

CATEGORY MEASURES, THE DUAL OF $C(K)^\delta$ AND HYPER-STONEAN SPACES

JAN HARM VAN DER WALT

ABSTRACT. For a compact Hausdorff space K , we give descriptions of the dual of $C(K)^\delta$, the Dedekind completion of the Banach lattice $C(K)$ of continuous, real-valued functions on K . We characterize those functionals which are σ -order continuous and order continuous, respectively, in terms of Oxtoby's category measures. This leads to a purely topological characterization of hyper-Stonean spaces.

1. INTRODUCTION

It is well known that the Banach lattice $C(K)$ of continuous, real-valued functions on a compact Hausdorff space K fails, in general, to be Dedekind complete. In fact, $C(K)$ is Dedekind complete if and only if K is Stonean, i.e. the closure of every open set is open, see for instance [25]. A natural question is to describe the Dedekind completion of $C(K)$; that is, the unique (up to linear lattice isometry) Dedekind complete Banach lattice containing $C(K)$ as a majorizing, order dense sublattice. Several answers to this question have been given, see for instance [3, 8, 10, 14, 21, 22, 27] among others. A further question arises: what is the dual of $C(K)^\delta$? As far as we are aware, no direct answer to this question has been stated explicitly in the literature.

We give two answers to this question; one in terms of Borel measures on the Gleason cover of K [13], and another in terms of measures on the category algebra of K [23, 24]. We use the latter description to obtain natural characterizations of strictly positive, σ -order continuous, and order continuous functionals on $C(K)^\delta$. In particular, we show that $C(K)^\delta$ admits a strictly positive, σ -order continuous functional if and only if K admits a category measure in the sense of Oxtoby [23]. Specialising to the case of a Stonean space K , we obtain a purely topological characterization of hyper-Stonean spaces. Although the class of hyper-Stonean spaces was introduced by Dixmier [11] in 1951, no such characterization has been given to date [7, page 197].

The paper is organised as follows. In Section 2 we introduce notation and recall definitions and results from the literature to be used throughout the rest of the paper. Section 3 contains two representations of $C(K)^\delta$ which are amendable to the problem of characterizing the dual of this space; these characterizations of $(C(K)^\delta)^*$ are given in Section 4. Strictly positive, σ -order continuous and order continuous functionals on $C(K)^\delta$ are discussed in Section 5 which ends with a characterization of hyper-Stonean spaces.

2010 *Mathematics Subject Classification*. Primary 46E05, 54G05; Secondary 46E27, 54F65.

Key words and phrases. Hyper-Stonean spaces, category measures, Banach lattices, continuous functions.

2. PRELIMINARIES

2.1. Vector and Banach lattices. Let E be a real vector lattice. We assume that E is Archimedean; that is, $\inf\{\frac{1}{n}f : n \in \mathbb{N}\} = 0$ for every $f \geq 0$. Denote by E_+ the positive cone in E , $E_+ = \{f \in E : 0 \leq f\}$. For $f \in E$ the positive part, negative part and modulus of f is given by

$$f^+ = f \wedge 0, \quad f^- = (-f) \wedge 0 \quad \text{and} \quad |f| = f^+ + f^-,$$

respectively. For $D \subseteq E$ and $f \in E$ we write $D \downarrow f$ if D is downward directed and $\inf D = f$.

E is *Dedekind complete* if every subset of E which is bounded above (below) has a least upper bound (greatest lower bound). We call E *order separable* if every subset A of E contains a countable set with the same set of upper bounds as A .

Let F be a vector lattice subspace of E . F is *order dense* in E if for every $0 < f \in E$ there exists $g \in F$ so that $0 < g \leq f$. If for every $f \in E_+$ there exists $g \in F$ so that $f \leq g$ then F is *majorizing* in E . An *ideal* in E is a vector lattice subspace F of E so that if $f \in F$ and $|g| \leq |f|$ then $g \in F$. A σ -ideal in E is an ideal F in E so that $\sup A \in F$ whenever A is a countable subset of F and $\sup A$ exists in E . The *principal ideal* generated by $f \in E$ is $I_f = \{g \in E : |g| \leq \alpha|f| \text{ for some } \alpha \in \mathbb{R}_+\}$.

