Exercises

1. Suppose $R(K)$ is a Dirichlet algebra and $a \in \operatorname{int} K$. Define S on $R^2(K, \omega_a)$ by $Sf = zf$. Find $\sigma(S)$ and $\sigma_e(S)$.
2. Repeat Exercise 1, but this time assume that K is finitely connected.
3. Let ω_α, ω_β, $\mathscr{H}$, and Z be as in Lemma 16.3 and show that if f and g are functions in $\mathscr{H} \cap L^\infty(\omega_\alpha)$, then $fg \in \mathscr{H}$ and $\int fg\, d\omega_\beta = (\int f\, d\omega_\beta)(\int g\, d\omega_\beta)$.

§**17 Bands of measures.** In this section we will develop the elementary properties of bands of measures. This concept will be the basis for an abstract F. and M. Riesz Theorem which will be proved in the next section.

17.1 Definition. If X is a compact space, a *band of measures* is a norm closed linear subspace $\mathscr{B}$ of $M(X)$ such that if $\mu \in \mathscr{B}$ and ν is a measure on X that is absolutely continuous with respect to μ, then $\nu \in \mathscr{B}$.

Perhaps a word to the cautious is worth uttering here. Let's agree (as all sane mathematicians do) that to say that two complex-valued measure ν and μ satisfy $\nu \ll \mu$ means that $|\nu| \ll |\mu|$. That is, $|\nu|(\Delta) = 0$ for every Borel set E with $|\mu|(\Delta) = 0$.

17.2 Examples. (a) $M(X)$ and (0) are bands. Call these the *trivial bands.*

(b) If μ is a positive measure on X, then $L^1(\mu)$ can be identified with a closed subspace of $M(X)$ by means of the Radon-Nikodym Theorem. That is, $L^1(\mu) = \{\nu \in M(X) : \nu \ll \mu\}$. With this identification, $L^1(\mu)$ is a band.

(c) The collection of purely atomic measures on X is a band.

(d) The collection of completely nonatomic measures is a band.

The proof of the first result is an easy exercise.

17.3 Proposition. *If $\mathscr{B}$ is a band of measures on X and $\mu \in \mathscr{B}$, then the following statements hold.*

(a) *If Δ is any Borel set, $\mu|\Delta \in \mathscr{B}$.*

(b) *If $\mu = (\mu_1 - \mu_2) + i(\mu_3 - \mu_4)$ is the Jordan decomposition of μ, then $\mu_j \in \mathscr{B}$ for $1 \le j \le 4$.*

(c) $|\mu| \in \mathscr{B}$.

The next theorem is a generalization of the Lebesgue Decomposition Theorem. Indeed, the proof is the same.

17.4 Theorem. *If $\mathscr{B}$ is a band of measures on X and $\nu \in M(X)$, then $\nu = \nu_a + \nu_s$, where $\nu_a \in \mathscr{B}$ and $\nu_s \perp \mu$ for every μ in $\mathscr{B}$. The measures ν_a and ν_s are unique.*

Proof. If $\nu \perp \mu$ for every μ in $\mathscr{B}$, then we are done. So assume the contrary. Thus there is a Borel set F such that $|\nu|(F) > 0$ and $\nu|F \ll \mu$ for some μ in $\mathscr{B}$. Note that this implies that $\nu|F \in \mathscr{B}$. Let $c = \sup\{|\nu|(F) : F$ is a Borel subset of X and $\nu|F \in \mathscr{B}\}$; by our assumption,

$c > 0$. It follows that there is an increasing sequence $\{F_n\}$ of Borel sets such that $\nu|F_n \in \mathscr{B}$ and $|\nu|(F_n) \to c$. If $F = F_1 \cup F_2 \cdots$, then the fact that $\mathscr{B}$ is norm closed implies $\nu|F \in \mathscr{B}$ and $|\nu|(F) = c$.

Let $\nu_a = \nu|F$ and $\nu_s = \nu - \nu_A = \nu|(X\backslash F)$. Clearly $\nu_a \in \mathscr{B}$. If E is any Borel set disjoint from F and $\nu|E \ll \mu$ for some μ in $\mathscr{B}$, then $\nu|E \in \mathscr{B}$ and $|\nu|(F \cup E) = c + |\nu|(E)$. By the definition of c, it must be that $|\nu|(E) = 0$. Therefore $\nu_s \perp \mathscr{B}$.

The proof of uniqueness is left to the reader. □

Call the above decomposition of ν the *Lebesgue decomposition* of ν with respect to $\mathscr{B}$.

For convenience, let's agree that for a nonempty subset $\mathscr{S}$ of $M(X)$, the notation $\mu \perp \mathscr{S}$ means that $\mu \perp \sigma$ for every σ in $\mathscr{S}$.

17.5 Proposition. *If $\mathscr{S}$ is a nonempty subset of measures on X and $\mathscr{S}' \equiv \{\nu \in M(X) : \nu \perp \mathscr{S}\}$, then $\mathscr{S}'$ is a band of measures.*

In fact this proof is an immediate consequence of the definitions. If $\mathscr{B}$ is a band of measures, then the band $\mathscr{B}'$ is called the *complementary band* to $\mathscr{B}$. Note that $\mathscr{B} \cap \mathscr{B}' = (0)$. This terminology is justified by the next proposition.

17.6 Proposition. *Let $\mathscr{B}$ be a band of measures on X.*

(a) $(\mathscr{B}')' = \mathscr{B}$.

(b) $M(X) = \mathscr{B} \oplus_1 \mathscr{B}'$, *where $\oplus_1$ denotes the Banach space l^1-direct sum. (That is, $\|\nu \oplus \mu\| = \|\nu\| + \|\mu\|$.)*

Proof. (a) From the definition we have that $\mathscr{B} \subseteq (\mathscr{B}')'$. If $\nu \in (\mathscr{B}')'$, then the preceding theorem implies that $\nu = \mu + \eta$, where $\mu \in \mathscr{B}$ and $\eta \in \mathscr{B}'$. But $(\mathscr{B}')'$ is a band, so μ and $\eta \in (\mathscr{B}')'$. But then $\eta \in \mathscr{B}' \cap (\mathscr{B}')'$ and hence $\eta = 0$. Thus $\nu = \mu \in \mathscr{B}$.

(b) This is a straightforward reformulation of Theorem 17.4. □

Note that the intersection of any nonempty collection of bands in $M(X)$ is again a band. Thus for any nonempty subset $\mathscr{S}$ of $M(X)$, define the *band generated by S* to be the intersection of all bands that contain $\mathscr{S}$. So the band generated by $\mathscr{S}$ is the smallest band containing $\mathscr{S}$.

