Banach Limits

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We are now in a position to prove the convergence of $\sum S_i z^i$. We will show that $\|S_i\| \leqq N^i M(M+1)^{i-1}$. For $i=1$, $(1-\tilde{A}_0+\tilde{A}_0')S_1=A_1$ so by Lemma 3, $\|S_1\| \leqq M\|A_1\| \leqq MN$. Now suppose that the statement holds for $i=1, \cdots, n-1$. Then since:

$$(k\tilde{I} - \tilde{A}_0 + \tilde{A}_0')S_n = A_1 S_{n-1} + \cdots + A_n$$

$$\begin{aligned}
\|S_n\| &\leqq M\|A_1 S_{n-1} + \cdots + A_n\| \\
&\leqq M\{NM(M+1)^{n-2}N^{n-1} + \cdots + N^n\} \\
&= MN^n\{M(M+1)^{n-2} + \cdots + 1\} \\
&= MN^n\{1 + M + M(M+1) + \cdots + M(M+1)^{n-2}\} \\
&= MN^n(1+M) + M^2N^n\left\{\frac{(M+1) - (M+1)^{n-1}}{1-(M+1)}\right\} \\
&= N^n M(M+1)^{n-1}
\end{aligned}$$

as asserted. Then trivially $\sum S_n Z^n$ converges if $|z| < 1/(N(M+1))$. Furthermore $\sum S_n z^n$ is nonsingular at $z=0$ and hence by continuity is nonsingular in some neighborhood of $z=0$.

REMARKS. Since the proof of the theorem clearly hangs on Lemmas 2 and 3, it is probably a good idea in presenting the proof to students to take a matrix A_0 in Jordan form and exhibit the matrices $\tilde{A}_0'$ and $\tilde{A}_0$ and $k\tilde{I}-\tilde{A}_0+\tilde{A}_0'$, since this makes the two lemmas completely obvious.

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BANACH LIMITS

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The purpose of this note is didactic; we show how Banach limits, together with the identification of their extremal values [5] and a theorem of Lorentz [4] giving conditions for their uniqueness, may be simply introduced to an audience armed only with an old-fashioned version of the Hahn-Banach theorem. An elementary proof of Lorentz's theorem may be found in the book by Goffman and Pedrick [2], the chief difference of our approach being in the use of a simpler expression for the functional p.

DEFINITION. *Banach limits (or Banach-Mazur limits, or generalized limits) are linear functionals L on the space B of bounded sequences of real numbers* $(x_n)_{n=0}^{\infty}$, *satisfying the conditions:*

$$\text{(i)} \quad L(x_n) \geqq 0 \quad \textit{if} \quad x_n \geqq 0, \qquad n = 0, 1, \cdots,$$

(ii) $L(x_{n+1}) = L(x_n)$,

(iii) $L(1) = 1$.

A word about notation: $L(x_n)$ is of course L at $(x_n)=(x_0, x_1, \cdots)$. $L(x_{n+1})$ is L at $(x_1, x_2, \cdots)$. $L(1)$ is L at $(1, 1, \cdots)$.

THEOREM. (α) *Banach limits exist.* (β) *The maximal value of Banach limits on a sequence* (x_n) *is*

$$M(x_n) \overset{\text{def}}{=} \lim_{n\to\infty}\left(\sup_j\ n^{-1}\sum_{i=0}^{n-1} x_{i+j}\right). \tag{1}$$

(γ) *The minimal value of Banach limits on a sequence* (x_n) *is*

$$\lim_{n\to\infty}\left(\inf_j\ n^{-1}\sum_{i=0}^{n-1} x_{i+j}\right) \tag{2}$$

(δ) *A necessary and sufficient condition in order that all Banach limits on a sequence* (x_n) *agree and equal* s *is that*

$$\lim_{n\to\infty} n^{-1}\sum_{i=0}^{n-1} x_{i+j} = s$$

uniformly in j. *Hence on convergent sequences Banach limits agree with limits.*

Proof. First we prove that the limit as $n\to\infty$ in the expression (1) for M exists. (Another proof of this is in [**5**] incorporated in the proof that M equals the expression used for p in the original argument of Banach, [**1**], p. 34.) Set

$$c_n = \sup_j \frac{1}{n}\sum_{i=0}^{n-1} x_{i+j}.$$

We are to show that $\lim c_n$ exists. While (c_n) is not monotone, for each k, m one has $c_{km}\leqq c_m$. Thus

$$(r+km)c_{r+km} \leqq rc_r + kmc_{km} \leqq rc_r + kmc_m.$$

Dividing by $r+km$ and letting $k\to\infty$ with r, m fixed, we obtain

$$\limsup_{k\to\infty} c_{r+km} \leqq c_m.$$

Since this holds for $r=1, 2, \cdots, m$, $\limsup c_n\leqq c_m$ for each m, and hence $\limsup c_n\leqq\liminf c_m$ which implies that $\lim c_n$ exists.

