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GERARD BUSKES

The Hahn–Banach Theorem surveyed

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Gerard Buskes
Department of Mathematics
The University of Mississippi
University, MS 38677
U.S.A.

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Dedicated to the 100th birthday
of Stefan Banach

1. Prerequisites

Confusing as it is, there are two theorems that are called *the* Hahn–Banach Theorem. In order to clear the air right away, we use this section to distinguish them and to introduce the necessary definitions. In this paper, E will be a vector space over the reals, unless otherwise specified.

DEFINITION 1. A map $p : E \to \mathbb{R}$ is called *sublinear* if

$$p(x+y) \le p(x) + p(y) \qquad \text{for all } x, y \in E$$

and

$$p(\alpha x) = \alpha p(x) \qquad \text{for all } x \in E \text{ and all } \alpha \in \mathbb{R}^+ .$$

p is called a *seminorm* if also

$$p(\alpha x) = |\alpha| p(x) \qquad \text{for all } \alpha \in \mathbb{R} .$$

p is called a *norm* if in addition

$$p(x) = 0 \;\Leftrightarrow\; x = 0 .$$

THEOREM 2 (Hahn–Banach Theorem — First Version). *Let p be a norm on E and let E_0 be a vector subspace of E. Let f_0 be a continuous linear function from E_0 to $\mathbb{R}$. Then there exists a continuous linear function from E to $\mathbb{R}$ that extends f_0 such that*

$$\|f_0\| = \|f\| .$$

THEOREM 3 (Hahn–Banach Theorem — Second Version). *Let p be a sublinear function from E to $\mathbb{R}$. Let E_0 be a vector subspace of E and let f_0 be a linear function from E_0 to $\mathbb{R}$ for which*

$$f_0(x) \le p(x) \qquad \textit{for all } x \in E_0 . \tag{1}$$

Then there exists a linear function f from E to $\mathbb{R}$ that extends f_0 and for which

$$f(x) \le p(x) \qquad \textit{for all } x \in E . \tag{2}$$

Instead of (1), we will sometimes say that f_0 is *dominated* by p on E_0. Similarly, in (2), f is said to be dominated by p (on E).

2. The history

The history of the Hahn–Banach Theorem (or Theorems, if you so like) closely parallels the history of functional analysis. The road was paved in the late nineteenth century and the first precursor of the Hahn–Banach Theorem can be found in a 1907 paper by Riesz [275]. The development of functional analysis is beautifully described by Monna in [235] and Dieudonné in [85], while more personal, but equally charming reminiscences can be found in Young's book [348]. Hochstadt in [140] analyzes the history of the Hahn–Banach Theorem itself. Hahn [123] presented a proof of the First Version of the Hahn–Banach Theorem in 1927, but an essential trick had already been used by Helly [136] in a proof of the so called Hamburger problem. The Second Version of the Hahn–Banach Theorem first appears in Banach [31]. In 1932 Banach's classic [28] came out. In fact, it had appeared one year earlier in a Polish edition, "Teorja Operacyj". A kernel of the idea for the theorem goes back as far as 1923 (see [32]). Banach [30] credits Helly [136] and Hahn [123] for their roles in the First Version of the Hahn–Banach Theorem. Hahn [123] does not refer to Helly's contribution [136]. Hochstadt calls Helly the father of the Hahn–Banach Theorem. It is more accurate to call Helly the father of the First Version and contribute the Second Version to Banach alone. The generalization to sublinear functions shows much more ingenuity and foresight from the father of Banach space theory than can be seen from analyzing the proofs of these results alone. Bohnenblust and Sobczyk in 1938 (see [46]) coined the term *Hahn–Banach Theorem*.

The bibliography of this paper chronicles in many different ways the history of the Hahn–Banach Theorem. The list of papers is undoubtedly far from complete. For most of the publications a Mathematical Reviews number is provided. To describe everything that is in the bibliography is an ambitious project. The choices that I have made are a result of personal taste, knowledge or lack thereof. Particularly motivating were the talks by Kalton [160], Neumann [248], Neumann [250], and van Rooij [332].

We will first present several different proofs of the Hahn–Banach Theorem (either version). Initially there will be one proof per section and citations only to the instigators of that particular proof. These early proofs will return later in disguise of wide generalizations and with loads of additional references. As a consequence, one does not find the utmost generality in the earlier but rather in the later sections.

3. Helly's Part

Of course, Helly never proved the Hahn–Banach Theorem. He did prove Theorem 4 below for $C[a,b]$ rather than a general normed space and with $I = \mathbb{N}$. Though the former has nothing to do with the generality of Helly's argument (normed spaces had not been introduced yet in 1912), the latter cannot be separated from the role of the Axiom of Choice in the development of functional analysis. Historically, Helly's Part connects the Hahn–Banach Theorem (First Version) with the older Hamburger problem.

THEOREM 4. *Let $(E, \| \, \|)$ be a normed linear space. Let $\{\tau_\alpha : \alpha \in I\}$ be a subset of $\mathbb{R}$ and let $\{x_\alpha : \alpha \in I\}$ be a subset of E. Then the following are equivalent.*

(1) *There exists a continuous linear function on E such that $f(x_\alpha) = \tau_\alpha$.*
(2) *There exists $M > 0$ such that*

$$\Big|\sum_{\alpha \in J} \mu_\alpha \tau_\alpha\Big| \leq M \Big\|\sum_{\alpha \in J} \mu_\alpha x_\alpha\Big\|$$

for all finite subsets J of I and all μ_α.

The First Version of the Hahn–Banach Theorem follows by taking $I = E_0$, $x_\alpha = \alpha$ and $\tau_\alpha = f_0(\alpha)$ for all $\alpha \in E_0$. The condition (2) above is nothing else than the continuity of f_0. Conversely, the theorem itself follows from the Hahn–Banach Theorem (First Version) and even easier from the Mazur–Orlicz Theorem that we are to discuss in Section 13. It can be found as Theorem 4 on page 34 of the 1987 translation of Banach's book [28]. Banach does give some credit to Helly in his notes to Chapter IV on page 143 of the same book, as he had done before in [30]. In those very notes he credits Hahn for the First Version of the Hahn–Banach Theorem. We note that our Theorem 4 is in Hahn's paper [123] as the main step to prove the Hahn–Banach Theorem and he does refer to Helly (though not to Helly's 1912 paper) before he proves it.

4. Banach's proof

The Hahn–Banach Theorem to which the proof of the title of this section alludes is the Second Version of the theorem with that name. (We will, for reasons of convenience, no longer always specify which of the two versions we have in mind.) The introduction of the sublinear function was a marvelous find. Analyzing the role of the sublinear function became one of the main motives in subsequent developments and the applicability and flexibility of the theorem were greatly enhanced by using sublinear functions. The following is Banach's proof on page 20 of the English translation of his book [28].

We may assume that E_0 does not equal E. Take any x_0 in $E \setminus E_0$. Any value A of any extension of f_0 at x_0, dominated by p, will satisfy

$$-p(-x - x_0) - f_0(x) \leq A \leq p(x + x_0) - f_0(x) \quad \text{for all } x \in E_0\,. \tag{3}$$

From the linearity of f_0, the sublinearity of p and the domination of f_0 by p, we get

$$\sup_{x\in E_0}\{-p(-x-x_0)-f_0(x)\} \le \inf_{x\in E_0}\{p(x+x_0)-f_0(x)\}\,. \tag{4}$$

In other words, an A as above is not an impossibility. Choose any such A and define an extension of f_0 to the vector subspace E_1 of E generated by $E_0 \cup \{x_0\}$ by

$$f(x+\alpha x_0) = f_0(x) + \alpha A \quad \text{for all } x \in E_0 \text{ and all } \alpha \in \mathbb{R}\,. \tag{5}$$

Assuming $\alpha \ne 0$, put x/α instead of x in (1). Multiply the left inequality by α when α *is negative* and the right inequality by α when α *is positive*. It follows that

$$f(x) \le p(x) \quad \text{for all } x \in E_1\,.$$

Banach finishes his proof by well-ordering $E \setminus E_0$ and transfinite induction.

A few comments are in order. The contributions of all four, Riesz, Helly, Hahn and Banach, to the Hahn–Banach Theorem deal with inequalities of the type (1) or (2) above. Therefore, the Hahn–Banach Theorem got off the mark with a heavy emphasis on the order properties of the reals. It is not surprising that order continued to play a role in later directions. One may speculate, however, as to how much it hindered research into other avenues. A form of the Hahn–Banach Theorem for vector spaces over $\mathbb{C}$ came surprisingly late (by Sukhomlinov in [320]) and a version for non-Archimedean valued fields had to wait until 1952 (by Ingleton in [146]). How influential Banach's proof has been, can be seen from opening almost any textbook in functional analysis where invariably his proof is copied almost verbatim. (Almost, because most authors seem to prefer Zorn's Lemma rather than transfinite induction. For the novice in functional analysis transfinite induction by well-ordering may seem more direct.) There now are many equally elegant proofs available. It would be advisable for textbook writers to choose a version that suits their coverage of topics best. We continue to present some of the options.

5. The shortest proof

We first set the stage for a theorem that is inducive to accept the method behind the shortest proof of the Hahn–Banach Theorem.