17.7 Proposition. *If $\mathscr{S}$ is a nonempty subset of $M(X)$ and $\mathscr{B} = \{\mu \in M(X) : \mu \perp \mathscr{S}'\}$, then $\mathscr{B}$ is the band generated by $\mathscr{S}$.*

Proof. Let $\mathscr{A}$ be any band containing $\mathscr{S}$. It is easy to see that $\mathscr{A}' \subseteq \mathscr{S}'$ and so $\mathscr{B} = (\mathscr{S}')' \subseteq (\mathscr{A}')' = \mathscr{A}$. □

17.8 Lemma. *Let $\{\nu_n\}$ be a sequence in $M(X)$ such that $\sum_n \nu_n$ converges in norm to a measure ν. If $\eta \in M(X)$ and $\eta \ll \nu$, then $\eta = \sum_n \eta_n$ where the measures $\{\eta_n\}$ are pairwise singular and $\eta_n \ll \nu_n$ for every $n \geq 1$.*

Proof. Let $\eta = \eta_1 + \sigma_1$ be the Lebesgue decomposition of η with respect to ν_1. So there is a Borel partition $\{E_1, F_1\}$ of X such that $\eta_1 = \eta|E_1$, $\sigma_1 =$

$\eta|F_1$, and $|\nu_1|(F_1) = 0$. Let $\sigma_1 = \eta_2 + \sigma_2$ be the Lebesgue decomposition of σ_1 with respect to ν_2. This produces a Borel partition $\{E_2, F_2\}$ of the set F_1 such that $\eta_1 = \eta|E_2$, $\sigma_2 = \eta|F_2$, and $|\nu_2|(F_2) = 0$. Continue and we obtain a sequence $\{E_n\}$ of pairwise disjoint Borel sets and a decreasing sequence $\{F_n\}$ of Borel sets having the following properties:

(i) $\eta_n = \eta|E_n \ll \nu_n$;
(ii) $E_1 \cup \cdots \cup E_n \cup F_n = X$;
(iii) $(E_1 \cup \cdots \cup E_n) \cap F_n = \varnothing$;
(iv) $|\nu_n|(F_n) = 0$

If $\sigma_n = \eta|F_n$, then $\{\sigma_n\}$ converges in norm to a measure σ in $M(X)$. In fact, $\sigma = \eta|F$, where $F = \bigcap_n F_n$. But $|\nu_n|(F) = 0$ for every $n \geq 1$. Thus $|\nu|(F) = 0$ and so $|\eta|(F) = 0$; that is, $\sigma = 0$ and so $\eta = \sum_n \eta_n$. □

17.9 PROPOSITION. *If $\mathscr{S}$ is a nonempty subset of $M(X)$ and $\mathscr{B}$ is the band generated by $\mathscr{S}$, the following statements are equivalent for a measure ν in $M(X)$.*

(a) $\nu \in \mathscr{B}$.
(b) $\nu = \sum_n \nu_n$, *where this series is norm convergent,* $\nu_n \perp \nu_m$ *for* $n \neq m$, *and for each n there is a μ_n in $\mathscr{S}$ with $\nu_n \ll \mu_n$.*
(c) $\nu = \sum_n \nu_n$, *where this series is norm convergent and for each n there is a μ_n in $\mathscr{S}$ with $\nu_n \ll \mu_n$.*

PROOF. Clearly (b) implies (c) and the preceding lemma gives that (c) implies (b). Let $\mathscr{A}$ be the set of measures described in (b) (or (c)). It will be shown that $\mathscr{A}$ is a band. Once this is established, the equivalence of (a) and (b) will follow. Indeed, $\mathscr{S}$ is clearly a subset of $\mathscr{A}$ and so $\mathscr{A}' \subseteq \mathscr{S}'$. On the other hand, if $\eta \in \mathscr{S}'$ and $\nu = \sum_n \nu_n$ with $\nu_n \ll \mu_n$ for some μ_n in $\mathscr{S}$, then $\eta \perp \mu_n$ for every n and hence $\eta \perp \nu_n$ for every n. Thus $\eta \perp \nu$ and so $\eta \in \mathscr{A}'$. That is, $\mathscr{A}' = \mathscr{S}'$ and so $\mathscr{B} = (\mathscr{S}')' = (\mathscr{A}')' = \mathscr{A}$.

To show that $\mathscr{A}$ is a band we first establish that $\mathscr{A}$ is a closed subspace of $M(X)$. The fact that $\mathscr{A}$ is a linear space follows easily by using (c). Now suppose that $\{\nu^k\} \subseteq \mathscr{A}$ and $\nu^k \to \nu$. Let $\nu^k = \sum_n \nu_n^k$ where $\nu_n^k \ll \mu_n^k$ and $\mu_n^k \in \mathscr{S}$. Then $\nu = \nu^1 + \sum_k (\nu^{k+1} - \nu^k) = \sum_n \nu_n^1 + \sum_k \sum_n (\nu_n^{k+1} - \nu_n^k)$. From here the proof that ν belongs to $\mathscr{A}$ becomes a test of expository skills; this test is left to the reader.

If $\nu \in \mathscr{A}$ and $\eta \ll \nu$, then Lemma 17.8 shows that η satisfies the necessary conditions to belong to $\mathscr{A}$. Therefore $\mathscr{A}$ is a band and the proof is complete. □

Before proceeding, we must have another measure theoretic interlude. This lemma will also be used later in this book in a different context.

17.10 LEMMA (Chaumat [**1974**]). *Let (X, Ω, μ) be a finite measure space and let C be a closed bounded convex subset of $L^p(\mu)$, $1 \leq p \leq \infty$. If $h \in C$ and $\varepsilon > 0$, then there is a function f in C such that $|g|\mu \ll |f|\mu$ for every g in C and $\|f - h\|_p < \varepsilon$.*

PROOF. It suffices to assume that $C \subseteq \text{ball}\, L^p(\mu)$. At first we will ignore the requirement that f be close to h and wait until the end to take care of that.

CLAIM 1. If f_1 and $f_2 \in C$ and $\varepsilon > 0$, then there is an α with $0 < \alpha < \varepsilon$ such that $|f_1|\mu$ and $|f_2|\mu$ are absolutely continuous with respect to $|f_1 + \alpha f_2|\mu$. To see this let $f_1\mu = kf_2\mu + h\mu$ be the Lebesgue decomposition of $f_1\mu$ with respect to $f_2\mu$. So $f_2 h = 0$ a.e. $[\mu]$. If $E_\alpha = \{x \colon k(x) = -\alpha\}$, then $\mu(E_\alpha) > 0$ for at most a countable number of α. Pick an α with $0 < \alpha < \varepsilon$ and $\mu(E_\alpha) = 0$. Thus $f_1 + \alpha f_2 = (\alpha + k)f_2 + h$ a.e. $[\mu]$. It follows that $|f_1|\mu$ and $|f_2|\mu$ are absolutely continuous with respect to $|f_1 + \alpha f_2|\mu$.

CLAIM 2. If $f_0, f_1, \ldots, \in C$, then there is an f in C with $|f_n|\mu \ll |f|\mu$ for every $n \geq 0$. For any function g put $S(g) = \{x \colon g(x) \neq 0\}$. The reason for the introduction of this set is the observation that $|g_1|\mu \ll |g_2|\mu$ if and only if $\mu(S(g_2)\backslash S(g_1)) = 0$. Let $g_0 = f_0$. There are positive numbers ε_0 and A_1 such that if $U_1 = \{x \colon |g_0(x)| > \varepsilon_0$ and $|f_1(x)| < A_1\}$, then $\mu(S(g_0)\backslash U_1) < 1/2$. By Claim 1 there is an α_1 with $0 < \alpha_1 < \min\{1/2, \varepsilon_0/4A_1\}$ such that if $g_1 = g_0 + \alpha_1 f_1 = f_0 + \alpha_1 f_1$, then $|f_0|\mu$ and $|f_1|\mu \ll |g_1|\mu$. Note that this last condition is equivalent to $\mu([S(f_0) \cup S(f_1)]\backslash S(g_1)) = 0$.

