Notes to "Operator Theory" tutorials 19.X'21

In the previous tutorials we have checked some properties of ran-one operators on a Hilbert space H. Such operators can be written as $u \otimes w$ for some $u, w \in H \setminus \{0\}$, where $u \otimes w$ maps a vector $x \in H$ into $(u \otimes w)(x) := \langle x, w \rangle u$. The representation is not unique (take $u_1 = 2u, w_1 = \frac{1}{2}w$ for example. We have shown that:

- $\|u \otimes w\| = \|u\| \cdot \|w\|$
- $(u \otimes w)^* = w \otimes u$. Hence $u \otimes u$ is self-adjoint (and positive).
- any rank-one bounded linear operator is of this form
- any finite -rank bounded linear operator is a finite sum of the form $\sum_{j=1}^{n} u_j \otimes w_j$
- If A is a linear operator then $A \circ w \otimes u = (Aw) \otimes u$,
- $(w \otimes u) \circ A = w \otimes (A^* u)$
- finite-rank bounded linear operators are a two-sided ideal in $\mathcal{B}(H)$

We also know that finite-rank operators from $\mathcal{B}(H)$ are compact. This can be checked in a number of ways (try it!)

0.1 Next tutorials

Given a basis (= algebraic basis!) $(e_j)_{j \in J}$ the dual system of functionals $(e_j^*)_{j \in J}$ is defined by

$$e_j^*(x) := \alpha_j, \ \texttt{so that} \ x = \sum_{j=1}^{m} e_j^*(x) e_j, \tag{1}$$

where the summation ranges through some finite subset $\{j_1, \ldots, j_m\}$ of the set J of indices. From linear algebra we know that these functionals e_j^* are linear. They just describe the coordinates of a vector x with respect to the given basis.

1. In the Euclidean space $\mathbb{K}^n$ the norm is given by $\|\vec{x}\| = \sqrt{|x_1|^2 + \ldots + |x_n|^2}$. Here $x_j = \epsilon_j^*(\vec{x})$ are the coordinates of $\vec{x}$ in the canonical 0-1 basis $(\epsilon_1, \ldots, \epsilon_n)$. Any linear operator $T : \mathbb{K}^n \to Y$ can be represented as $T = \sum_{j=1}^{n} \epsilon_j^* T \epsilon_j$. Deduce the continuity of such T. Express the matrix entries a_{jk} in terms of the basis vectors, T and the dual basis functionals only.

2. Assume for a while that $\dim(X) < \infty$. Show that if $(e_1, \ldots, e_n)$ form a basis of X, then the dual system: $(e_1^*, \ldots, e_n^*)$ is a basis of X^*, hence $\dim(X) = \dim(\mathcal{L}(X, \mathbb{K}))$.

3. In the infinitely dimensional case let $G = (e_j : \ j \in J)$ be a basis of X. Show that the dual system is linearly independent, but it fails to span the algebraic dual space $\mathcal{L}(X, \mathbb{K})$. In this case $\dim(\mathcal{L}(X, \mathbb{K})) = 2^{\dim(X)}$. Here $\mathcal{L}(X, \mathbb{K}) = \{\varphi : X \to \mathbb{K} : \varphi \ \text{ is linear } \}$.

4. Orthonormal sequences are linearly independent (but fail to span algebraically the whole ∞-dimensional Hilbert pace H. We can add some set $\{e_\alpha : \alpha \in A\}$, where $A \cap \mathbb{N} = \emptyset$ to obtain a Hamel (=algebraic) basis in H. Then show that the coordinate functionals for such "newly added" e_α, namely e_α^* cannot be continuous.

5. The diagonal operator defined by a scalar sequence $\{\alpha_n\}$ in an orhonormal basis $\{e_n\}_{n \in \mathbb{N}}$, denoted $diag(\alpha_n)$ is defined as the only bounded linear operator T such that $Te_n = \alpha_n e_n$ for any $n \in \mathbb{N}$. Show that this operator has the norm equal to $\|\{\alpha_n\}\| := \sup_n |\alpha_n$ and find when it is compact (resp. when it is a Hilbert-Schmidt operator).

We are going to prove that: the norm closure of finite-rank operators in $\mathcal{B}(H)$ is precisely the set of all compact operators. This set is a two-sided (and norm-closed) ideal in $\mathcal{B}(H)$. This happens not only in Hilbert spaces, but in all separable Banach spaces admitting a Schauder basis (call it $\{e_n\}_{n \in \mathbb{N}}$).

Hints: Let us define $S_n := \sum_{j=1}^{n} e_j \otimes e_j$. Then $S_n \to I$ as $n \to \infty$.

- norm -limits of compact operators are compact
- the identity operator, $I : H \ni x \mapsto x \in H$ is a strong limit of finite rank operators S_n
- If $A_n \to A$ in (SOT) and T is a compact operator, check that $\|A_n T - AT\| \to 0$
- If $A_n \to A$ in (SOT) and T is a compact operator, then $\|TA_n - TA\| \to 0$. (You may need:

Theorem 0.1 *(J.P.Schauder) The adjoint T^* of a compact operator (in a Banach space) is compact (there is even the equivalence here)*