An element e of E_+ is called an *order unit* if for every $f \in E_+$ there exists a real number $\alpha > 0$ so that $f \leq \alpha e$. A *weak order unit* is any $e \in E_+$ so that if $f \wedge e = 0$ then $f = 0$. Observe that $e \in E_+$ is an order unit if and only if $I_e = E$; e is a weak order unit if and only if I_e is order dense in E .

A linear functional φ on E is *order bounded* if it maps order bounded subsets of E to bounded subsets of \mathbb{R} , and *positive* if $\varphi(f) \geq 0$ whenever $f \in E_+$. Every positive functional on E is order bounded. Conversely, every order bounded functional is the difference of two positive functionals. The space E^\sim of all order bounded functionals on E is a Dedekind complete vector lattice with respect to the ordering

$$\varphi \geq \psi \text{ if and only if } \varphi - \psi \text{ is positive.}$$

Therefore every $\varphi \in E^\sim$ can be expressed as the difference of positive functionals, $\varphi = \varphi^+ - \varphi^-$.

If E is Banach lattice then a functional φ on E is order bounded if and only if it is norm bounded. Therefore the norm dual E^* and order dual E^\sim of E coincide and is itself a Banach lattice.

A functional $\varphi \in E^\sim$ is σ -*order continuous* if $\inf\{|\varphi(f_n)| : n \in \mathbb{N}\} = 0$ whenever $f_n \downarrow 0$ in E . If $\inf\{|\varphi(f)| : f \in D\} = 0$ for every $D \downarrow 0$ in E , we say that φ is *order continuous*. A functional φ is (σ) -order continuous if and only if $|\varphi|$ is (σ) -order continuous, if and only if both φ^+ and φ^- are (σ) -order continuous.

Lemma 2.1. *Let E be an Archimedean vector lattice and F an order dense, majorizing vector lattice subspace of E .*

- (i) *If $\varphi \in E^\sim$ then φ is order continuous on E if and only if the restriction of φ to F is order continuous.*
- (ii) *If $\varphi \in E^\sim$ is σ -order then the restriction of φ to F is σ -order continuous.*
- (iii) *If $\varphi \in F^\sim$ is order continuous, then φ has a unique order continuous extension to E .*

Proof. We prove the results for positive functionals. The case of a general $\varphi \in E^\sim$ follows from the decomposition $\varphi = \varphi^+ - \varphi^-$.

Proof of (i). Let $D \subseteq F_+$. Because F is order dense in E , $D \downarrow 0$ in E if and only if $D \downarrow 0$ in F . Therefore, if φ is order continuous on E then its restriction to F is order continuous.

Conversely, assume that the restriction of φ to F is order continuous. Let $D \downarrow 0$ in E . For each $g \in D$ let $D_g = \{f \in F : g \leq f\}$, and let $D' = \bigcup \{D_g : g \in D\}$. The fact that F is majorizing in E guarantees that $D_g \neq \emptyset$ for each $g \in D$. We observe that $D \downarrow 0$ in E implies that D' is downward directed. In particular, if $f, h \in D'$ then $f \wedge h \in D'$. Because F is an order dense sublattice of E , $g = \inf D_g$ for every $g \in D$. Therefore $D' \downarrow 0$ in E , hence also in F . It follows from the positivity and order continuity of φ on F that $0 \leq \inf \varphi[D] \leq \inf \varphi[D'] = 0$. Therefore $\inf \varphi[D] = 0$.

Proof of (ii). The proof is similar to the proof of (i).

Proof of (iii). This is a special case of [2, Theorem 4.12]. \square

2.2. $C(K)$ and its dual. Let K be a compact Hausdorff space. $C(K)$ is the real Banach lattice of continuous, real-valued functions on K ; the algebraic operations and order relation are defined pointwise as always, and the norm on $C(K)$ is $\|f\| = \max\{|f(x)| : x \in K\}$. The symbols $\mathbf{0}$ and $\mathbf{1}$ denote the constant functions with values 0 and 1, respectively. For $A \subseteq K$, the characteristic function of A is denoted $\mathbf{1}_A$.