Continue; by induction we can choose positive constants ε_n, A_n, and α_n and functions g_n in $L^1(\mu)$ such that:

(i) $\varepsilon_{n+1} < \varepsilon_n/2$, $\alpha_n < \min\{2^{-n}, \varepsilon_{n-1}/4A_n\}$;
(ii) if $U_{n+1} = \{x \colon |g_n(x)| > \varepsilon_n$ and
$|f_{n+1}(x)| < A_{n+1}\}$ and $g_{n+1} = g_n + \alpha_{n+1} f_{n+1}$, then
$\mu(S(g_n)\backslash U_{n+1}) < 2^{-(n+1)}$ and
$\mu([S(g_n) \cup S(f_{n+1})]\backslash S(g_{n+1})) = 0$.

Define $g = f_0 + \sum_{n\geq 1} \alpha_n f_n = \lim g_n$. (Because $\alpha_n < 2^{-n}$, this limit exists in $L^p(\mu)$.) It is left as an exercise to show that $S(g) \supseteq \bigcup_{n\geq 1} \bigcap_{k\geq n} U_k$ and for $m < n$, $\mu(S(g_m)\backslash \bigcap_{k\geq n+1} U_k) < 2^{-n}$. From these relations it follows that $\mu(S(g_m)\backslash S(g)) = 0$ and hence $|f_m|\mu \ll |g|\mu$. Put $\alpha_0 = 1$; if $f = (\sum \alpha_n)^{-1} \sum \alpha_n f_n = (\sum \alpha_n)^{-1} g$, then $f \in C$ and Claim 2 is established.

Now let $\gamma = \sup\{\mu(S(f)) \colon f \in C\}$ and choose $\{f_n\}$ contained in C such that $\mu(S(f_n)) \to \gamma$. Let $f \in C$ that f satisfies Claim 2 for this sequence. If $g \in C$, then Claim 1 implies there is a g_1 in C such that $\mu([S(f) \cup S(g)]\backslash S(g_1)) = 0$ and g_1 is a convex combination of f and g. Hence $S(g_1) = S(f) \cup S(g)$. Also, $\mu(S(f)) = \gamma \geq \mu(S(g_1)) = \mu(S(f)) + \mu(S(g)\backslash S(f)) \geq \mu(S(f))$. So $\mu(S(g)\backslash S(f)) = 0$ and hence $|g|\mu \ll |f|\mu$.

Now to adjust f so that it is close to h. In fact, Claim 1 implies that there is an α, $0 < \alpha < \varepsilon/2$, such that $|f|\mu$ and $|h|\mu \ll |h + \alpha f|\mu$. Put $f_1 = (1+\alpha)^{-1}(h + \alpha f)$. So $f_1 \in C$ and hence $[|f|\mu] = [|f_1|\mu]$ so that $|g|\mu \ll |f_1|\mu$ for all g in C. Also, $\|f_1 - h\|_p = \alpha(1+\alpha)^{-1}\|f - h\|_p < 2\alpha < \varepsilon$. □

Also see Theorem 5 in Helson [**1983**] for a result related to the preceding lemma.

17.11 PROPOSITION. *If $\mathcal{S}$ is a closed convex set of measures and $\mathcal{B}$ is the band generated by $\mathcal{S}$, then $\nu \in \mathcal{B}$ if and only if there is a η in $\mathcal{S}$ such that $\nu \ll \eta$.*

PROOF. It suffices to show that if $\nu \in \mathcal{B}$, then $\nu \ll \eta$ for some η in $\mathcal{S}$. By Proposition 17.9, $\nu = \sum_n \nu_n$ where $\nu_n \ll \mu_n$ for some $\mu_n \in \mathcal{S}$ for each n. Let $\mu = \sum_n |\mu_n|$ and put $C = \{f \in L^1(\mu): f\mu \in \mathcal{S}$ and $\|f\mu\| \le 1\}$. It is easy to deduce from the hypothesis that C is a closed bounded convex subset of $L^1(\mu)$. Moreover, $\mu_n = f_n\mu \in C$ for every $n \ge 1$. Thus Lemma 17.10 implies that there is an f in C such that $\mu_n \ll f\mu \equiv \eta$ in $\mathcal{S}$. Clearly $\nu \ll \eta$. □

17.12 PROPOSITION. *If M is a weak* closed convex set of probability measures on X and if $\nu \perp M$, then there is a Borel set E such that E carries ν and $|\mu|(E) = 0$ for every μ in M.*

The proof of this proposition requires a result from general functional analysis. For a proof see Gamelin [**1969**] page 40.

17.13 THE MINIMAX THEOREM. *Let $\mathcal{V}$ be a vector space over $\mathbb{R}$ and let $\mathcal{X}$ be a real topological vector space. If C is a convex subset of $\mathcal{V}$, M is a compact convex subset of $\mathcal{X}$, and $F: C \times M \to \mathbb{R}$ is a function such that:*

(a) $\inf\{F(c, m): c \in C$ *and* $m \in M\} > -\infty$;

(b) *for every m in M, $c \to F(c, m)$ is a convex function;*

(c) *for every c in C, $m \to F(c, m)$ is a continuous concave function;*

then

$$\sup_m \inf_c F(c, m) = \inf_c \sup_m F(c, m).$$

PROOF OF PROPOSITION 17.12. Let $C = \{u \in C_{\mathbb{R}}(X): 0 < u < 1\}$ and let M be as in the statement of the proposition. Define $F: C \times M \to \mathbb{R}$ by $F(u, m) = \int u\,dm + \int (1-u)\,d|\nu|$. It is routine to check that F is convex linear in each variable and $F \ge 0$. Thus the Minimax Theorem applies. Since $\nu \perp m$ for every m in M, $\inf_u F(u, m) = 0$. Hence $\inf_u \sup_m F(u, m) = 0$. This says that there is a sequence $\{u_n\} \subseteq C_{\mathbb{R}}(X)$ with $0 < u_n < 1$ such that

$$\sup_{m \in M} \left\{ \int u_n\,dm + \int (1 - u_n)\,d|\nu| \right\} < n^{-2}.$$

Let $E = \{x: u_n(x) \to 1\}$. If $m \in M$, then $n^{-2} \ge \int_E u_n\,dm + \int_E (1 - u_n)\,d|\nu| \to m(E)$. Hence $m(E) = 0$ for every m in M. But $\sum_n \int (1 - u_n)\,d|\nu| \le \sum_n n^{-2}$ and so $u_n \to 1$ a.e. $[|\nu|]$. Thus $\nu = \nu|E$. □

This concludes this introduction to the theory of bands. Now let's prove a result about subnormal operators that is a direct consequence of Lemma 17.10 and is of use in the theory of subnormal operators. Recall [**ACFA**] page 288, that if N is a normal operator on $\mathcal{K}$, a vector f in $\mathcal{K}$ is a *separating vector* for N if the only operator A in $W^*(N)$ that satisfies $Af = 0$ is $A = 0$. If $N = \int z\,dE$, then f is a separating vector if and only if $\langle E(\cdot)f, f\rangle$ is a scalar-valued spectral measure for N.