Suppose we do not have one but two sublinear functions, p and p^*. Suppose furthermore that f_0 is dominated by p and by p^*, in other words that

$$f_0(x) \le p(x) \quad \text{and} \quad f_0(x) \le p^*(x) \quad \text{for all } x \in E_0\,. \tag{6}$$

For any extension f of f_0 which is dominated by p and p^* we have

$$f_0(x) = f(x-y) + f(y) \le p(y) + p^*(x-y) \quad \text{for all } x \in E_0 \text{ and all } y \in E\,. \tag{7}$$

We better take (7) as an assumption, that is,

$$f_0(x) \le p(y) + p^*(x-y) \quad \text{for all } x \in E_0 \text{ and all } y \in E\,. \tag{8}$$

By taking $x = 0$, it follows that for all $y \in E$

$$-p^*(-y) \le p(y)\,, \tag{9}$$

and by taking $y = 0$ or $y = x$ we see that (6) follows from (8). Notice that any linear function f from E to $\mathbb{R}$ for which

$$-p^*(-y) \le f(y) \le p(y) \quad \text{for all } y \in E \tag{10}$$

is dominated by p and by p^* (and conversely). Thus the existence of an extension of f_0 dominated by p and p^* is equivalent to the existence of an extension f that satisfies (10). That brings us to a curious question. If $E_0 = \{0\}$ and f_0 is the only functional on E_0, condition (8) is equivalent to condition (9). That is, if we start off with almost nothing (E_0 and f_0 are trivial), can we create a linear function f that satisfies (10)?

It can immediately be checked that the function q defined by

$$q(x) = -p^*(-x)$$

has the following properties:

$$q(x) + q(y) \le q(x+y) \quad \text{for all } x, y \in E \tag{11}$$

and

$$q(\alpha x) = \alpha q(x) \quad \text{for all } x \in E \text{ and all } \alpha \in \mathbb{R}^+\,. \tag{12}$$

A function q with properties (11) and (12) is called *superlinear*. It is time to formulate a theorem.

THEOREM 5 (Sandwich Theorem). *Let p be a sublinear and q a superlinear function on E. If*

$$q(x) \le p(x) \quad \textit{for all } x \in E$$

then there exists a linear function f on E such that

$$q(x) \le f(x) \le p(x) \quad \textit{for all } x \in E\,.$$

The theorem has a variety of interesting aspects. It does not appear to be an extension theorem. Indeed, we only have the superlinear and superadditive function to work with. In Section 13 we will see that the theorem is an easy corollary of the Mazur–Orlicz Theorem (see [229]) and so is a theorem that deals with a pair of seminorms p and p^*. Much of the attention in the previous section was directed to finding a suitable value to be assigned to the next element in E. Is it possible to sand p to make it more linear instead of building up f_0? The answer is yes: Simply order the set of all sublinear functions that dominate f_0 on E_0 and that are dominated by p on E. By Zorn's Lemma choose a minimal element. That minimal element itself is a sublinear function to which the Hahn–Banach

Theorem applies. But then it must be linear and coincide with f_0 on E_0. Even if one writes down the details of the last step, the proof is shorter than Banach's.

6. Luxemburg's proof

Banach's proof may well be the most natural. One is trying to extend a linear function and thus searches for what value to assign to an element which is not in the subspace yet. The short proof in Section 4 focused on the sublinear function instead. The idea in Luxemburg's proof is to allow an extension to take values in a space bigger than $\mathbb{R}$.

Let $\mathcal{D}$ be the set of linear subspaces D of E each containing E_0 and such that $E_0 \subset D$ has finite codimension. Let for each D in $\mathcal{D}$ be given a nonempty set $\Re(D)$ contained in

$$\{f \in \mathbb{R}^D : f \text{ is linear, dominated by } p \text{ and extends } f\}\,.$$

Define

$$A = \{(D, f) : D \in \mathcal{D} \text{ and } f \in \Re(D)\}\,.$$

Let $\wp$ be a filter of subsets of A such that for every finite subset X of E we have that

$$U_X := \{(D, f) \in A : X \subset D\}$$

is an element of $\wp$. Define

$$\mathbb{R}_1 = \{f \in \mathbb{R}^A : f(A) \text{ is bounded}\}$$

and

$$\mathbb{R}_0 = \{f \in \mathbb{R}_1 : f = 0 \ \ \wp\text{-almost everywhere}\}\,.$$

Now we start working on an extension of f_0. Define $\tau : E \to \mathbb{R}^A$ by

$$\tau(x)(D, f) = \begin{cases} f(x) & \text{if } x \in D\,, \\ 0 & \text{if } x \notin D\,. \end{cases}$$

For every $x \in E$ and all $(D, f) \in A$ we have $|\tau(x)(D, f)| \le p(x) \vee p(-x)$, that is, τ maps E into $\mathbb{R}_1$. If $x \in E$ then $\tau(x)(D, f) = f(x)$ for all $(D, f) \in U_{\{x\}}$. Thus $\tau(x + y)(D, f) = (\tau(x) + \tau(y))(D, f)$ for all $(D, f) \in U_{\{x\}} \cap U_{\{y\}} \cap U_{\{x+y\}}$. A linear mind then introduces

$$\mathbb{R}_\wp = \mathbb{R}_1/\mathbb{R}_0\,.$$

We denote the equivalence classes by using $\widetilde{\ }$. Define

$$f_\wp(x) = \widetilde{\tau(x)}\,.$$

We list three easy consequences:

(1) $f_\wp$ is linear,

(2) $f_\wp(x) = \widetilde{f_0(x)}$ for $x \in E_0$,

(3) $f_\wp(x) \le \widetilde{p(x)}$ for all $x \in E$.

For (3) it should be understood that on $\mathbb{R}_1$ we have the pointwise ordering, which makes it a vector lattice, and $\mathbb{R}_0$ is an ideal (see [219]) in $\mathbb{R}_1$, which makes $\mathbb{R}_\wp$ a vector lattice. Diagrammatically we have

$$\begin{array}{ccc} E_0 & \subset & E \\ f\downarrow & & \downarrow f_\wp \\ \mathbb{R} & \subset & \mathbb{R}_\wp \end{array}$$

For the inclusion of $\mathbb{R}$ in $\mathbb{R}_\wp$, we remark that any $\alpha \in \mathbb{R}$ can be associated with the function $i(\alpha)$, that is, identically equal to α, in $\mathbb{R}_1$. The inclusion is shorthand for the map $\alpha \to \widetilde{i(\alpha)}$. Note that we *assumed* that for each D in $\mathcal{D}$ we have a nonempty $\Re(D)$. However, by using only the initial part of Banach's proof of the Hahn–Banach Theorem we *know* that extensions of the desired type exist and we could take $\Re(D)$ to be the set of all extensions. We do not use any well-ordering or Zorn's Lemma and that is the advantage of this approach. The disadvantage is that the "extension" does not take its values in $\mathbb{R}$ but in a bigger space. The proof constructs an extension of $\mathbb{R}$ and an "extension" of f_0. Its motivation was, of course, Robinson's invention of infinitesimals and it can rightfully be called the nonstandard proof of the Hahn–Banach Theorem. Luxemburg's proof first appeared in [216]. Before that, it had been discovered independently by Łoś and Ryll-Nardzewski in [210]. Luxemburg's papers [213] and [214] deal with the same topic. Luxemburg's elegant paper [214] contains much valuable information and we will return to it later.

Note that the actual Hahn–Banach Theorem can be recovered from the above by using the Ultrafilter Theorem. Indeed, one is then enabled to take an ultrafilter $\wp$ in the above and the map $\alpha \to \widetilde{i(a)}$ becomes an isomorphism. It is, in fact, such an ultrafilter that is used by Luxemburg. We will return to this observation later.

7. Nachbin's proof

This is going to be a proof of the First Version of the Hahn–Banach Theorem. Take an a in $E \backslash E_0$. For every $z \in E_0$ define

$$B_z = \{\alpha \in \mathbb{R} : |\alpha - f_0(z)| \leq \|z - a\|\}\,.$$

A reinterpretation of Banach's classical proof yields that $B_z \cap B_{\tilde{z}} \neq \emptyset$ whenever $z \neq \tilde{z}$. In $\mathbb{R}$ that means that $\bigcap B_z \neq \emptyset$. Any element in the latter intersection will serve as an appropriate value for the value of an extension of f_0 at a. Nachbin's Theorem shows that such an intersection property is exactly what makes the Hahn–Banach Theorem work. First some terminology.

Definition 6. A Banach space F is said to have the *binary intersection property* if every collection of closed balls in F, each pair of which has nonempty intersection, has nonempty intersection.

Definition 7. A normed vector space F is said to be 1-*injective* if it can take the role of $\mathbb{R}$ in the first version of the Hahn–Banach Theorem, i.e. for every normed space E and every subspace E_0 and every continuous linear map f_0 from E_0 to F there exists a linear continuous extension $f : E \to F$ such that $\|f\| = \|f_0\|$.

Theorem 8. *For a normed vector space F the following are equivalent.*

(1) *F is* 1*-injective.*
(2) *F has the binary intersection property.*

Nachbin proved this characterization of 1-injective Banach spaces in [244]. We will return to Theorem 8 in Section 15.

8. Mazur's geometric Hahn–Banach Theorem

Like each of the previous sections is one aspect, one way of looking the Hahn–Banach Theorem in the eyes, so is this one. We interpret the Hahn–Banach Theorem in a geometric way. It was first proved by Mazur in [227]. We follow the approach by Day (see [79], pages 23–25). In this section E is a normed space.

Theorem 9. *Let C be a convex set in E for which* $\mathrm{int}(C) \neq \emptyset$. *Let V be a linear variety* (*that is*, *a subset of the form $x + E_0$ where E_0 is a linear subspace of E*) *for which* $\mathrm{int}(C) \cap V = \emptyset$. *Then there exists a closed hyperplane H of E such that*

(1) $\mathrm{int}(C) \cap H = \emptyset$ *and*
(2) $V \subset H$.

There are many variations on that theme. The main technique is the making of gauges or Minkowski functionals, as they are called, from certain convex sets. More precisely, if C is an absorbing set in E (that is, for every $x \in E$ there exists an $\alpha > 0$ such that $x \in \alpha C$) then we can define

$$p_C(x) = \inf\{\alpha > 0 : x \in \alpha C\}\,.$$

Such a p_C is a sublinear function and that is the connection with the Hahn–Banach Theorem. Of course, in the proof of the above theorem one has to translate the set C so that it contains 0 in its interior (an absorbing set contains 0). Also, the interior points of C in the above are the ones for which the value of p_C is less than one. The rest of the proof easily follows from the Hahn–Banach Theorem (Second Version). The conclusion can also be stated as follows: There exists a continuous linear function $f : E \to \mathbb{R}$ and a number α such that $f(x) = \alpha$ for all $x \in V$ and $f(x) < \alpha$ for all $x \in \mathrm{int}(C)$ (that is, the interior of C is *on one side of H*). We state two corollaries, which are of interest in themselves, in order to show how the Hahn–Banach Theorem can be recovered.