The space K satisfies the countable chain condition (ccc) if every pairwise disjoint collection of nonempty open sets in K is countable. $C(K)$ is order separable if and only if K satisfies ccc [9, Theorem 10.3].

$\Sigma_K^{\mathcal{B}}$ denotes the σ -algebra of Borel sets in K and $M(K)$ the space of regular signed Borel measures on K ; that is, σ -additive, real valued regular measures on $\Sigma_K^{\mathcal{B}}$. As is well known, $M(K)$ is a Banach lattice with respect to the total variation norm

$$\|\varphi\| = |\varphi|(K), \quad \varphi \in M(K)$$

and pointwise ordering. In fact, $M(K)$ is isometrically lattice isomorphic to $C(K)^*$. We will not make a notational distinction between a measure φ and the associated functional. A measure $\varphi \in M(K)$ is called *normal* [7, 11] if the associated functional on $C(K)$ is order continuous. We denote the set of normal measures on by $N(K)$. Normal measures can be characterised as follows: $\varphi \in M(K)$ is normal if and only if $|\varphi|(N) = 0$ for every meagre set $N \in \Sigma_K^{\mathcal{B}}$, see for instance [7, Theorem 7.4.7].

Recall that K is *Stonean* if the closure of every open set is open. As mentioned in the introduction, K is Stonean if and only if $C(K)$ is Dedekind complete. If K is Stonean and, in addition, the union of the supports of the normal measures on K is dense in K , we call K *hyper-Stonean*, see [11] and [7, Section 4.7]. We recall the following result which contains the main motivation for studying hyper-Stonean spaces, see for instance [7].

Theorem 2.2. *A space K is hyper-Stonean if and only if any one of the following conditions hold.*

- (i) *For every $\mathbf{0} < f \in C(K)$ there exists a positive order continuous functional φ on $C(K)$ so that $\varphi(f) > 0$.*
- (ii) *$C(K)$ is isometrically isomorphic to the dual of a Banach space.*
- (iii) *$C(K)$ is isometrically lattice isomorphic to the dual of a Banach lattice.*
- (iv) *The C^* -algebra of complex-valued continuous functions on K is a von Neumann algebra.*

2.3. Category measures. Recall that a subset A of K has the property of Baire if there exists an open set U and a meagre set N in K so that $A = U\Delta N$. Let

$$(2.1) \quad \begin{aligned} \mathcal{N}_K &= \{N \subset K : N \text{ is meagre}\}, \\ \Sigma_K^c &= \{U\Delta N : U \text{ is open in } K, N \in \mathcal{N}\}. \end{aligned}$$

The collection Σ_K^c is a σ -algebra [24, Theorem 4.3] and \mathcal{N}_K is a σ -ideal in Σ_K^c . In fact, \mathcal{N}_K is a σ -ideal in the powerset of K . The quotient algebra Σ_K^c/\mathcal{N}_K is a complete Boolean algebra. In fact, Σ_K^c/\mathcal{N}_K is isomorphic to the algebra \mathcal{R}_K of regular open subsets of K , see [21, 23].

A *category measure* on K is positive, σ -additive measure $\mu : \Sigma_K^c \rightarrow \mathbb{R}$ with the property that for any $A \in \Sigma_K^c$, $\mu(A) = 0$ if and only if $A \in \mathcal{N}_K$, see for instance [23]. There is a bijective correspondence between category measures on K and strictly positive, σ -additive measures on \mathcal{R}_K . Oxtoby [23, 24] investigated the problem of characterizing those topological spaces which admit a category measure. A solution to this problem was given by J. M. Ayerbe Toledano [26]¹. In order to formulate this result we recall the following definitions, see [4, 16, 26].