17.14 PROPOSITION. *If S is a subnormal operator on $\mathscr{H}$ with minimal normal extension N acting on $\mathscr{K}$, $h \in \mathscr{H}$, and $\varepsilon > 0$, then there is a vector f in $\mathscr{H}$ that is separating for N and satisfies $\|f - h\| < \varepsilon$.*

PROOF. As in the proof of (17.10), we initially ignore the requirement that f be found close to h. Let $N = \int z\,dE$ be the spectral decomposition of N and let e be a separating vector for N in $\mathscr{K}$ with $\|e\| = 1$. Put $\mu(\Delta) = \langle E(\Delta)e, e\rangle$ and for each f in $\mathscr{H}$ let $\mu_f(\Delta) = \langle E(\Delta)f, e\rangle$. So $C = \{\mu_f : f \in \text{ball}\,\mathscr{H}\}$ is a bounded convex subset of ball $L^1(\mu)$. Moreover, $f \to \mu_f$ is a contractive linear map of $\mathscr{H}$ into $L^1(\mu)$ and is thus weakly continuous. Therefore C is weakly compact and thus norm closed. By Lemma 17.10, there is an f in ball $\mathscr{H}$ such that $\mu_g \ll |\mu_f|$ for every g in $\mathscr{H}$. We claim that f is a separating vector for N. In fact, suppose Δ is a Borel set such that $E(\Delta)f = 0$; we want to show that $E(\Delta) = 0$. If $\Delta_1 \subseteq \Delta$, then $|\mu_f|(\Delta_1) = 0$ and so $\mu_g(\Delta_1) = 0$ for every vector g in $\mathscr{H}$. Hence $|\mu_g|(\Delta) = 0$ for every g in $\mathscr{H}$. If n and k are non-negative integers, then $0 = \int_\Delta z^n \bar{z}^k\,d\mu_g = \langle N^n N^{*k} g, e\rangle = \langle N^n N^{*k} g, E(\Delta)e\rangle$. But $\bigvee\{N^n N^{*k} g : g \in \mathscr{H} \text{ and } n, k \geq 0\} = \mathscr{K}$. Thus it must be that $E(\Delta)e = 0$. Since e is a separating vector for N, $E(\Delta) = 0$.

To get a separating vector for N that is close to the given vector h, we proceed as in the proof of Claim 1 of the proof of (17.10). Since $\mu_h \ll \mu_f$, there is a Borel function ϕ such that $\mu_h = \phi\mu_f$. Now select α with $0 < \alpha < \varepsilon$ such that the function $\phi + \alpha \neq 0$ a.e. $[\mu_f]$; put $f_1 = h + \alpha f$. Now $\mu_{f_1} = (\phi+\alpha)\mu_f$ and $|\mu_{f_1}|$ and $|\mu_f|$ are mutually absolutely continuous. Thus f_1 is a separating vector for N and $\|f_1 - h\| \leq \alpha < \varepsilon$. □

Actually, a fact is contained in the last paragraph of the preceding proof that is worth recording because it will be used later in this book.

17.15 COROLLARY. *If f is a separating vector for N and h is any vector, then for all but a countable number of scalars α, $h + \alpha f$ is a separating vector for N.*

Exercises

1. Show that there is a band $\mathscr{B}$ and a measure ν in $\mathscr{B}'$ such that no Borel set E exists with the property that $\nu = \nu|E$ and $|\mu|(E) = 0$ for every μ in $\mathscr{B}$.
2. Let $\mathscr{B}$ be a band and define $P : M(X) \to M(X)$ by $P\nu = \nu_a$ as in Theorem 17.4. Show that P is a linear idempotent with $\|P\| = 1$, $\operatorname{ran} P = \mathscr{B}$, and $\ker P = \mathscr{B}'$.
3. Show that the map Θ in Theorem II.12.6 is injective.

§18 Annihilating measures. In this section we will study measures that annihilate a function algebra. In particular, we will prove an abstract version of the F. and M. Riesz Theorem, for which the central concept is the following.

18.1 DEFINITION. If A is a function algebra on X and $\mathscr{B}$ is a band of measures on X, then $\mathscr{B}$ is a *reducing band* (for A) if for every μ in $A^\perp$ with Lebesgue decomposition $\mu = \mu_a + \mu_s$, μ_a in $\mathscr{B}$ and μ_s in $\mathscr{B}'$, it follows that μ_a and μ_s both belong to $A^\perp$.

18.2 THE ABSTRACT F. AND M. RIESZ THEOREM. *If A is a function algebra and $\rho \in \mathscr{M}_A$, then the band generated by the representing measures for ρ is a reducing band.*

We won't try to trace the history of the abstract F. and M. Riesz Theorem, but it is a result that passed through several evolutionary stages before reaching the above form. The above formulation is due to Koenig and Seever **[1969]**, which was strongly influenced by Glicksberg **[1967]**. For more on reducing bands, the reader can see Cole and Gamelin **[1982** and **1985]** and Gamelin **[1973]**.

The classical F. and M. Riesz Theorem can be deduced from this abstract version, once an additional result about T-invariant algebras is proved. This will be done after (18.5) below.

PROOF OF THEOREM18.2. Let $\rho \in \mathscr{M}_A$ and let $\mathscr{B}$ be the band generated by M_ρ. Let $\mu \in A^\perp$ and let $\mu = \mu_a + \mu_s$ be the Lebesgue decomposition of μ with respect to $\mathscr{B}$. By (17.11), $\nu \in \mathscr{B}$ if and only if there is an η in M_ρ such that $\nu \ll \eta$. Thus Proposition 17.12 implies there is a Borel subset E of X such that $|\nu|(E) = 0$ for all ν in $\mathscr{B}$ and E carries μ_s. Without loss of generality it may be assumed that $E = \bigcup_n E_n$, where each E_n is compact and $E_n \subseteq E_{n+1}$.

By Lemma 11.6 we get that for every $n \geq 1$ there is a function f_n in A such that $\operatorname{Re} f_n > n\chi_{E_n}$ and $0 < \rho(f_n) < \delta_n$, where the numbers δ_n will be specified later. (Actually, the function f_n obtained in Lemma 11.6 must be replaced by $f_n - i \operatorname{Im} \rho(f_n)$.) Let $g_n = e^{-f_n}$. So $g_n \in A$, $\|g_n\| \leq 1$, $|g_n| < e^{-n}$ on E_n, and $|1-\rho(g_n)| < 1/n^2$ if the δ_n are chosen appropriately. Since $\sum_n |1-\rho(g_n)| < \infty$, Lemma 15.10 implies $g_n \to 1$ a.e. $[\mu_a]$. However, $|g_n| < e^{-n}$ on E_n and so $g_n \to 0$ a.e. $[\mu_s]$. Therefore, $g_n\mu \to \mu_a$ weak* in $M(X)$. So if $f \in A$, $\int f\, d\mu_a = \lim_n \int f g_n\, d\mu = 0$, since $\mu \perp A$ and $fg_n \in A$. Thus $\mu_a \in A^\perp$. □

Before giving some corollaries of this theorem and further information about annihilating measures, let's take a little time to rephrase some of our previous results in terms of bands.