THEOREM 10 (Support Theorem). *If x is not an interior point of a convex set C for which* $\mathrm{int}(C) \neq \emptyset$ *then there is a hyperplane H such that $x \in H$ and C is on one side of H.*

THEOREM 11 (Eidelheit Separation Theorem). *Let C_1 and C_2 be convex sets in E. Suppose that*

$$\mathrm{int}(C_1) \neq \emptyset \quad \text{and} \quad C_2 \cap \mathrm{int}(C_1) = \emptyset\,.$$

Then there exists a continuous linear function from E to $\mathbb{R}$ such that

$$\sup f(C_2) \leq \inf f(C_1)\,.$$

Let us now sketch how to prove the Second Version of the Hahn–Banach Theorem from these results.

Define $F = E \times \mathbb{R}$ and $A = \{(x, f_0(x)) : x \in E_0\}$. Also, define $C = \{(x, r) : r \geq p(x)\}$ and $W = C - A$. W is a wedge in F, that is, a convex set which contains with each element all the positive multiples of that element. A wedge orders a space by defining

$$x \leq_W y \quad \text{if} \quad y - x \in W\,.$$

Applying the Support Theorem and the Eidelheit Theorem one can then find a nontrivial monotone linear function g (that is, $x \leq_W y$ implies $g(x) \leq g(y)$) on F. Define the desired extension f of f_0 by

$$f(x) = r \quad \text{if} \quad g(x, r) = 0\,.$$

There are other approaches to the geometric Hahn–Banach Theorem. In particular, we would like to draw attention to [330]. Also, note the geometric approach to the Hahn–Banach Theorem in [284].

The remaining sections contain more special and more general results. We will list all relevant references from our bibliography.

9. The complex numbers

The analogs of the Hahn–Banach Theorem for vector spaces over $\mathbb{C}$ had to wait until 1938 (see e.g. [46] by Bohnenblust–Sobczyk). If we consider functionals that take their values in $\mathbb{C}$, we see that the Second Version, as it stands, no longer makes sense. The First Version still holds word for word. It is worthwhile to remember the trick that is used here. Write $f_0 = f_1 + i f_2$ for two real valued continuous functionals f_1 and f_2 on E_0. Take an extension $\widetilde{f_1}$ of f_1 such that $\|\widetilde{f_1}\| \leq \|f_0\|$. For the latter, remember that

$$|f_1(x)| = |\mathrm{Re}[f_0(x)]| \leq |f_0(x)| \leq \|f_0\|\|x\|$$

and thus we can apply the Second Version of the Hahn–Banach Theorem, considering E as a linear space over $\mathbb{R}$. Now define the desired extension of f_0 by

$$f(x) = \widetilde{f_1}(x) - i\widetilde{f_1}(x)\,.$$

It is easy to check that f is linear and indeed is an extension of f_0. For the calculation of the norm of f observe the following. For $x \in E$ there exist $r, \vartheta \in \mathbb{R}$ such that $f(x) = re^{i\vartheta}$. Then

$$\begin{aligned}|f(x)| &= |e^{-i\vartheta} f(x)| = |f(e^{-i\vartheta} x)| \\ &= f(e^{-i\vartheta} x) = \widetilde{f}_1(e^{-i\vartheta} x) \leq \|\widetilde{f}_1\| \|e^{-i\vartheta x}\| = \|\widetilde{f}_1\| \|x\|\end{aligned}$$

and thus $\|f\| \leq \|\widetilde{f}_1\| \leq \|f_0\|$.

Murray had obtained the above Hahn–Banach Theorem for $\mathbb{C}$ a little earlier for L_p-spaces (see [239]). For the real case the defense of the role of Helly has come up in the literature. The complex Hahn–Banach Theorem, however, is continuously credited to Bohnenblust and Sobczyk ([46]), even where Murray's proof indeed was completely general. The name Soukhomlinoff *is* almost always mentioned as having obtained the result independently and at about the same time in [320]. In fact, he proved the Hahn–Banach Theorem for vector spaces over the quaternions in the same paper. It is well known, and wittily documented, that theorems almost never get associated with their inventors. Why certain results, that do not carry the burden of being name-theorems, consistently get credited to some mathematicians in a certain order (in this case, always first Bohnenblust and Sobczyk, perhaps Murray at the side and, if at all, Soukhomlinoff in the last place) is a mystery. It should be mentioned that there also is a Complex Second Version of the Hahn–Banach Theorem:

THEOREM 12 (Complex Hahn–Banach Theorem, Second Version). *Let E be a vector space over $\mathbb{C}$. Let p be a seminorm on E (i.e. a sublinear function $E \to \mathbb{R}$ for which $p(\alpha x) = |\alpha| p(x)$ for all $\alpha \in \mathbb{C}$ and all $x \in E$). Let E_0 be a vector subspace of E and let f_0 be a linear function $E_0 \to \mathbb{C}$ with*

$$|f_0(x)| \leq p(x) \quad \textit{for all } x \in E_0 \,.$$

Then there exists a linear functional f from E to $\mathbb{C}$ that extends f_0 and such that

$$|f(x)| \leq p(x) \quad \textit{for all } x \in E \,.$$

Of course, the First Version of the Hahn–Banach Theorem for $\mathbb{C}$ follows from the Second Version above.

10. Ingleton's Theorem

We mentioned before how order played a premier role in the first eight sections of this paper. It is easy to downplay the proof in the previous section as a mere trick and it is hard to remember how we reacted to it when we ourselves saw it for the first time. If we leave the field of the complex or real numbers altogether and enter the area of non-Archimedean fields, many functional analysts feel more uneasy. Indeed, even though non-Archimedean functional analysis has seen quite a rise in the number of practitioners, its findings have not yet become common knowledge. It bears resemblance in that respect, but in a different direction, with nonstandard analysis. Nonstandard analysis is sometimes tarnished by the layman as "yet another approach to what I already know". Often, non-Archimedean functional analysis gets the stamp of pathology. In the context of this paper, both of those views seem naive. The nonstandard proof clarifies how one can enlarge the range space and extend the functional f_0 without using any transfinite induction. Order is far away from non-Archimedean valued fields. Therefore, there is hope that a study of the Hahn–Banach Theorem in non-Archimedean setting clarifies the role of order. And yes it does. We follow the standard reference [331] by van Rooij in this section and first meet an old acquaintance.

DEFINITION 13. A metric space E is called *spherically complete* if the collection of its closed balls has the binary intersection property (see Section 7).

The non-Archimedean Hahn–Banach Theorem (First Version) is the following:

THEOREM 14. *Let K be a spherically complete non-Archimedean valued field. Let E be a normed vector space over K. Let E_0 be a vector subspace of E and let f_0 be a continuous linear functional from E to K. Then there exists a continuous linear function $f : E \to K$ that extends f_0 and for which*

$$\|f\| = \|f_0\| .$$

Notice that we have seen in Section 7 how spherical completeness of the reals plays an important role in the Hahn–Banach Theorem. Also consider that $\mathbb{C}$ is not spherically complete. Spherical completeness was studied by Krull before the Hahn–Banach Theorem even got its name (see [188]). One wonders about the effect that general knowledge about this field of fascinating results would have within the larger community of functional analysis. As in Nachbin's characterization of injective Banach spaces over the reals, more than in the above theorem is true.

THEOREM 15. *Let E, F be normed vector spaces over a non-Archimedean field K. Let E_0 be a vector subspace of E and let f_0 be a continuous linear map $E_0 \to E$. If either E_0 or F is spherically complete then there exists a continuous linear map $f : E \to F$ that extends f_0 and for which*

$$\|f\| = \|f_0\| .$$

DEFINITION 16. A normed vector space F over a non-Archimedean valued field K is said to be *injective* if it can take the role of K in Theorem 14, i.e. for every normed space E over K and every subspace E_0 of E and every continuous linear map f_0 from E_0 to F there exists a continuous linear extension $f : E \to F$ such that $\|f\| = \|f_0\|$.

The previous theorem implies half of Ingleton's Hahn–Banach Theorem.

THEOREM 17. *A normed vector space over a non-Archimedean valued field is injective if and only if it is spherically complete.*

An injective normed vector space has to be complete here as well as in Section 7. The similarity with Section 7 is too close for there not to be a common generalization to both, Nachbin's Theorem and Ingleton's Theorem. Indeed, see [281] by Rodríguez-Salinas and Bou.

Not only does non-Archimedean functional analysis clarify the roles of $\mathbb{R}$ and $\mathbb{C}$ in classical functional analysis, it also has a life of its own. There is, for instance, the following theorem.

THEOREM 18. *In the situation of Theorem* 15 *there exists a linear isometry* $f_0 \to \bar{f}_0$ *from the space of all linear continuous maps* $E_0 \to F$ *to the space of all linear continuous maps* $E \to F$ *such that for every* f_0, $\bar{f}_0$ *is an extension of* f_0.

That this is very different from the classical situation will be seen when we talk about simultaneous Hahn–Banach extensions. Much more can be said, but we refer the interested reader to the following papers and the references therein: [331] and [332] by van Rooij, [281] by Rodríguez-Salinas and Bou, [258] and [259] by Pérez-García, [224] by Martínez-Maurica and Pérez-García and [81] by De Grande - De Kimpe and Pérez-García.