Definition 2.3. Let X be a nonempty set and \mathcal{F} a collection of nonempty subsets of X . For any $n \in \mathbb{N}$ and $\bar{F} = \langle F_1, \dots, F_n \rangle \in \mathcal{F}^n$ let

$$i(\bar{F}) = \max\{|J| : J \subseteq \{1, \dots, n\}, \bigcap_{j \in J} F_j \neq \emptyset\}.$$

The Kelley intersection number of \mathcal{F} is defined as

$$k(\mathcal{F}) = \inf\left\{\frac{i(\bar{F})}{n} : n \in \mathbb{N}, \bar{F} \in \mathcal{F}^n\right\}.$$

Definition 2.4. Let K be a compact Hausdorff space and \mathcal{T} the collection of nonempty, open subsets of K . We say that K satisfies property (***) if there exists a partition $\{\mathcal{T}_n : n \in \mathbb{N}\}$ of \mathcal{T} such that the following conditions are satisfied for every $n \in \mathbb{N}$.

- (i) $k(\mathcal{T}_n) > 0$.
- (ii) If (U_m) is an increasing sequence of open sets so that $\bigcup\{U_m : m \in \mathbb{N}\} \in \mathcal{T}_n$ then there exists $m_0 \in \mathbb{N}$ so that $U_{m_0} \in \mathcal{T}_n$.
- (iii) If $U \in \mathcal{T}_n$ and V is an open set so that $U\Delta V$ is meagre then $V \in \mathcal{T}_n$.

Ryll-Nardzewski obtained a necessary and sufficient condition of the existence of a σ -additive, strictly positive measure on a complete Boolean algebra, see [16, Adendum]. It is easy to see that property (***) is equivalent to \mathcal{R}_K satisfying Ryll-Nardzewski's condition. The following theorem of Ayerbe Toledano [26] therefore follows immediately from Ryll-Nardzewski's result.

Theorem 2.5. *A compact Hausdorff space K admits a category measure if and only if it satisfies property (***)*.

Remark 2.6. Cambern and Greim [6], see also [7], define a category measure to be a (not necessarily finite) positive Borel measure μ on a Stonean space K which satisfies the following conditions:

- (i) μ is regular on closed sets of finite measure,
- (ii) $\mu(N) = 0$ for every meagre Borel set N , and,

¹We formulate the result for compact Hausdorff spaces, but it is valid for arbitrary Baire spaces.

- (iii) for every nonempty clopen set A there exists a clopen set $A_0 \subseteq A$ so that $0 < \mu(A_0) < \infty$.

Such measures are called *perfect* in [5], and are distinct from the category measures discussed here.

3. CHARACTERIZATIONS OF $C(K)^\delta$

3.1. The Gleason cover and the Maeda-Ogasawara theorem. Let K be a compact Hausdorff space. A seminal result of Gleason [13] associates, in a canonical way, with K a Stonean space G_K , its projective cover. In order to formulate this result we recall the following. If L is compact Hausdorff space, a continuous surjection $f : K \rightarrow L$ is called *irreducible* if $f[C] \neq L$ for every proper closed subset C of K .

Theorem 3.1. *Let K be a compact Hausdorff space. There exists a Stonean space G_K , unique up to homeomorphism, and an irreducible map $\pi_K : G_K \rightarrow K$.*

Because π_K is onto, the induced map $T_{\pi_K} : C(K) \ni f \mapsto f \circ \pi_K \in C(G_K)$ is an isometric linear lattice isomorphism onto a closed vector lattice subspace of $C(G_K)$. Moreover, since π_K is irreducible, $\pi_K^*[C(K)]$ is a order dense sublattice of $C(G_K)$. Clearly, $\pi_K^*[C(K)]$ is majorising in $C(G_K)$, seeing as it contains the order unit $\mathbf{1}_{G_K}$ of $C(G_K)$. Lastly we note that, G_K being Stonean, $C(G_K)$ is Dedekind complete. Hence we have the following.

Theorem 3.2. *Let K be a compact Hausdorff space. Then $C(K)^\delta$ is isometrically lattice isomorphic to $C(G_K)$.*

Let $\bar{\mathbb{R}}$ be the two-point compactification of \mathbb{R} . Denote by $C^\infty(K)$ the space of continuous functions $f : K \rightarrow \bar{\mathbb{R}}$ so that $f^{-1}[\mathbb{R}]$ is dense (hence open and dense) in K . If K is a Stonean space then $C^\infty(K)$ is a universally complete vector lattice, see for instance [1, Theorem 7.27], and $C(K)$ is the ideal in $C^\infty(K)$ generated by $\mathbf{1}$.