Let Q be a nontrivial Gleason part for the algebra A and let $\rho \in Q$. From Corollary 3.5 we have that the band generated by M_ρ is the same as the band generated by the representing measures for any other element of Q.

Define the *band generated by* Q to be the band of measures on X generated by M_ρ for any ρ in Q. Denote this band by $\mathscr{B}_Q$.

Let $\{Q_i\}$ be the nontrivial Gleason parts for A. (In general, there may be an uncountable number of these parts, but not, of course, for T-invariant algebras.) According to Theorem 15.9, $\mathscr{B}_{Q_i} \subseteq \mathscr{B}'_{Q_j}$ for $Q_i \neq Q_j$. We therefore arrive at the following result whose proof is straightforward.

18.3 Proposition. *If $\{Q_i\}$ are the nontrivial Gleason parts of the function algebra A on X and $\mathscr{B}_{Q_i}$ is the band of measures generated by Q_i, then*

$$M(X) = \mathscr{S} \oplus \bigoplus_i \mathscr{B}_{Q_i},$$

where $\mathscr{S}$ is the band $\bigcap_i \mathscr{B}'_{Q_j}$ consisting of the measures that are singular to every representing measure for every homomorphism belonging to a nontrivial Gleason part, and the direct sum is an l^1 direct sum. Thus every measure ν in $M(X)$ can be written as $\nu = \nu_0 + \sum_i \nu_i$, where $\nu_0 \in \mathscr{S}$, $\nu_i \in \mathscr{B}_{Q_i}$ for each i, and $\|\nu\| = \|\nu_0\| + \sum_i \|\nu_i\|$.

Call the band $\mathscr{S}$ that appears in the preceding proposition the *singular band* for A. Note that $\mathscr{S}$ at least contains the point masses δ_a for each peak point a for A. The presence of this singular band is something of a nuisance, but, as we shall see, for T-invariant algebras and measures in the annihilator of A, it can be ignored. But first an explicit combination of Proposition 18.3 and the abstract F. and M. Riesz Theorem.

18.4 Corollary. *If $\mu \in A^\perp$, then $\mu = \mu_0 + \sum_i \mu_i$, where for all i, $\mu_0 \perp \mathscr{B}_{Q_i}$ and $\mu_i \in A^\perp \cap \mathscr{B}_{Q_i}$, $\|\mu\| = \|\mu_0\| + \sum_i \|\mu_i\|$, and $\mu_i \perp \mu_j$ for $i \neq j$.*

Now focus your attention on T-invariant algebras.

18.5 Wilkin's Theorem. *If A is a T-invariant algebra on K, then there is no nonzero annihilating measure that belongs to the singular band for A.*

Proof. Suppose $\nu \in A^\perp$ and $\nu \perp M_a$ for every nonpeak point a in K; it must be shown that $\nu = 0$. But since $R(K) \subseteq A$, $\hat{\nu}$, the Cauchy transform of ν, vanishes off K. If $a \in K$ such that $\tilde{\nu}(a) < \infty$ and $\hat{\nu}(a) \neq 0$, then a is not a peak point and there is a μ in M_a such that $\mu \ll \nu$ (8.10). So it must be that $\hat{\nu} = 0$ a.e. [Area] and hence $\nu = 0$. □

We can now derive the classical F. and M. Riesz Theorem as a consequence of the results of this section. Let A be the disk algebra, the uniform closure of the polynomials in $C(\partial\mathbb{D})$. So $\mathscr{M}_A = \text{cl}\,\mathbb{D}$. If ρ is evaluation at 0 and M_ρ = the representing measures for ρ that are supported on $\partial\mathbb{D}$, then $M_\rho = \{m\}$. Thus the band generated by M_ρ is precisely $L^1(m) = L^1$. If μ is a measure on $\partial\mathbb{D}$ that annihilates A, then Theorem 18.2 implies that if $\mu = \mu_a + \mu_s$ is the Lebesgue decomposition of μ with respect to m, then μ_a and $\mu_s \in A^\perp$. On the other hand, Wilkin's Theorem implies that $\mu_s = 0$. Thus $\mu = \mu_a$ and is thus absolutely continuous.

18.6 COROLLARY. *If A is a T-invariant algebra on K and Q is the set of nonpeak points for A, then $\mathscr{B}_Q$, the band generated by the representing measures for points in Q, is the same as the band generated by the set of annihilating measures of A.*

PROOF. Let $\mathscr{C}$ be the band generated by the annihilating measures of A. By Wilkins's Theorem, $\mathscr{C} \subseteq \mathscr{B}_Q$. On the other hand, if $a \in Q$, then there is a representing measure μ for a such that $\mu \neq \delta_a$. Thus $\mu - \delta_a \perp A$ and both μ and δ_a are absolutely continuous with respect to $\mu - \delta_a$. Hence $\mathscr{C}$ contains all representing measures for nonpeak points and so $\mathscr{B}_Q \subseteq \mathscr{C}$. □

In light of Wilkin's Theorem, Proposition 18.3 can be combined with Theorem 15.16 to produce a good structure theorem for the annihilator of a T-invariant algebra.

18.7 THEOREM. *Let A be a T-invariant algebra on K with nontrivial Gleason parts $Q_1, Q_2, \ldots$ and carriers $E_1, E_2, \ldots$. If ν is an annihilating measure of A, then $\nu = \sum_n \nu|E_n$, $\nu|E_n \in A^\perp$, and for each $n \geq 1$ and for every choice of a_n from Q_n, there is a μ_n in M_{a_n} such that $\nu|E_n \ll \mu_n$ (and thus $\nu|E_n \in \mathscr{B}_{Q_n}$). Also the Cauchy transform of $\nu|E_n$ is the function $\chi_{Q_n}\hat{\nu}$ and $\nu_n \perp R(K, Q_n)$.*

PROOF. The carriers E_n exist from Theorem 15.16 and for each choice of a_n in Q_n, every representing measure for a_n is carried by E_n. Therefore, by Proposition 17.11, every measure in $\mathscr{B}_{Q_n}$ is carried by E_n. Let ν_n be the projection of ν into $\mathscr{B}_{Q_n}$. Proposition 18.3 and Wilkin's Theorem imply that $\nu = \sum_n \nu_n$ and $\nu_n \in A^\perp$. Since the sets $\{E_n\}$ are pairwise disjoint, it must be that $\nu_n = \nu|E_n$. This establishes the first part of the theorem.

To see that $\hat{\nu}_n = \chi_{Q_n}\hat{\nu}$, observe that if w is a point with $\infty > \tilde{\nu}(w) = \sum_n \int |z-w|^{-1}\, d|\nu_n|(z)$, then $\hat{\nu}(w) = \sum_n \hat{\nu}_n(w)$. But $\nu_n \in \mathscr{B}_{Q_n}$ and so Corollary 8.10 implies that $\{w : \tilde{\nu}_n(w) < \infty \text{ and } \hat{\nu}_n(w) \neq 0\} \subseteq Q_n$. Hence $\hat{\nu}_n(w) = 0$ a.e. [Area] off Q_n. Thus $\hat{\nu}_n = \chi_{Q_n}\hat{\nu}$. Finally, since $\hat{\nu}_n = 0$ a.e. off Q_n, $\nu_n \perp R(K, Q_n)$ by Proposition 3.14. □

The following corollary is a shorthand formulation of the preceding theorem though, like all shorthand versions, it contains less information.