11. Constructive analysis and unique extensions

Those who have been reluctant to use the Axiom of Choice or its associates have come up with a variety of remedies to save the Hahn–Banach Theorem. One is to only allow a restricted form of the Axiom of Choice, usually the Countable Axiom of Choice or the Axiom of Dependent Choices. Generally speaking, that restricts the game to separable spaces and saves all the classical methods of proving the Hahn–Banach Theorem. We refer to [111] by Garnir – de Wilde – Schmetz and [339] by de Wilde. In the style of Bishop type analysis (see [56]) the best theorem around is Theorem 21 below by Ishihara (see [149]). Before we phrase it, some terminology has to be introduced.

DEFINITION 19. Let E be a normed vector space with norm $\| \; \|$. The norm is said to be *Gateaux differentiable* at $x \in E$ if for all $y \in E$

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t}$$

exists.

DEFINITION 20. The normed space $(E, \| \; \|)$ is said to be *uniformly convex* if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that for all $x, y \in E$ with $\|x\| = \|y\| = 1$ we have

$$\|x - y\| \geq \varepsilon \Rightarrow \left\| \frac{x + y}{2} \right\| \leq 1 - \delta \, .$$

THEOREM 21 (Ishihara's Hahn–Banach Theorem). *Let E_0 be a vector subspace of a uniformly convex Banach space E with a Gateaux differentiable norm. Let f_0 be a normable, nonzero continuous linear function $E_0 \to \mathbb{R}$. Then there exists a unique linear normable continuous extension f of f_0 from E to $\mathbb{R}$ such that $\|f\| = \|f_0\|$.*

The word *normable* in the above emerges because of the constructivist's way of doing functional analysis and we refer once more to [56] for its understanding. However, if you disregard that word in Ishihara's Theorem, then you see a theorem that you can also read through your classical glasses. Naturally, the theorem is still correct with those glasses. In fact, the differences with the First Version of the Hahn–Banach Theorem are these: There are the extra assumptions of uniform convexity and Gateaux differentiability and the strengthened conclusion of a unique extension. What classical theorems deal with uniqueness of Hahn–Banach extensions?

THEOREM 22 (Phelps in [260]). *f_0 in E_0^* has a unique norm preserving extension in E^* if and only if the annihilator*

$$H^{\perp} = \{f \in E^* : f|_{E_0} = 0\}$$

has the Haar property, i.e. for each f in E^* there exists a unique g in $H^{\perp}$ such that

$$\|f - g\| = \inf\{\|f - h\| : h \in H^{\perp}\}\,.$$

The latter theorem is the best result for an individual f_0. If you wish to investigate all f_0 and all E_0 simultaneously, the result is older:

THEOREM 23 (Taylor in [325] and Foguel in [100]). *There is unicity of extension for all E_0 and all f_0 if and only if E^* is uniformly convex.*

The words "uniformly convex" are the link to Ishihara's Theorem. The Gateaux differentiability had arisen even earlier, in 1932.

THEOREM 24 (Ascoli in [21]). *A linear functional f_0 on the one-dimensional subspace generated by x_0 can be extended uniquely if the norm on E is Gateaux differentiable in x_0 (or, if $f_0 \in E_0^*$ assumes its norm at x_0 where the norm is Gateaux differentiable then there exists a unique linear norm preserving extension in E^*).*

By using a famous theorem by James (see e.g. [98]) and combining the above, one can formulate some results where the conditions are on E and not on E^*.

THEOREM 25. *If the unit ball in E is weakly compact and if the norm in E is Gateaux differentiable then every continuous linear functional on every vector subspace of E can be uniquely extended to a continuous linear functional on E with the same norm.*

And in particular:

THEOREM 26. *If E is a reflexive Banach space and the norm on E is Gateaux differentiable then Hahn–Banach extensions are unique.*

Looking at Ishihara's Theorem the reader will notice that the above results are only slightly stronger than Ishihara's viewed-through-classical-glasses.

To look at classical results with the eye of a constructivist is more difficult than the other way around though. Constructive interpretations of classical notions may not be available. Does uniqueness in classical mathematics (that is, in ZFC) ensure existence in constructive mathematics? The following is a test case for that question. Denote by $B(\ell_2)$ the Hilbert space of all continuous linear operators $\ell_2 \to \ell_2$. By multiplication we have

$$\ell^\infty \subset B(\ell_2) \,.$$

The elements of $\ell^{\infty *}$ are complicated but the extreme points of its unit ball are easily identified with the elements of $\beta\mathbb{N}$. Using the Hahn–Banach Theorem and just a little of the Krein–Milman Theorem one extends such extreme points to extreme points of the unit ball of $B(\ell_2)^*$. In 1959 (see [155]) Kadison and Singer investigated the following:

CONJECTURE 27. *Such extensions are unique.*

The question is still open and recent studies (see Akeman–Anderson in [3] and Bourgain–Tzafriri in [53]) favor a positive answer.

We should mention one more constructive extension theorem, even though it is a special case of Theorem 26: In Hilbert spaces life is easy because of the Riesz–Fischer Theorem. Other references to uniqueness results are [82] by de Guzmán, [138] by Hennefield, [177] by Kolumban, [195] by Lima, [267] by Poulsen and [302] by Shukla.

12. The Axiom of Choice and the Ultrafilter Theorem

How effective is the Hahn–Banach Theorem? The previous section investigated parts of mathematics where the Hahn–Banach Theorem can be applied with no or little choice. Here, we want to know what is needed to prove the full Hahn–Banach Theorem.

The Ultrafilter Theorem states that for every Boolean algebra B and for every filter F in B there is an ultrafilter U that contains F. Like the Hahn–Banach Theorem, the Ultrafilter Theorem cannot be proved in ZF-set theory. As we mentioned in Section 6, Łoś and Ryll-Nardzewski were the first to show that the Ultrafilter Theorem suffices to prove the Hahn–Banach Theorem (see [210]). Of course, this gets more interesting as soon as one knows that the Ultrafilter Theorem does not imply the Axiom of Choice. This was proved by Halpern in [125]. Next we may ask: Is the Hahn–Banach Theorem equivalent to the

Ultrafilter Theorem? Luxemburg originally conjectured that it might, but Pincus (see [264] and [265]) showed that the answer is negative. Pincus uses model-theory and the present paper does not permit us to discuss his idea in detail. There are other ways, though, in which one gets an inkling of the difference between the Ultrafilter Theorem and the Hahn–Banach Theorem. We discuss two of such results. For the first of them we refer to page 179 of [290].

THEOREM 28. *The following are equivalent.*

(1) *The Axiom of Choice.*
(2) *The Krein–Milman Theorem and the Ultrafilter Theorem.*

To relate the previous theorem to the Hahn–Banach Theorem, we need a definition.

DEFINITION 29. A subset A of a topological vector space X is called *convex -compact* if whenever $\wp$ is a collection of closed convex subsets of A with the finite intersection property then $\bigcap \wp \neq \emptyset$.

One gets the statement Convex Krein–Milman Theorem by replacing the word *compact* in the Krein–Milman Theorem by *convex-compact*. For the next theorem we again refer to page 179 of [290].

THEOREM 30. *The following are equivalent.*

(1) *The Axiom of Choice.*
(2) *The Convex Krein–Milman Theorem and the Hahn–Banach Theorem.*

The distinction between the Ultrafilter Theorem and the Hahn–Banach Theorem is further explored in the next two theorems.

THEOREM 31 (Tarski in [324]). *The following are equivalent.*

(1) *If μ_0 is a measure on a subalgebra B_0 of a Boolean algebra B then there exists a measure μ on B with $\mu = \mu_0$ on B_0 and $\mu(B)$ is contained in the closure of the range of $\mu_0(B_0)$.*
(2) *The Ultrafilter Theorem.*

THEOREM 32 (Luxemburg in [214]). *The following are equivalent.*

(1) *If μ_0 is a measure on a subalgebra B_0 of a Boolean algebra B then there exists a measure μ on B with $\mu = \mu_0$ on B_0 and the range $\mu(B)$ is contained in the closure of the convex hull of the range of the convex hull of $\mu_0(B_0)$.*
(2) *The Hahn–Banach Theorem.*

There exist more special Hahn–Banach-like theorems that are equivalent to the Ultrafilter Theorem. In the context of vector lattices, for instance, the following setup exists.

DEFINITION 33. Let E be a vector lattice. A sublinear map $p : E \to \mathbb{R}$ is called a $\vee$-*homomorphism* if

$$p(x \vee y) = p(x) \vee p(y) \quad \text{for all } x, y \in E\,.$$

THEOREM 34. *Let E and p be as in the previous definition. Let E_0 be a vector sublattice of E and let f_0 from E_0 to $\mathbb{R}$ be a linear lattice homomorphism such that*

$$f_0(x) \le p(x) \quad \textit{for all } x \in E_0\,.$$

Then there exists a linear lattice homomorphism $f : E \to \mathbb{R}$ that extends f_0 and for which

$$f(x) \le p(x) \quad \textit{for all } x \in E\,.$$

Theorem 34 was first proved by Schmidt in [293] (also see [65] and [133]) using Alaoglu's Theorem and the Krein–Milman Theorem, a combination that implies the Axiom of Choice. It was then shown by Buskes and van Rooij in [64] to be equivalent to the Ultrafilter Theorem.

There still are many open problems connected with this section. We mention three of them. In non-Archimedean functional analysis we discussed Ingleton's Hahn–Banach Theorem in Section 10. Can Luxemburg's approach to the ordinary Hahn–Banach Theorem (see Section 6) be used to prove Ingleton's Theorem as well? Van Rooij answers that question positively for the case of spherically complete locally compact non-Archimedean valued fields in [332]. Since there are many more non-Archimedean valued fields it is not unreasonable to conjecture (it is a question in [332]) the following:

CONJECTURE 35. *Ingleton's Theorem is equivalent to the Axiom of Choice.*

On the other hand, there is

CONJECTURE 36. *Ingleton's Theorem for locally compact fields is equivalent to the Ultrafilter Theorem.*

Related to the above Hahn–Banach Theorem for linear lattice homomorphisms is Sikorski's Extension Theorem for Boolean algebras. For the many papers in that direction see [24] by Bacsich, [40] by Bell, [62] by Buskes, [71] by Cignoli, [215] by Luxemburg, [236] by Monteiro, [305] and [306] by Sikorski. Bell has shown that Sikorski's Extension Theorem is not equivalent to the Ultrafilter Theorem and Luxemburg on many occasions has asked whether it is equivalent to the Axiom of Choice. Notice that there is a Hahn–Banach Theorem (Second Version) of Sikorski's Extension Theorem in [236]. The relations between various type Hahn–Banach Theorems in the context of foundations have not been studied very extensively. Lately, in the theory of topoi, advances have been made, without using the Axiom of Choice, by Banaschewski in [34] and Mulvey–Pelletier in [238]. Of course, such results might belong more to Section 11 than to the present section.

