1. Example of strange sequences of Hilbetr space operators

Obviously, the multiplication in any normed algebra is jointly continuous and the algebra $\mathcal{B}(H)$ of bounded linear operators is no exception. But there are two other operator topologies:

- the *strong operator topology* (denoted $\mathtt{SOT}$) defined by the family of seminorms $\{p_x : x \in H\}$, where $p_x(T) := \|T(x)\|$,
- the *weak operator topology* (denoted $\mathtt{WOT}$) defined by the family of seminorms $\{p_{x,y} : x, y \in H\}$, where $p_{x,y}(T) := |\langle T(x), y\rangle|$.

Neither of them is metrizable (but their restrictions to bounded sets of operators are). We shall consider only the corresponding convergences of sequences. Clearly, for $T_n, T \in \mathcal{B}(H)$

- the uniform convergence, meaning $\lim \|T_n - T\| = 0$ implies
- the strong convergence: $T_n \to T$ $(\mathtt{SOT})$, meaning $\|T_n x - Tx\| \to 0\ \forall_x$, which implies
- the weak convergence: $T_n \to T$ $(\mathtt{WOT})$, meaning $\langle T_n x, y\rangle \to \langle Tx, y\rangle\ \forall_{x,y\in H}$.

and none of the implications is reversible. It turns out, that **the multiplication is discontinuous -but is sequentially continuous!**

Example 1. Let $\{e_n,\, n \in \mathbb{N}\}$ be the canonical 0-1 basis for ℓ^2 and put

$$H_k = \mathtt{span}\{e_1, \dots, e_k\}, \qquad R_k = H_k^{\perp} = \{f \in \ell^2 : \forall_{j\leqslant k} f \perp e_j\}.$$

Here we construct a sequence $T_n \in \mathcal{B}(\ell^2)$ convergent pointwise to I (the identity operator)on the the linear span of the basis $(= \bigcup_{k=1}^{\infty} H_k)$ and such that $T_n^2 = 0$.

Define $T_n(e_j) = e_j + \frac{1}{n} e_{j+n}$ for $1 \leqslant j \leqslant n$ and $T_n(e_j + \frac{1}{n} e_{j+n}) = 0$ for $j > n, j \leqslant 2n$. Then T_n extends to a linear operator $T_n : H_{2n} \to H_{2n}$. Indeed, note that the systems: $(e_1, \dots, e_n)$ and $(e_j + \frac{1}{n} e_{j+n})_{1\leqslant j\leqslant n}$ are jointly linearly independent. If we define T_n to be zero on R_{2n}, then this is a direct sum of a finite- dimensional (hence continuous) -and zero -operators. Clearly, $T_n^2 = 0$. Moreover, $\|(T_n - I)x\|$ is zero for $x \in H_n$. To get the strong convergence to I (which actually fails here) one would need the following.

Lemma *If a uniformly bounded sequence of operators $T_n \in \mathcal{B}(H)$ satisfies $T_n y \to Ty$ for any y from a linearly dense set Y, then $T_n \to T$ (SOT).*

Indeed, The assumed convergence can be extended to the linear span of S, since finite linear combinations can be "interchanged" with the limit operation, due to the linearity of the T_n's. Let us fix an arbitrary $\epsilon > 0$. Hence we may assume "w.l.o.g." that

$$\forall_{y\in Y} \exists_{N(y)} \forall_{n\geqslant N(y)} \|T_n y - Ty\| < \frac{\epsilon}{3},$$

where Y is dense in H. Given a fixed $x \in H$ we find $y \in Y$ sufficiently close to x to estimate:

$$\|T_n x - Tx\| \leqslant \|T_n x - T_n y\| + \|T_n y - Ty\| + \|Ty - Tx\| < \epsilon$$

provided that $\|x - y\| \leqslant \frac{\epsilon}{3M}$, where $M = \max(\|T\|, \sup_n \|T_n\|)$ and $n \geqslant N(y)$.

Note that the same method is used to show the Riemann-Lebesgue Lemma -using as Y the space $L^2[-\pi, \pi]$ dense in $L^1[-\pi, \pi]$.

Exercises: (0) Check the SOT continuity of multiplication on bounded sets of operators(1) Conclude, that $\|T_n\|$ are unbounded! (2) By starting from a generic SOT-neigbourhood Uof I :

$$U(I, f_1, \dots, f_k; \epsilon) := \{S : \|(S - I)f_j\| < \epsilon \forall_{j\leqslant k}\}$$

note that one can assume "w.l.o.g." that the f_j are orthonormal. Slightly modifying our example, get the discontinuity. (Hint:) work on (2) first, its conclusion is needed for solving (1).

The operation of **taking the adjoints is also strongly -discontinuous!** (but it is (WOT)-continuous).

Example 2. This time one uses the unilateral shift S acting on the above basis $(e_n)_{n\geqslant 1}$ by the formula: $Se_n = e_{n+1}$. If $P_k =$ the orthoprojection onto the defined above subspace R_2, so that e.g. $P_1 e_1 = 0$ and $\forall_{j\geqslant 2} P_1 e_j = e_j$, then $T_k := (S^*)^k = P_k (S^*)^k$ is zero on H_k and satisfies $T_k(e_n + k) = e_n$ (mapping isometrically R_k onto ℓ^2). From the above lemma, or otherwise, we easily obtain the strong convergence of T_n to 0. But $T_n^* = S^n$ sends e_1 to the orthonormal sequence (e_{n+1}) -without any convergent subsequence (it fails the Cauchy condition as $\|e_n - e_m\| = \sqrt{2}$ for any $n \neq m$.) Hence we see that $T_n \to 0(\mathtt{SOT}) \not\Rightarrow T_n^*$ is (SOT)-convergent.

If $T_n = T_n^*$ and $T_n \geqslant T_{n+1} \geqslant \cdots \geqslant 0$, then we call it a monotone *decreasing operator sequence* . Such sequences are strongly convergent and muliplying term -by term two such sequences, we have $S_n T_n \to ST$ $(\mathtt{SOT})$, if $S_n \to S$ and $T_n \to T$ strongly and monotonically. (Hint: $\|S_n x - Sx\|^2 \leqslant \|S_n - S\| \langle (S_n - S)x, x\rangle$.)