18.8 COROLLARY. *If A is a T-invariant algebra on K with nontrivial Gleason parts $Q_1, Q_2, \ldots$, then*

$$A^\perp = \bigoplus_i [A^\perp \cap \mathscr{B}_{Q_i}].$$

The next result generalizes the classical F. and M. Riesz Theorem.

18.9 COROLLARY. *If $R(K)$ is a Dirichlet algebra and ν is an annihilating measure of $R(K)$, then $\nu|\partial K$ is absolutely continuous with respect to harmonic measure for K.*

PROOF. Adopt the notation of Theorem 18.7 for the T-invariant algebra $R(K)$. So $\nu_n \ll \mu_n$ for some representing measure μ_n of a_n in Q_n. It follows that $\hat{\mu}_n$, the sweep of μ_n, is also a representing measure for a_n that is concentrated on ∂K. Since $R(K)$ is a Dirichlet algebra, $\hat{\mu}_n = \omega_n$, harmonic measure for K at a_n. But $\hat{\mu}_n = \mu_n|\partial K + (\mu|\operatorname{int} K)\hat{}$ so that $\mu_n|\partial K \ll \omega_n$. Hence $\nu_n|\partial K \ll \omega_n$ for every $n \geq 1$. □

18.10 COROLLARY. *If $R(K)$ is a Dirichlet algebra and E is a compact subset of ∂K, then E is a peak interpolating set for $R(K)$ if and only if E has zero harmonic measure.*

PROOF. Combine the preceding corollary with Theorem 11.3. □

§**19 Mergelyan's Theorem.** In this section some of the techniques that have been developed will be applied to determine sufficient conditions for $R(K)$ and $A(K)$ to coincide. Necessary and sufficient conditions for this can be found but this would take us too far from our goal of studying subnormal operators. The interested reader can find this material in Gamelin **[1969]** page 217.

We begin with one of the oldest theorems of this type.

19.1 MERGELYAN'S THEOREM. *If K is polynomially convex, then $P(K) = R(K) = A(K)$*

PROOF. Of course the first equality, $P(K) = R(K)$, is a direct consequence of Runge's Theorem and we were already aware of this. Since $P(K) \subseteq A(K)$, it remains to show that $A(K) \subseteq P(K)$. Actually we will prove that $A(K)|\partial K \subseteq P(K)|\partial K$ which will also complete the proof of the theorem. Let $\nu \in M(\partial K)$ such that $\nu \perp P(K)$; it must be shown that $\nu \perp A(K)$.

Let $Q_1, Q_2, \ldots$ be the nontrivial Gleason parts and let $E_1, E_2, \ldots$ be the corresponding carriers. Put $\nu_n = \nu|E_n$. By Theorem 18.7, $\nu_n \perp P(K)$. By Corollary 18.8, $\nu_n \ll \omega_n$, harmonic measure for a point a_n in Q_n. It must be shown that $\nu_n \perp A(K)$ for each $n \geq 1$.

Fix $n \geq 1$; also fix f in $A(K)$ and let $C = 2\|f\|$. Hence $\operatorname{Re}(f + C) > 0$ on K and so $g = \log(f + C) \in A(K)$. Since $P(K)$ is a Dirichlet algebra, there is a sequence of polynomials $\{p_k\}$ such that for every $k \geq 1$, $\|\operatorname{Re} p_k - \operatorname{Re} g\|_{\partial K} < 2^{-k}$. By the Maximum Principle we have that $\|\operatorname{Re} p_k - \operatorname{Re} g\|_K < 2^{-k}$ for all k. Also the polynomials p_k can be chosen so that $p_k(a_n) = g(a_n)$. Now $(p_k - g)^2$ is harmonic on $\operatorname{int} K$ and vanishes at a_n, so $0 = \int (p_k - g)^2 \, d\omega_n$. Since the real part of this integral must vanish, we have that

$$\int (\operatorname{Im} p_k - \operatorname{Im} g)^2 \, d\omega_n = \int (\operatorname{Re} p_k - \operatorname{Re} g)^2 \, d\omega_n$$
$$< (2^{-k})^2.$$

Hence

$$\int \sum_{k=1}^{\infty} |p_k - g|^2 \, d\omega_n = \sum_{k=1}^{\infty} \int |p_k - g|^2 \, d\omega_n < \infty.$$

Therefore $p_k(z) \to g(z)$ a.e. $[\omega_n]$ and hence a.e. $[\nu_n]$. Thus $e^{p_k} \to f + C$ a.e. $[\nu_n]$. But $|e^{p_k}| = e^{\operatorname{Re} p_k} \le e^{(\operatorname{Re} g + 1)}$. By the Lebesgue Dominated Convergence Theorem, $\int f \, d\nu_n = \int (f + C)\, d\nu_n = \lim_k \int e^{p_k} \, d\nu_n = 0$ since $e^{p_k} \in P(K)$ for each $k \ge 1$. □

Mergelyan's Theorem can be used, in conjunction with Bishop's Localization Theorem, to show that $R(K) = A(K)$ whenever K is finitely connected. The same proof technique can be used to obtain an even better result.

19.2 Theorem. *If K is a compact subset of $\mathbb{C}$ with the property that there is a $\delta > 0$ such that each component of $\mathbb{C} \backslash K$ has diameter at least δ, then $R(K) = A(K)$.*

Proof. Fix f in $A(K)$. Let $a \in K$ and put $U = B(a; \delta/3)$; by Bishop's Theorem (3.10), if it can be shown that $f|(K \cap \operatorname{cl} U) \in R(K \cap \operatorname{cl} U)$, then $f \in R(K)$. But clearly $f|(K \cap \operatorname{cl} U) \in A(K \cap \operatorname{cl} U)$, so if we can show that $K \cap \operatorname{cl} U$ is polynomially convex, then the result will follow by Mergelyan's Theorem. This is precisely what will be shown.

In fact, suppose to the contrary that $\mathbb{C} \backslash (K \cap \operatorname{cl} U)$ has a bounded component W. Now $\mathbb{C} \backslash (K \cap \operatorname{cl} U) = (\mathbb{C} \backslash K) \cup (\mathbb{C} \backslash \operatorname{cl} U)$ and the fact that $\mathbb{C} \backslash \operatorname{cl} U$ is connected and unbounded implies that $W \cap (\mathbb{C} \backslash \operatorname{cl} U) = \emptyset$. Hence $W \subseteq \operatorname{cl} U$ and so $\operatorname{diam} W \le 2\delta/3$. We also have that $W \subseteq \mathbb{C} \backslash K$ and so there is a component W_1 of $\mathbb{C} \backslash K$ that contains W. But $\mathbb{C} \backslash K \subseteq \mathbb{C} \backslash (K \cap \operatorname{cl} U)$ and so $W = W_1$. Since W has diameter $< \delta$, this is a contradiction. □

19.3 Corollary. *If K is finitely connected, $R(K) = A(K)$.*

§**20 The double dual of a T-invariant algebra.** In this section we will characterize the second dual space of a T-invariant algebra. If A is a function algebra on X, then general Banach space theory tells us that $A^* = M(X)/A^\perp$ and $A^{**} = A^{\perp\perp} =$ the weak* closure of A in $M(X)^{**}$. For T-invariant algebras we can improve this and relate A^{**} to the structure of the algebra. Clearly this will involve the annihilator of the algebra and Theorem 18.7 will be of value. We begin by determining the dual of a band of measures.