13. The Mazur–Orlicz Theorem

We have had occasion before to mention the Mazur–Orlicz Theorem. It is difficult to overstate its importance.

THEOREM 37 (Mazur–Orlicz). *Let E and p be as in the Second Version of the Hahn–Banach theorem. Let T be a set and let $\varphi : T \to E$ and $\alpha : T \to \mathbb{R}$ be mappings. Then the following are equivalent.*

(1) *There exists a linear $f : E \to \mathbb{R}$ with*
  (i) *$f \le p$ and*
  (ii) *$\alpha(t) \le f(\varphi(t))$ for all $t \in T$.*

(2) *For all $k \in \mathbb{N}$ and all $\tau_1, \ldots, \tau_k \in \mathbb{R}^+$ and all $t_1, \ldots, t_k \in T$ we have*

$$\sum_{n=1}^{k} \tau_n \alpha(t_n) \le p\Big(\sum_{n=1}^{k} \tau_n \varphi(t_n)\Big).$$

Not only has the theorem been influential, but equally significant is the proof that Pták provided. We present Pták's proof.

Assuming (2), define

$$q(x) = \inf\Big\{p\Big(x + \sum_{n=1}^{k} \tau_n \varphi(t_n)\Big) - \sum_{n=1}^{k} \tau_n \alpha(t_n) : \\ k \in \mathbb{N} \text{ and } \tau_1, \ldots, \tau_k \ge 0 \text{ and } t_1, \ldots, t_k \in T\Big\}.$$

Note that the elements in the above expression are bounded below by $-p(-x)$, which makes it possible to take the infimum. It is a fact that q is subadditive and positive-homogeneous and that $q \le p$. Also observe that

$$q(-\varphi(t)) \le p(-\varphi(t) + \varphi(t)) - \alpha(t) = -\alpha(t) \quad \text{for all } t \in T.$$

By the Hahn–Banach Theorem there exists a linear $f : E \to \mathbb{R}$ such that

$$f \le q.$$

But then by the inequality above

$$f(-\varphi(t)) \le -\alpha(t)$$

and therefore

$$\alpha(t) \le f(\varphi(t)).$$

That proves the implication (2)⇒(1) above. The implication (1) ⇒(2) is easy.

The original proof by Mazur and Orlicz in [228] is complicated. Pták's proof appeared in [268]. Notice how the theorem has some resemblance to Helly's Theorem 4 above. Before Pták, Sikorski had simplified the Mazur–Orlicz proof (see [306]) with a geometric argument which is reminiscent of the techniques in

Section 8. There are a few particular consequences of the Mazur–Orlicz Theorem that we present to exemplify its versatility.

THEOREM 38. *Let E and p be as in the Second Version of the Hahn–Banach Theorem. Let $e \in E$. Then there exists a linear $f : E \to \mathbb{R}$ with $f \le p$ and $f(e) = p(e)$. Moreover, p is the pointwise supremum of all the linear f that it dominates.*

Proof. Take in the Mazur–Orlicz Theorem $T = \{t\}$, a singleton set, $\varphi(t) = e$ and $\alpha(t) = p(e)$. The condition (2) in the Mazur–Orlicz Theorem is easily verified and we find a linear $f : E \to \mathbb{R}$ with

$$f \le p \quad \text{and} \quad p(e) = \alpha(t) \le f(\varphi(t)) = f(e)$$

and therefore $f(e) = p(e)$. The second part of the theorem follows from the first part.

THEOREM 39. *The Hahn–Banach Theorem (Second Version) follows from the Mazur–Orlicz Theorem.*

Proof. Take $T = E_0$, φ the identity on E_0 and $\alpha(t) = f_0(t)$ for all $t \in E_0$. The condition (2) in the Mazur–Orlicz Theorem now corresponds to the requirement that $f_0 \le p$ in the Hahn–Banach Theorem.

THEOREM 40. *The Sandwich Theorem (Theorem 5) follows from the Mazur–Orlicz Theorem.*

Proof. This time we take $T = E$, φ is the identity on E and $\alpha(t) = q(t)$ for all $t \in T$. Condition (2) in the Mazur–Orlicz Theorem then corresponds precisely to the condition $q \le p$ in the Sandwich Theorem.

It is an entertaining exercise to show (within ZF-set theory) that the Hahn–Banach Theorem, the Mazur–Orlicz Theorem, the geometric Hahn–Banach Theorem and the Sandwich Theorem are pairwise equivalent.

Returning more to the roots of the Hahn–Banach Theorem, there is the following moment theorem.

THEOREM 41. *Let E, p, T, α and φ be as in the Mazur–Orlicz Theorem. Then the following are equivalent.*

(1) *There exists a linear $f : E \to \mathbb{R}$ such that*

 (i) *$f \le p$ and*
 (ii) *$\alpha(t) = f(\varphi(t))$ for all $t \in T$.*

(2) *For all $k \in \mathbb{N}$, for all $\tau_1, \ldots, \tau_k \in \mathbb{R}^+$ and all $t_1, \ldots, t_k \in T$ we have*

$$\sum_{n=1}^{k} \tau_n \alpha(t_n) \le p\Big(\sum_{n=1}^{k} \tau_n \varphi(t_n)\Big).$$

If you look at the previous theorem in normed spaces, Helly's Theorem 4 of Section 3 appears. There are many papers devoted to the Mazur–Orlicz Theorem, its analogous versions and its applications. We refer to [104] by Fuchssteiner and Lusky, [105] by Fuchssteiner and König, [166] by Kaufman, [173] by Kindler, [180] and [181] by König, [183] and [185] by Kranz, [248], [249] and [250] by Neumann, [296] by Seever, [312] and [314] by Simons.

14. Simultaneous Hahn–Banach extensions

Hilbert spaces were behind the scenes all the time in Section 11. In the present section they become the forerunners. Dually (but not in the sense of the category theory in [297]) to injectivity, which we studied a little in Section 7, there is projectivity.

DEFINITION 42. A Banach space E is called *projective* if for every Banach space F and for every subspace E_0 of E and every continuous linear map f_0 from E_0 to F there exists a norm preserving linear extension $E \to F$.

THEOREM 43. *For a Banach space E the following are equivalent.*

(1) *E is projective.*
(2) *E is a Hilbert space or* $\dim(E) \leq 2$.

This theorem holds for real as well as complex vector spaces. For vector spaces over $\mathbb{R}$ the result is due to Kakutani in [156], for vector spaces over $\mathbb{C}$ the result was proved by Sobczyk (see [317]). We also wish to refer here for results about projectivity and injectivity to the beautiful survey article by Nachbin (see [246]).

Hilbert spaces also play a major role in Theorem 45, for which we need

DEFINITION 44. Let E and E_0 be like in the First Version of the Hahn–Banach Theorem. A linear map $T : E_0^* \to E^*$ is called a *linear extension operator* if for every f_0 in E_0^*

(1) $T(f_0)$ is an extension of f_0 and
(2) $\|T(f_0)\| = \|f_0\|$.

THEOREM 45. *For a Banach space E the following are equivalent.*

(1) *There exists a linear extension operator for every* (*two-dimensional*) *subspace of E.*
(2) *E is a Hilbert space.*

There now is much more information about so-called simultaneous extensions like T above. Important papers in this direction are [93] by Fakhoury, [134] by Heinrich and Mankiewicz and [347] by Yost and Sims. We want to say a little more about the last two papers. Using ultrapowers (nonstandard analysis for the working mathematician) Heinrich and Mankiewicz proved the next theorem, which says that for every subspace E_0 of E there exists a larger subspace $\widetilde{E}_0$ which is not much bigger and a linear extension operator $\widetilde{E}_0^* \to E^*$. We need to explain what it means to be not much bigger first.

Definition 46. The *density character* of a Banach space E is the smallest cardinal number for which there exists a dense subset of E with that cardinality. We denote the density character of E by $\mathrm{dens}(E)$.

Theorem 47. *Let E be a Banach space. For every vector subspace E_0 of E there exist a vector subspace $E_0 \subset \widetilde{E}_0$ of E such that* $\mathrm{dens}(E_0) = \mathrm{dens}(\widetilde{E}_0)$ *and a linear extension operator $\widetilde{E}_0^* \to E^*$.*

Yost and Sims recently (in [347]) proved the same result, in a straightforward way and without using ultrapowers, returning to earlier techniques as employed by Lindenstrauss in [200].

15. Injective Banach spaces and injective Banach lattices

We take up here where we left in Section 7. Nachbin characterized injective spaces in 1950 (also see [245]). Before we continue that discussion we remark that there are other ways of defining 1-injectivity.

Proposition 48. *Let E be a Banach space. Then the following are equivalent.*

(1) *E is 1-injective.*

(2) *For every Banach space $\widetilde{E}$ with $\widetilde{E} \supset E$ there exists a surjective linear projection $P : \widetilde{E} \to E$ with $\|P\| \leq 1$.*

(3) *If f is a linear isometry from E into a Banach space $\widetilde{E}$ then there exists a linear $g : \widetilde{E} \to E$ with $\|g\| \leq 1$ such that $g \circ f$ is the identity operator on E.*

The very first result about 1-injectivity of Banach spaces came from Phillips in 1940 (see [262]). He showed that $\ell^\infty(S)$ for any set S is injective by applying the classical Hahn–Banach Theorem coordinatewise. In fact, the following clarification of Theorem 8 is valid.

