μ be some measure on a sigma- algebra $\mathcal{B}$ of subsets of some space T. If we write $\mu(A)$ -this will also mean the measurability of A. Assume either that μ is σ-finite or that if $\mu(A) > 0$ implies that $0 < \mu(B) < \infty$ for some $B \subset A$. Let us recall that the multiplication by $\varphi \in L^\infty(\mu)$ operator M_φ is defined on $L^2(\mu)$ -spaces by the formula

$$(M_\varphi f)(t) := \varphi(t)f(t), \qquad f \in L^2(\mu).$$

Clearly, such M_φ are bounded, linear and normal operators:

$$\|M_\varphi\| = \|\varphi\|_\infty,\ \ M_{\bar{\varphi}} = (M_\varphi)^*,\ \ M_\psi M_\varphi = M_{\psi\cdot\varphi} = M_\varphi M_\psi,\ \ M_1 = I.$$

In other words, the mapping which assigns to $\varphi \in L^\infty(\mu)$ the operator M_φ is an isometric homomorphism between this Banach algebra and $\mathcal{B}(L^2(\mu)$ preserving the involutions and unit elements. Here we identify functions that are *equal almost everywhere* $[\mu]$ and 1 denotes the (class of) constant function equal 1 a.e.$[\mu]$.

(1) For which functions φ is M_φ a projection?
(2) For which functions φ is M_φ a self-adjoint operator?
(3) For a pair of orthogonal projections $P_j := M_{M_j}$ onto closed subspaces M_j of a given Hilbert space H show that P_1P_2 is a projection onto some subspace M iff the P_j commute. What is the relation between M_1, M_2 and M ?
(4) If $P_1 + P_2$ is a projection of the form P_L then we must have$L = M_1 \oplus M_2$,i.e.

$$M_1 \perp M_2 \quad \text{and} \quad M_1 + M_2 = L.$$

(5) $M_1 \subset M_2 \Leftrightarrow P_1 \leq P_2$ and in such case for $M_2 \ominus M_1 := M_2 \cap (M_1^\perp)$ we have

$$P_2 - P_1 = P_{M_2 \ominus M_1}.$$

(6) If M is the linear span of $\bigcup_{n\in\mathbb{N}} M_n$, where $\forall_n M_n \subset M_{n+1}$, show that $P_n := P_{M_n}$ strongly converge to P_M. (Hint: we may use here the fact that the intersection of the orthocomplements: $\bigcap_{n\in\mathbb{N}}(M_n^\perp)$ equals $M^\perp$).
(7) Show that from the strong convergences $S_n \to S$ and $T_n \to T$ we can deduce the strong convergence of the products: $S_nT_n \to ST$.
(8) One can show that for nets (=generalised sequences) of operators S_α converging strongly to 0 we may have S_α^2 **not converging** (SOT) to 0. Here the key observation is that 0 is in the weak -topology -closure of $\{\sqrt{n}e_n\}$ for any orthonormal sequence (e_n). `In fact, let` $U = \{x \in H : \langle x, y_j\rangle < \epsilon, j = 1, \ldots, k\}$ `be a basic weak neighbourhood of 0 in` H. `Here` $\epsilon > 0$ `and` $y_1, \ldots, y_k \in H$ `are fixed. Since` $(\sum_{j=1}^k |\langle e_n, y_j\rangle|)^2$ `are summable over` $n \in \mathbb{N}$, `we cannot have` $\sum_{j=1}^k |\langle e_n, y_j\rangle| \geq \frac{\epsilon}{\sqrt{n}}$ `for all` n `(this raised to power 2 would give the divergent harmonic series` $\sum_n \frac{\epsilon^2}{n}$. `Hence for some` m `we must have` $\sum_{j=1}^k |\langle e_m, y_j\rangle| < \frac{\epsilon}{\sqrt{m}}$, `implying that` $\forall_{j=1,\ldots,k}$ $|\langle e_m, y_j\rangle| \leq \frac{\epsilon}{\sqrt{m}}$, `so that` $\sqrt{m}e_m \in U$ `for our` m.

Now taking $\sqrt{n_\alpha}e_{n_\alpha}$- a net weakly convergent to zero, let

$$S_\alpha := \sqrt{n_\alpha}e_{n_\alpha} \otimes e_{n_\alpha}.$$

For any fixed vector $y \in H$ this weak convergence implies $\sqrt{n_\alpha}\langle y, e_{n_\alpha}\rangle \to 0$, hence $\|S_\alpha y\| \to 0$, showing the strong convergence to 0 of S_α.

But $S_\alpha^2 = n_\alpha e_{n_\alpha} \otimes e_{n_\alpha}$ and for $f := \sum_{n=1}^\infty \frac{1}{n}e_n$ we have $S_\alpha^2 f = e_{n_\alpha}$ -a net of vectors having norms 1, **not convergent in norm to 0**.

(9) $V \in \mathcal{B}(H)$ is a partial isometry[1] iff V^*V is a projection. Then so is VV^*.

[1]This means that V restricted to orthogonal complement of $\mathcal{N}(V)$ is an isometry