20.1 Definition. If X is a compact metric space and $\mathscr{B}$ is a band of measures on X, define $L^\infty(\mathscr{B})$ to be the collection of all $F = \{F_\mu\}$ in the Cartesian product $\prod\{L^\infty(\mu) \colon \mu \in \mathscr{B}\}$ such that if μ and $\nu \in \mathscr{B}$ and $\mu \ll \nu$, then $F_\mu = F_\nu$ a.e. $[\mu]$.

It is easy to see that $L^\infty(\mathscr{B})$ is a linear subspace of $\prod\{L^\infty(\mu) \colon \mu \in \mathscr{B}\}$. Also if f is a bounded Borel function on X and $F_\mu = f$ for all μ in $\mathscr{B}$, then this defines an element of $L^\infty(\mathscr{B})$. It will be shown that $L^\infty(\mathscr{B})$ is the Banach space dual of $\mathscr{B}$, but first we must attend to a few amenities like defining the norm on $L^\infty(\mathscr{B})$.

20.2 LEMMA. *If $\mathscr{B}$ is a band of measures on X and $F \in L^\infty(\mathscr{B})$, then* $\sup\{\|F_\mu\|_\infty : \mu \in \mathscr{B}\} < \infty$.

PROOF. If this supremum is infinite, then there is a sequence $\{\mu_n\}$ in $\mathscr{B}$ such that $\|F_{\mu_n}\|_\infty \to \infty$. We may assume that $\|\mu_n\| \leq 1$ for all $n \geq 1$ and so $\mu = \sum_n 2^{-n}|\mu_n| \in \mathscr{B}$. Since $\mu_n \ll \mu$, $F_{\mu_n} = F_\mu$ a.e. $[\mu_n]$ for all $n \geq 1$. Thus $\|F_\mu\|_\infty \geq \|F_{\mu_n}\|_\infty$ for all $n \geq 1$, contradicting the fact that $F_\mu \in L^\infty(\mu)$. □

The proof of the next proposition is left to the reader.

20.3 PROPOSITION. *If $\mathscr{B}$ is a band of measures on X and $\|F\| \equiv \sup\{\|F_\mu\|_\infty : \mu \in \mathscr{B}\}$ for F in $L^\infty(\mathscr{B})$, then $L^\infty(\mathscr{B})$ is an abelian C^*-algebra.*

20.4 EXAMPLES. (a) If μ is a positive measure on X and $\mathscr{B} = L^1(\mu)$, then for each f in $L^\infty(\mu)$ and ν in $L^1(\mu)$ define f_ν to be the element of $L^\infty(\nu)$ naturally associated with f. That is, the inclusion map $L^1(\nu) \to L^1(\mu)$ is an isometry (but possibly not surjective) and f_ν is the image of f under the dual of this map. In a certain sense f_ν is a restriction of f. It follows that $f \to \{f_\nu\}$ defines an isometric isomorphism of $L^\infty(\mu)$ onto $L^\infty(\mathscr{B})$.

(b) If $\mathscr{B}$ is the band of all purely atomic measures on X, then $L^\infty(\mathscr{B})$ "=" $l^\infty(X)$.

20.5 THEOREM. *If $\mathscr{B}$ is a band of measures on X and for F in $L^\infty(\mathscr{B})$, $\Phi_F : \mathscr{B} \to \mathbb{C}$ is defined by*

$$\Phi_F(\mu) = \int F_\mu \, d\mu,$$

then the map $F \to \Phi_F$ defines an isometric isomorphism of $L^\infty(\mathscr{B})$ onto $\mathscr{B}^*$.

PROOF. The fact that each Φ_F is linear is left to the reader. Also $|\Phi_F(\mu)| \leq \int |F_\mu| \, d|\mu| \leq \|F\| \, \|\mu\|$. Hence $\Phi_F \in \mathscr{B}^*$ and $\rho: L^\infty(\mathscr{B}) \to \mathscr{B}^*$ defined by $\rho(F) = \Phi_F$ is a linear contraction. It remains to show that ρ is isometric and surjective. As often happens in these situations, both these properties will be demonstrated simultaneously.

Fix Φ in $\mathscr{B}^*$. If $\mu \in \mathscr{B}$ and $g \in L^1(\mu)$, then $g\mu \in \mathscr{B}$. Thus $g \to \Phi(g\mu)$ is a well-defined linear functional on $L^1(\mu)$. Moreover, $|\Phi(g\mu)| \leq \|\Phi\| \, \|g\mu\| = \|\Phi\| \, \|g\|_1$; hence there is an F_μ in $L^\infty(\mu)$ with $\|F_\mu\|_\infty \leq \|\Phi\|$ such that $\Phi(g\mu) = \int gF_\mu \, d\mu$ for all g in $L^1(\mu)$.

Now suppose $\mu, \nu \in \mathscr{B}$ and $\mu \ll \nu$. Hence $\mu = g\nu$ for some g in $L^1(\nu)$. Thus for every h in $L^1(\mu)$, $hg \in L^1(\nu)$ and $h\mu = hg\nu$. This gives that $\int h F_\mu \, d\mu = \Phi(h\mu) = \Phi(hg\nu) = \int hg F_\nu \, d\nu$ for all h in $L^1(\mu)$. It follows that $F_\mu = F_\nu$ a.e. $[\mu]$. Therefore $F = \{F_\mu\} \in L^\infty(\mathscr{B})$. Also $\|F_\mu\|_\infty \leq \|\Phi\|$ and so $\|F\| \leq \|\Phi\|$. This completes the proof. □

20.6 Corollary. *The Banach space dual of* $M(X)$ *is isometrically isomorphic to* $L^\infty(M(X))$.

Now that we know that $L^\infty(\mathscr{B})$ is the dual of the Banach space $\mathscr{B}$, $L^\infty(\mathscr{B})$ has a weak* topology.

20.7 Proposition. *A net* $\{F_i\}$ *in* $L^\infty(\mathscr{B})$ *converges to* F *in the weak* topology if and only if* $(F_i)_\mu \to F_\mu$ *weak* in* $L^\infty(\mu)$ *for every* μ *in* $\mathscr{B}$.

The proof of this proposition is left as an exercise for the reader. Notice that this says that the weak* topology on $L^\infty(\mathscr{B})$ is the relative product topology it has as a subset of $\prod\{L^\infty(\mu)\colon \mu \in \mathscr{B}\}$, where each coordinate has its natural weak* topology.