Theorem 49. *Let E be a Banach space. Then the following are equivalent.*

(1) *E is 1-injective.*

(2) *E has the binary intersection property.*

(3) *E is a Dedekind complete vector lattice with strong order unit.*

(4) *E is isomorphic to $C(X)$ for an extremally disconnected compact Hausdorff space X.*

Not all in the above theorem is due to Nachbin. In fact, the characterization of 1-injectivity in (4) above, under the additional assumption that the unit ball in E has an extreme point, is due independently to Nachbin and Goodner (see respectively [244] and [117]). That the condition about an extreme point was unnecessary was shown by Kelley in 1952 (see [172]). To put this result in the perspective of its time we refer to [4] and [5] (by Akilov) and [319] (by Sobczyk). The condition (3) above once again emphasizes the connection with order. In fact, a reader may suspect that order will play an even more pronounced role in the Second Version of the Hahn–Banach Theorem when the range space is varied. We will satisfy that suspicion when we discuss the relation between the Hahn–Banach Theorem and the least upper bound property in the next section.

A few comments about injective Banach spaces over $\mathbb{C}$. Hasumi in [130] proved the equivalence of (1) and (4) above in that category. Since the main idea in the results of Nachbin is based on a study of the binary intersection property, positively absent in $\mathbb{C}$, the techniques in Hasumi's argument are understandably very different. He uses a selection (see [232]) argument. There are other places in the Hahn–Banach literature where selection theorems play a role (see e.g. Pełczyński in [257]). It is fascinating to watch when, so to speak, the Tietze Extension Theorem teams up with the Hahn–Banach Extension Theorem. With regard to Hasumi's result, [74] by Cohen and [165] by Kaufman should not be overlooked. As to injective Banach spaces in general, the reader should consult [297] with references first. Furthermore, there are [129] by Hasumi and Seever, [266] by Pothoven, [285] and [286] by Rosenthal and the appendix by Pełczyński and Bessaga in the English translation of Banach's book [28]. The connection to projective topological spaces is Gleason's [114].

In the various forms of the definition of 1-injective spaces above, the condition that certain operators have norms less than or equal to 1 can be found. If one replaces that condition with less than or equal to λ (and in Definition 7 one puts $\|f\| \leq \lambda\|f_0\|$) the result is a set of equivalent definitions for so-called λ-*injective* spaces. If a space is λ-*injective* for some λ, we say that the space is *injective.* Day in his book [79] on page 125 remarks that there is no characterization corresponding to Nachbin's Theorem above for λ-injective spaces. Unfortunately, there is little change today. The problem is still open.

PROBLEM 49A. *Characterize the λ-injective Banach spaces for $\lambda \neq 1$.*

Injectivity is a notion of category theory. As such there are objects and mappings. Our previous discussion was in the category of Banach spaces and continuous linear maps. Knowing of the intimate relation between the Hahn–Banach Theorem and partial order, the category of Banach lattices and positive linear maps needs attention. Notice that the objects and the maps are more restricted but that the extensions need to have a stronger property (i.e. positivity) as well. The outcome is surprising and most of the surprise is in the following theorem.

THEOREM 50. *$L^1(\mu)$ is a 1-injective Banach lattice.*

This theorem was proved by Lotz in [211]. Important contributions were made in [67] by Cartwright, where the influence of Nachbin's result and some of Lindenstrauss' papers (e.g. [196] and [202]) nicely combine into more examples of 1-injective Banach lattices. The focus of research in this direction was an investigation into L^∞-combinations of L^1's and L^1-combinations of L^∞'s as 1-injective Banach lattices. The results are too technical to record here. A conclusive paper is [132] by Haydon. For λ-injective Banach lattices, the situation is a little better than for λ-injective Banach spaces, but a characterization is still missing. However, for the isomorphic characterization of injective Banach lattices there are some partial results in [203] by Lindenstrauss and Tzafriri and in [221] by Mangheni.

We mention two more directions of research. One is about injective hulls of Banach spaces. The fundamental paper is [74] by Cohen and additional references are [73] by Cohen and Lacey, [77] by Daigenault, [148] by Isbell and [165] by Kaufman. The other is about separably-injective spaces; the final paper on the topic is [350] by Zippin. For Banach lattices see [63] by Buskes.

16. The interpolation property

In Banach's proof of the Hahn–Banach Theorem a number A is chosen such that

$$-p(-x-x_0)-f_0(x)\leq A\leq p(x+x_0)-f_0(x)\qquad \text{for all } x\in E_0\,.$$

In other words, it is used that $\mathbb{R}$ has the interpolation property or, equivalently, the least upper bound property. Early on there was an effort to understand the role of the least upper bound property in Banach's proof. Kantorovich was the first to investigate ordered spaces F instead of $\mathbb{R}$ in [162]. We consider a linear space F ordered with a wedge (see Section 8).

Definition 51. F is said to have the *interpolation property* if for each pair of subsets U,V of F with $a\leq b$ for all $a\in U$ and all $b\in V$ there exists $c\in F$ such that $a\leq c\leq b$ for all $a\in U$ and all $b\in V$.

Definition 52. F is said to have the *least upper bound property* if every set in F which has an upper bound has a least upper bound.

Clearly F has the least upper bound property if and only if it has the interpolation property. It is a matter of copying Banach's proof word by word to find that

Theorem 53. *The Hahn–Banach Theorem* (*Second Version*) *holds with* $\mathbb{R}$ *replaced by an ordered space* F *with the interpolation property.*

In the reverse direction, there is an interesting history. In 1959 Silverman and Yen announce that one needs an extra condition on the wedge for the converse (see [308]). They name the extra condition *lineally closed* (see page 25 of [79]). To summarize, they prove that an ordered space F can replace $\mathbb{R}$ in the Hahn–Banach Theorem (Second Version) if and only if it has the interpolation property and its wedge is lineally closed. In 1966 Bonnice and Silverman (see [48]) find that the counterexample to show that *lineally closed* could not be left out from the 1959 paper was wrong. Moreover, they prove that it can be left out if F is finite-dimensional. A year later Bonnice and Silverman in [47] prove that the condition was unnecessary altogether. Thus, the new status was that the interpolation property is necessary and sufficient for the Hahn–Banach Theorem. In 1969, Ting On To finds a mistake in the 1966 paper by Silverman and Bonnice (see [326]) and he fixes the finite-dimensional case. In 1970 (see [327]), he gives a proof of the general case (i.e., the interpolation property is equivalent to the extension property). All of the work in the history above is geometric in nature and technical in execution. After trying to figure out what is right and what is wrong in this sequence of events, there is a sigh of relief with the 1981 paper [147]. Ioffe proves in it the equivalence of the Hahn–Banach Extension Theorem for F to the interpolation property for F in an elegant fashion that fits well into the central themes of this paper. In the process he reproves the Hahn–Banach Theorem in the following way. Recall that Nachbin's investigations about the First Version of the Hahn–Banach Theorem relied on a certain intersection property of closed balls. We replace the role of the closed balls by a collection $\Re$ of subsets of F with certain invariance properties:

(i) If $A \in \Re$ and $f \in F$ then $f + A \in \Re$,
(ii) If $A_1 \in \Re$ and $A_2 \in \Re$ then $A_1 + A_2 \in \Re$,
(iii) If $A \in \Re$ and $\lambda \in \mathbb{R}$ then $\lambda A \in \Re$.

Furthermore, it is required that every element of $\Re$ is convex.

Definition 54. A map $\psi : E \to \Re$ is called a *fan* if

(i) $\psi(\alpha x) = \alpha\psi(x)$ for all $x \in E$ and all $\alpha \in \mathbb{R}^+$,
(ii) $\psi(x + y) \subset \psi(x) + \psi(y)$ for all $x, y \in E$,
(iii) $0 \in \psi(0)$,
(iv) $\psi(x) \neq \emptyset$ for all $x \in E$.

A fan ψ is called *odd* if $\psi(-x) = -\psi(x)$ for all $x \in E$. Fans are the analogs of sublinear functions.

Definition 55. A linear map $f : E \to F$ is called a *selection for the fan* ψ if $A(x) \in \psi(x)$ for every $x \in E$.

Theorem 56 (Ioffe). *The following are equivalent.*

(1) *Every selection for every odd fan on any subspace of E extends to a selection on all of E.*

(2) $\Re$ *has the binary intersection property.*

Ioffe mentions that the implication (2)⇒(1) was already proved in [282] (see also page 75 in [104]).

A commonly held conviction says that if a Hahn–Banach Theorem holds for $\mathbb{R}$ as a range space then it holds more generally for any Dedekind complete ordered space. That conviction turns out to hold true surprisingly often. However, one can *not* always copy the $\mathbb{R}$-valued *proof*. The difficulties in mimicking $\mathbb{R}$-valued proofs are often caused when a Hahn–Banach argument is combined with a topological technique, like the Krein–Milman Theorem. We also remark that not much research has been done to see which parts of Hahn–Banach Theorems hold if the range space is σ-Dedekind complete or has the σ-interpolation property. In [64] by Buskes and van Rooij one can find a theory of ultrapowers for Dedekind complete Riesz spaces which makes many nonstandard arguments accessible to a wider range of spaces.

17. Invariant extensions

In this section we follow the treatment by Larsen in [193] of some results by Silverman ([309], [310] and [311]) and Klee [175]. There are connections here with Banach limits, fixed point theorems, the Banach–Tarski paradox and amenability. Naturally, we also have to point in the direction of Greenleaf's book [119].