THE HAHN-BANACH THEOREM ([**1**], p. 27). *Let* B *be a linear space and* C *a subspace of* B. *Let* p *be a sublinear functional on* B: $p(x+y)\leqq p(x)+p(y)$ *and if* $a\geqq 0$, $p(ax)=ap(x)$. *Then if* f *is a linear functional on* C *with*

$$f(x) \leqq p(x) \tag{3}$$

for $x\in C$, f *can be extended to* B *so that* (3) *holds for* $x\in B$. *If* $y\in B-C$, *then the* *process of extension may begin by ascribing to* $f(y)$ *any value in the closed interval with left endpoint*

$$\sup_{x\in C}\,[-p(-x-y)-f(x)] \tag{4}$$

and right endpoint

$$\inf_{x\in C}\,[p(x+y)-f(x)]. \tag{5}$$

We let B be the space of bounded sequences $x=(x_n)$ and we let C be the subspace of convergent sequences. The Hahn-Banach theorem is now applied with $p=M$ and $f=\lim$ on C. It is easy to see that if $\lim x_n=s$ then the Cesàro means of x_n converge to s, i.e. $\lim_{n\to\infty} n^{-1}(x_0+\cdots+x_{n-1})=s$ and also $M(x_n)=s$.

Proof of (α). We show that linear functionals $f=L$ obtained on B by the extension procedure satisfy (i), (ii) and (iii). For (iii) this is obvious, since each L coincides with lim on C. (i) follows from

$$L(x_n)=-L(-x_n)\geqq -p(-x_n)=-M(-x_n)\geqq 0$$

if $x_n\geqq 0$ for all n.

Finally, (ii) follows from the "telescoping" property of M

$$\left|M(x_{n+1}-x_n)\right| = \left|\lim_n[\sup_j n^{-1}(x_{j+n}-x_j)]\right| \leqq \lim_n 2n^{-1}\sup_j |x_j| = 0,$$

$$\left|M(x_n-x_{n+1})\right| = 0,$$

and the relations $L(x_{n+1}-x_n)\leqq M(x_{n+1}-x_n)=0$ and

$$L(x_{n+1}-x_n)=-L(x_n-x_{n+1})\geqq -M(x_n-x_{n+1})=0.$$

Remark. If instead of M, lim sup x_n were used as p, then the generalized limits thus obtained would not satisfy (ii) because lim sup does not have the "telescoping" property. However, lim sup of Cesàro averages could be used for p, this construction yielding some, but not all, Banach limits.

Proof of (β). We first show that given a sequence (y_n), there is a Banach limit L_0 such that $M(y_n)=L_0(y_n)$. We may assume $(y_n)\in B-C$. We construct L_0 by beginning the extension procedure with (y_n), defining $L_0(y_n)$ by (5):

$$L_0(y_n)=\inf_{(x_n)\in C}\,[M(x_n+y_n)-\lim x_n]$$

which, equals $M(y_n)$, because the convergence of (x_n) implies $M(x_n+y_n)=\lim x_n+M(y_n)$. We now show that conversely, for each sequence (y_n) and each Banach limit L, $L(y_n)\leqq M(y_n)$. Note that $\sup_j z_j-z_n\geqq 0$, $n=0, 1, \cdots$ implies by (i) and (iii) that always $\sup_j z_j\geqq L(z_n)$. If for a fixed m we let

$$z_n=m^{-1}\sum_{i=0}^{m-1} y_{i+n},$$

we have $L(y_n)=L(z_n)\leqq \sup_j z_j$. On letting $m\to\infty$, we obtain $L(y_n)\leqq M(y_n)$, which completes the proof of (β).

Proof of (γ). Apply (β) to the sequence $(-x_n)$.

Proof of (δ). (δ) is an immediate corollary of (β) together with (γ).

Banach limits, their maximal values, and sequences (called "almost convergent") on which all Banach limits agree have found applications in Ergodic Theory (cf. [5] where also further references are given). In the opposite direction, an interesting application of Ergodic Theory to Banach limits is due to Jerison [3].

In conclusion, we wish to acknowledge our debt to the paper [4] of Lorentz, and to a conversation with Professor Alfred Rényi. The author's work is in part supported by NSF Grant GP-1458.

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A FURTHER EXTENSION OF OLIVIER'S THEOREM

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1. The extension. In [1], Štěpánek derived an extension of Olivier's theorem [2]. In this paper we shall use Štěpánek's method of proof to derive the following extension of Štěpánek's theorem.

THEOREM. *Let* $\sum_{n=1}^{\infty} a_n$ *be a convergent series of real numbers,* $\sum_{n=1}^{\infty} 1/\alpha_n$ *a divergent series of positive numbers, and* (β_n) *a sequence such that* $(\alpha_{n+1}-\alpha_n+\beta_n)$ *is a bounded monotone sequence and one of the following conditions holds*:

$$\text{(a) } \alpha_n a_n \geqq (\alpha_n+\beta_{n-1})a_{n-1}[n\geqq 2] \quad \text{or} \quad \text{(b) } \alpha_n a_n \leqq (\alpha_n+\beta_{n-1})a_{n-1}[n\geqq 2].$$

Then $\lim \alpha_n a_n=0$.

Proof. If the inequality (a) or the inequality (b) is multiplied by -1 it is immediately evident that we need only prove the theorem for the case (a) or for the case (b). We shall prove the theorem under the hypothesis that (a) holds.

Following Štěpánek, we define $r_1=s_1=0$, and for each integer $n\geqq 2$ we define:

$$r_n = \alpha_n a_n \text{ and } s_n = -\sum_{i=1}^{n-1}(\alpha_{i+1}-\alpha_i+\beta_i)a_i.$$

For each $n\geqq 1$ we define $h_n=r_n+s_n$.

Since $\sum_{n=1}^{\infty} a_n$ converges and $(\alpha_{n+1}-\alpha_n+\beta_n)$ is bounded and monotone,