If $\mathscr{S} \subseteq L^\infty(\mathscr{B})$, define $\mathscr{S}_\mu \equiv \{F_\mu\colon F \in \mathscr{S}\}$.

20.8 Proposition. *If* $\mathscr{S} \subseteq L^\infty(\mathscr{B})$ *and* $F \in L^\infty(\mathscr{B})$, *then* F *belongs to the weak* closure of* $\mathscr{S}$ *if and only if for every* μ *in* $\mathscr{B}$, F_μ *belongs to the weak* closure of* $\mathscr{S}_\mu$ *in* $L^\infty(\mu)$.

Proof. It is easy to check that if $F \in \text{wk}^*\text{-cl}\,\mathscr{S}$, then for every μ in $\mathscr{B}$, $F_\mu \in wk^* - \text{cl}\,\mathscr{S}_\mu$. So let's concentrate on the converse; so assume that $F \in L^\infty(\mathscr{B})$ and $F_\mu \in wk^* - \text{cl}\,\mathscr{S}_\mu$ for every μ in $\mathscr{B}$. Let $I =$ the collection of all pairs (ε, M), where $\varepsilon > 0$ and M is a finite subset of $\mathscr{B}$. Define an order on I as follows: $(\varepsilon, M) \le (\delta, N)$ if $\delta \le \varepsilon$, and $M \subseteq N$. Clearly with this definition of order, I becomes a directed set.

If $\alpha = (\varepsilon, \{\mu_1, \dots, \mu_m\}) \in I$, then $\mu = |\mu_1| + \cdots + |\mu_m| \in \mathscr{B}$ and $\mu_1, \dots, \mu_m \in L^1(\mu)$. By hypothesis there is a G is $\mathscr{S}$ such that $|\int (G_\mu - F_\mu)\,d\mu_j| < \varepsilon$ for $1 \le j \le m$. (Let's remark that $\int (G_\mu - F_\mu)\,d\mu_j = \int (G_{\mu_j} - F_{\mu_j})\,d\mu_j$.) Denote any such element G of $\mathscr{S}$ by F_α. Thus $\{F_\alpha\colon \alpha \in I\}$ is a net in $\mathscr{S}$. It is claimed that $F_\alpha \to F$ weak* in $L^\infty(\mathscr{B})$.

By Proposition 20.7 it must be shown that $(F_\alpha)_\mu \to F_\mu$ weak* in $L^\infty(\mu)$ for every μ in $\mathscr{B}$. That is, it must be shown that for μ in $\mathscr{B}$ and g in $L^1(\mu)$, $\int (F_\alpha)_\mu g\,d\mu \to \int F_\mu g\,d\mu$. So fix μ in $\mathscr{B}$, g in $L^1(\mu)$, and $\varepsilon > 0$. Let $\alpha_0 = (\varepsilon, \{\mu, g\mu\})$ in I. If $\alpha \ge \alpha_0$, then $\alpha = (\delta, M)$, where $\delta \le \varepsilon$ and $\{\mu, g\mu\} \subseteq M$. By definition of F_α, we have $|\int ((F_\alpha)_\mu - F_\mu) g\,d\mu| < \delta \le \varepsilon$, precisely what we had to show. □

Note that $\mathscr{B}$ is a module over $L^\infty(\mathscr{B})$. That is, if $F \in L^\infty(\mathscr{B})$ and $\mu \in \mathscr{B}$, then define $\mu F \equiv (F_\mu)\mu \in \mathscr{B}$. It is easy to check that the desired distributive laws are satisfied and also $\|\mu F\| \le \|\mu\|\,\|F\|$. (Write μF rather than $F\mu$ to avoid the possible confusion of this product with the usual notation of F_μ for the μ th coordinate of the element F of $L^\infty(\mathscr{B})$.)

The algebra $L^\infty(\mathscr{B})$ has several idempotents. For example, if E is a Borel set and $F_\mu = \chi_E$ for every μ in $\mathscr{B}$, then $F \in L^\infty(\mathscr{B})$ and $F^2 = F$. If F is any idempotent of $L^\infty(\mathscr{B})$, not just one of the preceding type, then it can be verified by the reader that $\mathscr{B}F$ is a norm closed subband of $\mathscr{B}$.

The converse is also true. (Recall that the set of idempotents in any abelian ring forms a lattice.)

20.9 PROPOSITION. *If $\mathscr{B}$ is a band of measures on X, then the map $F \to \mathscr{B}F$ defines a lattice isomorphism between the lattice of idempotents of $L^\infty(\mathscr{B})$ and the lattice of subbands of $\mathscr{B}$.*

PROOF. Only a sketch of the proof is given here. The reader can fill in the details. Suppose $\mathscr{A}$ is a subband of $\mathscr{B}$ and for μ in $\mathscr{B}$ let $\mu = \mu_a + \mu_s$ be the Lebesgue decomposition of μ with respect to the band $\mathscr{A}$. Define F_μ by letting it be 1 a.e. $[\mu_a]$ and 0 a.e. $[\mu_s]$. Then $F \in L^\infty(\mu)$, $F^2 = F$, and $\mathscr{B}F = \mathscr{A}$. □

20.10 DEFINITION. If A is a function algebra on X and $\mu \in M(X)$, let $A^\infty(\mu)$ be the weak* closure of A in $L^\infty(\mu) \equiv L^\infty(|\mu|)$. If $\mathscr{B}$ is a band of measures on X, then there is a natural inclusion of A inside $L^\infty(\mathscr{B})$. Let $A^\infty(\mathscr{B})$ be the weak* closure of A in $L^\infty(\mathscr{B})$.

20.11 PROPOSITION. *The double dual of the function algebra A on X is naturally isometrically isomorphic to $A^\infty(M(X))$.*

REMARK. The word "naturally" in the preceding proposition means that the diagram

$$\begin{array}{ccc} A & \longrightarrow & L^\infty(M(X)) \\ \downarrow & \nearrow & \\ A^{**} & & \end{array}$$

is commutative, where the horizontal and vertical arrows are the natural embeddings of A.

The proof of this proposition is just a specific instance of a Banach space phenomenon. Namely, if $\mathscr{X}$ is a Banach space and $\mathscr{Y}$ is a closed subspace of $\mathscr{X}$, then $\mathscr{Y} \subseteq \mathscr{X} \subseteq \mathscr{X}^{**}$ and $\mathscr{Y}^{**}$ "is" the closure of $\mathscr{Y}$ in the weak* topology of $\mathscr{X}^{**}$.

For the remainder of this book we will identify A^{**} with $A^\infty(M(X))$.

20.12 PROPOSITION. *If A is a function algebra on X and $F \in L^\infty(M(X))$, the following statements are equivalent.*

(a) $F \in A^\infty(M(X))$.

(b) *If $\nu \in A^\perp$, then $\int F_\nu \, d\nu = 0$.*

(c) $A^\perp F \subseteq A^\perp$.

PROOF. The equivalence of (a) and (b) is the consequence of general Banach space theory and is left to the reader.

(a) *implies* (c). Let $\nu \in A^\perp$; we want to show that $\nu F \in A^\perp$. Since $F \in A^\infty(M(X))$, there is a net $\{f_i\}$ in A such that $f_i \to F$ weak* in $L^\infty(M(X))$.