THEOREM 57. *Let E, E_0, f_0 and p be as in the Hahn–Banach Theorem (Second Version). Let G be an Abelian semigroup of transformations $E \to E$ such that $T(E_0) \subset E_0$ for all $T \in G$. Suppose furthermore that*

$$f_0(T(x)) = f_0(x) \quad \textit{and} \quad p(T(x)) \leq p(x)$$

for all $x \in E_0$ and for all $T \in G$. Then there exists an extension $f : E \to \mathbb{R}$ of f_0 such that

$$f(T(x)) = f(x) \quad \textit{for all } x \in E \textit{ and all } T \in G\,.$$

The proof is ingenious. A new subadditive function is defined by

$$p_0(x) = \inf\left\{\frac{p(\sum_{k=1}^{n} T_k(x))}{n} : n \in \mathbb{N} \text{ and } \{T_1, \ldots, T_n\} \subset G\right\} \quad \text{for all } x \in E.$$

Each term in the right hand side is not smaller than $-p(-x)$ and a proof of the subadditivity uses the fact that G is Abelian. The gain is in the following property:

$$p_0(x - T(x)) = p_0(T(x) - x) = 0 \qquad \text{for all } x \in E \text{ and all } T \in G\,.$$

Using the Hahn–Banach Theorem with p_0 instead of p we find an extension f of f_0 with

$$-p_0(T(x) - x) \leq f(x - T(x)) \leq p_0(x - T(x)) \quad \text{for all } x \in E \text{ and all } T \in G\,.$$

In other words, $f(T(x)) = f(x)$ for all $x \in E$ and all $T \in G$.

The ordinary Hahn–Banach Theorem follows from the previous theorem by considering the group G with the identity operator only. The aforementioned results in [309], [310], [175] and [193] contain more information and applications. More in particular, Klee in [175] relaxes the conditions on the semigroup G. In his paper, G is not necessarily Abelian. His definition of an auxiliary subadditive function bears resemblance to both p_0 above as well as q in Pták's proof of the Mazur–Orlicz Theorem. His main result covers the above theorem, but also the situation in which each finite subset of G generates a finite subgroup of G. Silverman in [309], [310] and [311] continued the study of which semigroups admit invariant extensions. It is interesting to compare the five classes that admit such invariant extensions in Silverman's work with the five classes that occur in 10 of the beautiful book [335] by Wagon.

This paper would not be complete without at least one proof of the existence of Banach limits. For each convergent sequence of real numbers (a_n) we define

$$f_0((a_n)) = \lim_{n\to\infty} a_n .$$

This defines a linear function on $\mathbf{c}$, the linear subspace of all convergent sequences of ℓ^∞. We define $p((a_n)) = \|(a_n)\|_\infty$ and $T((a_n)) = (a_{n+1})$. Furthermore, $G = \{T^n : n \in \mathbb{N}\}$. Applying the above theorem now yields a Banach limit. Banach proved the existence of such a limit in his book [28] (see page 20 of the English translation) as an application of the Hahn–Banach Theorem.

18. Locally convex spaces

A particular consequence of the Hahn–Banach Theorem is the fact that E^* separates the points of E for normed vector spaces E. Actually, the First Version of the Hahn–Banach Theorem holds for a seminorm just as well as for a norm. This implies that the dual of a locally convex space separates the points.

THEOREM 58. *In a locally convex topological space E the following hold.*

(1) *If E_0 is a vector subspace of E and f_0 is a continuous linear functional on E then there exists $f \in E^*$ such that $f|_{E_0} = f_0$.*

(2) *E^* separates the points of E.*

We will say that a topological vector space E has the Hahn–Banach Extension Property (in short HBEP) if it has property (1) of the theorem above. For non-locally convex topological vector spaces the situation is rather different. Some spaces do have HBEP and some do not. The one and only result (that I know of) which characterizes HBEP for a large class of spaces is due to Kalton. Kalton proved in [160] the following result for F-spaces (also see Kalton–Peck–Roberts in [159] and Kalton in [161]).

THEOREM 59 (Kalton). *For every F-space E the following are equivalent.*

(1) *E is locally convex.*

(2) *E has the Hahn–Banach Extension Property.*

In [161], Kalton remarks that the following problem is still unsolved.

PROBLEM 60A. *Is every metrizable topological vector space with the Hahn–Banach Extension Property automatically locally convex*?

Partial results and references to more partial results to the above problem can be found in [161] and [272].

19. Non-commutative Hahn–Banach Theorems

One of the amazing features of the Hahn–Banach Theorem is its flexibility. On first sight it seems to depend totally on order. Nonetheless, the Hahn–Banach Theorem for $\mathbb{C}$ and Ingleton's Theorem for non-Archimedean valued fields depend on a closer study of its proof. Studying non-commutative settings, one realizes that the range space is to be an anti-lattice, once again as remote from $\mathbb{R}$ and the comfort of its order structure as one can get. We remind the reader that a subset E_0 of a partially ordered set E is called cofinal if for every $e \in E$ there exists an $f \in E_0$ such that $e \leq f$. The analogy in form between the following theorems should be striking and anyone whose fancy it strikes should consult the references.

Theorem 60 (Kantorovich in [162]). *Let E be an ordered vector space and E_0 a cofinal (i.e. majorizing) subspace of E. Let F be an order complete vector space. Then each positive linear map E_0 to F can be extended to a positive linear map $E \to F$.*

Theorem 61 (Lipecki–Plachky–Thomsen in [206] and Luxemburg–Schep in [212]). *Let E be a vector lattice and E_0 a cofinal vector sublattice of E. Let F be an order complete vector lattice. Then each vector lattice homomorphism $E_0 \to F$ can be extended to a vector lattice homomorphism $E \to F$.*

Theorem 62 (Arveson in [20]). *Let E be a C^*-algebra and let E_0 be a cofinal $*$-invariant subspace of E. Let H be a Hilbert space and $B(H)$ the space of all bounded operators on H. Then each completely positive linear map $E_0 \to B(H)$ extends to a completely positive linear map $E \to B(H)$.*

Obviously, the condition of cofinality in these theorems is an analog of a sublinear functional, which indeed in the case of the first two theorems can be readily introduced by taking an appropriate infimum in the range space. All three theorems know variations with regard to the cofinality. In the first two, sometimes E_0 contains an order unit from E, and in the third, E_0 might have a unit instead of being majorizing. All three theorems can be formulated as E being a (positive, lattice homomorphic, completely positive, respectively) retract of any bigger space in which it majorizes, which should be compared with (2) of Proposition 48.

The last of these theorems requires more care. Though Arveson proved it in 1969, it was only in 1981 (see [342]) that Wittstock brought the theorem into the realm of Hahn–Banach Theorems. Arveson's theorem becomes even more surprising if one realizes that just "positive" instead of "completely positive" doesn't do when the range space is not $\mathbb{C}$. Wittstock's contribution was to go from "completely positive" to "completely bounded" and he proved a beautiful Hahn–Banach Theorem in First Version fashion.

Theorem 63 (Wittstock). *Let E be a unital C^*-algebra and let E_0 be a subspace of E. Let f_0 be a completely bounded linear map $E_0 \to B(H)$. Then there* *exists a completely bounded map $f : E \to B(H)$ which extends f_0 and such that the completely bounded norm of f, $\|f\|_{\mathrm{cb}}$, equals the completely bounded norm of f_0, $\|f_0\|_{\mathrm{cb}}$.*

Wittstock's approach is based on sublinear functionals with values in the subsets of $B(H)$, i.e. on a Second Version Hahn–Banach Theorem. It is amusing to see how many of the theorems and techniques that we have seen in this paper come together in a recent elegant paper by Schmitt (see [294]). The subject of completely bounded maps on C^*-algebras is popular and there is an extensive literature. Much is known about injective objects and injective hulls in this category. Indispensable papers in that direction are [69] by Choi and Effros, [126] and [127] by Hamana, [289] by Ruan, [341] and [342] by Wittstock. I have not come across any attempts to study λ-injectivity for operator algebras.

20. The strength of the Hahn–Banach Theorem

Section 12 (and perhaps Section 11) is part of an investigation of the strength of the Hahn–Banach Theorem. In Section 12 we measured its strength by comparing it to some giants (that is, stronger set-theoretic results). In this section we see what consequences it has in the absence of such giants. While the Axiom of Choice was a topic of heated debate right from its earliest discovery, the emphasis in critique on AC never was on the Hahn–Banach Theorem. There were more likely candidates, in particular the Banach–Tarski paradox and the existence of nonmeasurable subsets of $\mathbb{R}$, to stir doubts about the truth of AC. Interestingly, the Hahn–Banach Theorem has turned out to imply both.

Theorem 64 (Foreman–Wehrung in [101]). *The Hahn–Banach Theorem effectively implies the existence of a nonmeasurable subset of $\mathbb{R}$.*

Theorem 65 (Pawlikowski in [255]). *The Hahn–Banach Theorem effectively implies the Banach–Tarski paradox.*

We discuss some of the techniques in [101]; the proof of Pawlikowski follows similar lines. We remark that before [101] there were proofs of the existence of nonmeasurable subsets (Vitali's and less known Bernstein's), using the Axiom of Choice or at best the Ultrafilter Theorem (Sierpiński's proof in [303]).

Foreman and Wehrung consider a set Ω, the collection of its subsets $\wp(\Omega)$ and a probability measure μ on $\wp(\Omega)$. G is a group that acts on Ω

(i) measure preservingly and
(ii) free (i.e. $g\omega = \omega \ \Rightarrow \ g$ is the identity element of G).

For an element $[x]$ in Ω/G, they consider the direct sum of the subsets of $[x]$, $\wp[x]$, as a Boolean algebra. With the aid of the Hahn–Banach Theorem one then produces a measure $\mu_{[x]}$ on each $\wp[x]$. For $A \subset G$ define a function a on Ω by

$$a(x) = \mu_{[x]}(Ax)$$

and define a measure λ on $\wp(G)$ that is invariant under G, by

$$\lambda(A) = \int a(x)\, d\mu(x)\,.$$

In proving the additivity of λ, condition (i) is used; the invariance under G uses (ii). Then the crux of the matter: F_2, the free group on two generators, is not amenable (see [335]). If one assumes that there exists a probability measure on the sphere S^2 that is invariant under the rotation group $SO(3)$, then one almost gets a free and measure preserving action of F_2 on S^2. The "almost" in the previous sentence is because of some fixed points for each action, a small technicality because only countably many points are involved.

Conclusion: The Hahn–Banach Theorem implies that there does not exist a probability measure on S^2 that is invariant under $SO(3)$.

Amenability is a widely studied subject (see [79] by Day and [254] by Paterson). It has branched out into many areas of mathematics and so has the Hahn–Banach Theorem. A study of the relationship between the two in ZF might turn out profitable in other directions than the above.

21. Other categories

There are many categories in which a Hahn–Banach Theorem is available. We number the subsections, followed by the category that we discuss.

21.1. Groups and semigroups. The first to consider the category of commutative semigroups as a setting for the Hahn–Banach Theorem were Aumann in [23] and Halperin in [124]. There are many other papers now. A nice survey is given in [104] by Fuchssteiner and Lusky. "Forgetting" some of the structure in the Hahn–Banach Theorem (in this case scalar multiplication or more) brings with it at least as many complications as adding structure (like extension of vector lattice homomorphisms in Theorem 34). For instance, the natural analog of the Hahn–Banach Theorem (Second Version) need *not* hold. A natural Sandwich Theorem (Theorem 1.1.2 in [104]) does hold. Important contributions came from Dinges in [86], Fuchssteiner in [106], [107] and [108], Kaufman in [163] and [166], Kranz in [183], [184] and [185], Seever in [296], Simons in [313] and [314], and Topsøe in [328]. On its turn, this renewed interest in variations on the Hahn–Banach Theorem, which took place in the late sixties and early seventies (not counting the precursors), influenced a new look at the Hahn–Banach Theorem itself. Particularly, the German School, in which Anger and Lembcke ([15], [17] and [18]), Fuchssteiner and König [105], König ([178], [179], [180], [181] and[182]), Neumann ([248], [249] and [250]) and Rodé [278] produced a string of papers approaching the Hahn–Banach Theorem from various kinds of directions, with an emphasis on convex analysis. The diversity of applications to their results is wide. We present one of their results, due to König.

Theorem 66 (König). *Let E and p be as in the Hahn–Banach Theorem (Second Version). Let T be a nonempty subset of E and $\tau : T \to \mathbb{R}$ be such that*

(i) *there exist* $\alpha, \beta > 0$ *with* $\inf\{p(w - \alpha u - \beta v) - \tau(w) + \alpha\tau(u) + \beta\tau(v) \leq 0 : w \in E\} \leq 0$ *for all* $u, v \in T$ *and*

(ii) $\tau \leq p$.

Then there exists a linear function $f : E \to \mathbb{R}$ such that $f \leq p$ and $\tau \leq f$ on T.

If T is convex and q is superlinear then any α, β with $\alpha + \beta = 1$ satisfies the inequality in (i) and that is the relation between this theorem and the Mazur–Orlicz Theorem. (A proof for that special situation can be given using exactly Pták's argument for the Mazur–Orlicz Theorem.) Neumann in [248] gives a proof for the more general case where the range space is a Dedekind complete vector lattice. We refer the reader to his paper for many interesting insights. One of the most general Hahn–Banach Theorems in this direction is Rodé's Abstract Hahn–Banach Theorem ([278]). By the best of my knowledge it has not been investigated how close generalizations of the Hahn–Banach Theorem like Rodé's are to the Axiom of Choice.

21.2. Vector lattices. We mentioned a Hahn–Banach Theorem for homomorphisms on vector lattices in Section 12. Many Hahn–Banach Theorems (Banach's proof that $\mathbb{R}$ is amenable, Klee's Invariant Extension Theorem in [175], Pták's argument for the Mazur–Orlicz Theorem of Section 13, Nakano's generalization of the Hahn–Banach Theorem in [247], the Musielak–Orlicz generalization of the Hahn–Banach Theorem in [243] and many, many more) are proved by adapting a given sublinear function to a somewhat smaller one that still majorizes whatever needs to be majorized; the ordinary Hahn–Banach Theorem is then applied to yield the required result. The resulting theorems invariantly turn out to be effectively equivalent to the Hahn–Banach Theorem. There is one such technique which is easily and often applied in the theory of vector lattices. If p is the sublinear functional involved, the adaptation defines

$$q(x) = p(x^+)\,.$$

Notice that a linear f dominated by q is automatically positive: if $x \geq 0$ then

$$-f(x) = f(-x) \leq q(-x) = p(0) = 0\,.$$

This technique gives an immediate proof of the following Hahn–Banach Theorem for normed vector lattices.

Theorem 67. *Let E be a normed Riesz space. Let E_0 be a positive linear functional on E_0. Then there exists a positive linear extension f of f_0 such that $\|f_0\| = \|f\|$.*

With respect to the Hahn–Banach Theorem for Vector Lattice Homomorphisms or the Lipecki–Luxemburg–Schep Extension Theorem the reader may marvel at the many ways in which these results can be proved (see Aron–Hager –Madden in [19], Bernau in [41], Buskes – van Rooij in [64] and [65], Lipecki in [205] and [208], Luxemburg–Schep in [212]).

21.3 Algebras. The area of extension theorems for algebras is not well developed. The situation just might be inhospitable for Hahn–Banach Theorems as is shown by the following example.

EXAMPLE 68. Take $E = \mathbb{C}$, $E_0 = \mathbb{R}$, $f_0 : \mathbb{R} \to \mathbb{C}$ the identity and $p(x) = |x|$ $(x \in \mathbb{C})$. p is sublinear and even multiplicative. However, any homomorphic extension $f : E \to \mathbb{R}$ of f_0 will satisfy $f(i)^2 = f(i^2) = -1$, which is impossible.

Nonetheless, positive results have been obtained by Grilliot in [120]. In more particular situations, say Banach algebras or C^*-algebras, there is no difficulty (see Kadison and Singer in [155] and Żelazko in [349]). We believe that the most common reason for the relative ease in such circumstances lies in an extreme behavior of algebra homomorphisms in classes of simpler mappings. The problem is then adequately dealt with by applying the result on maximal extensions by Andenaes in [13] or by combining the ordinary Hahn–Banach Theorem with the Krein–Milman Theorem as in [107] by Fuchssteiner. Such combinations of the Hahn–Banach Theorem and extreme point arguments tend to be equivalent to the Axiom of Choice (see also [141], where it is shown that Krull's Theorem about the existence of maximal ideals implies the Axiom of Choice).

21.4. Distributive lattices and Boolean algebras. For Boolean algebras there is the following Hahn–Banach Theorem.

THEOREM 69 (Monteiro in [236]). *Let E be a Boolean algebra and E_0 a Boolean subalgebra. Let F be a complete Boolean algebra and $p : E \to F$ a map with $p(x \vee y) = p(x) \vee p(y)$ for all $x, y \in E$ and $p(1) = 1$. Then every homomorphism $E_0 \to F$ that is dominated by p can be extended to a homomorphism $E \to F$ that is dominated by p.*

In fact, Monteiro also proves a Sandwich Theorem for homomorphisms on Boolean algebras. The classical result by Sikorski that we mentioned in Section 12 is a consequence and so are other facts from the theory of Boolean algebras (for instance Stone's Theorem proving that if I is an ideal disjoint from a filter F then there exists a prime ideal $\wp$ disjoint from F with $\wp \supset I$). Cignoli transferred these results to the more general setting of distributive lattices (see Cignoli in [71]) and Bacsich in [24] investigated the relation with Sikorski's Extension Theorem more closely. Bernau in his proof [41] of the Lipecki–Luxemburg–Schep Theorem for vector lattice homomorphisms used Cignoli's Theorem (without quoting Cignoli) while Luxemburg and Schep [212] employed Stone's Theorem. Notice that there is no Sandwich Theorem for vector lattice homomorphisms (see [65]). Recently, a special type maps called orthomorphisms have become increasingly more important in the study of vector lattices; for a study of extension theorems for orthomorphisms see [338] by Wickstead.

21.5 Module versions of the Hahn–Banach Theorem. Instead of considering vector spaces over a field, we now consider modules. To not confuse the issue by technicalities, we assume that

(i) $A = C(X)$ is a σ-Dedekind complete space,
(ii) E is an A-module,
(iii) F is a Dedekind complete A-module,
(iv) E_0 is an A-submodule of E,
(v) $p : E \to F$ is an A-sublinear mapping (i.e. p is subadditive and $p(\alpha x) = \alpha p(x)$ for all $x \in E$ and all $\alpha \in A^+$).

In the light of the remarks at the end of Section 16, the reader does not miss much by putting $F = \mathbb{R}$ in the next theorem.

THEOREM 70. *In the situation above, let $f_0 : E_0 \to F$ be an A-linear (i.e. additive and preserving the action of A) map dominated by p. Then there exists an A-linear map $f : E \to F$ that extends f_0 and that is dominated by p.*

The proof is significantly different from Banach's proof of the Hahn–Banach Theorem in one place only. Choosing an element of F to be the value of an additional element x_0 from $E \backslash E_0$ under an extension can be copied. But to prove that the extension is still dominated by p is more difficult, for we now have to show that

$$f(x + ax_0) \leq p(x + ax_0) \quad \text{for all } x \in E_0 \text{ and all } a \in A$$

and you may recall that in the case $A = \mathbb{R}$ the invertibility of any $a \neq 0$ played a role in that part of the proof. (To be honest, one has to check that the map f is well defined, but that also follows from the inequality above.) Theorem 70 is not the most general at all. In Ghika's [112] A is an F-ring and in Vuza's [334] A is a G-ring . The situation with $A = C(X)$ is rather special and was investigated in [252] by Ohron. The paper [333] by Vincent-Smith also contains extreme point arguments in the style of [13] and [261] as well as some results about averaging operators. Vuza's paper [334] also has a section on Banach limits.