A classical result in the representation theory for vector lattices is due to Maeda and Ogasawara [20], see for instance [1] for a more recent presentation.

Theorem 3.3. *Let E be an Archimedean vector lattice with weak order unit e . There exists a Stonean space Ω_E and a linear lattice isomorphism $T : E \rightarrow C^\infty(\Omega_E)$ onto an order dense sublattice of $C^\infty(\Omega_E)$ so that $Te = \mathbf{1}$.*

In Theorem 3.3, let $E = C(K)$. Then $T(\mathbf{1}_K) = \mathbf{1}_{\Omega_E}$. For $f \in C^\infty(\Omega_E)$, $f \in C(\Omega_E)$ if and only if there exists $c > 0$ so that $|f| \leq c\mathbf{1}_{\Omega_E}$. Therefore $T[C(K)]$ is an order dense and majorizing vector lattice subspace of $C^\infty(\Omega_E)$. Hence we have the following.

Theorem 3.4. *Let K be a compact Hausdorff space. Then $C(K)^\delta$ is isometrically lattice isomorphic to $C(\Omega_{C(K)})$.*

A compact Hausdorff space is uniquely determined, up to homeomorphism, by its lattice of real-valued continuous functions. The Dedekind completion of a vector lattice is unique up to a linear lattice isomorphism. Therefore $\Omega_{C(K)}$ and G_K are homeomorphic. This can be seen directly by recalling the constructions of $\Omega_{C(K)}$ and G_K , respectively. G_K may be constructed as the Stone space of \mathcal{R}_K . On the other hand, $\Omega_{C(K)}$ is the Stone space of the Boolean algebra of bands in $C(K)$ which is isomorphic to \mathcal{R}_K .

3.2. Category measurable functions. Denote by $B(\Sigma_K^c)$ the Archimedean vector lattice consisting of all real-valued, bounded and Σ_K^c -measurable functions on K . The subset

$$N(\Sigma_K^c, \mathcal{N}_K) = \{f \in B(\Sigma_K^c) : K \setminus f^{-1}[\{0\}] \in \mathcal{N}_K\}$$

of $B(\Sigma_K^c)$ is a σ -ideal in $B(\Sigma_K^c)$. Therefore

$$L^\infty(\Sigma_K^c, \mathcal{N}_K) = B(\Sigma_K^c)/N(\Sigma_K^c, \mathcal{N}_K)$$

is an Archimedean vector lattice [19, Theorem 60.3]. For each $f \in B(\Sigma_K^c)$ we denote by \hat{f} the equivalence class in $L^\infty(\Sigma_K^c, \mathcal{N}_K)$ generated by f . We note that for $\hat{f}, \hat{g} \in L^\infty(\Sigma_K^c, \mathcal{N}_K)$,

$$(3.1) \quad \hat{f} \leq \hat{g} \text{ if and only if } \{x \in K : f(x) > g(x)\} \in \mathcal{N}_K.$$

For $\hat{f} \in L^\infty(\Sigma_K^c, \mathcal{N}_K)$ set

$$\|\hat{f}\|_\infty = \inf\{c \geq 0 : |\hat{f}| \leq c\hat{\mathbf{1}}\}.$$

Note that for $\hat{f} \in L^\infty(\Sigma_K^c, \mathcal{N}_K)$,

$$\|\hat{f}\|_\infty = \inf\{c \geq 0 : \{x \in K : |f(x)| > c\} \in \mathcal{N}_K\}.$$

With respect to this norm the vector lattice $L^\infty(\Sigma_K^c, \mathcal{N}_K)$ is a Banach lattice. Because Σ_K^c/\mathcal{N}_K is complete, $L^\infty(\Sigma_K^c, \mathcal{N}_K)$ is Dedekind complete, see for instance [12, 363M & 363N]. We remark that it is possible to prove this result directly. Every equivalence class in $L^\infty(\Sigma_K^c, \mathcal{N}_K)$ contains a bounded lower semi-continuous function, and, every bounded lower semi-continuous function belongs to $B(\Sigma_K^c)$. Dedekind completeness of $L^\infty(\Sigma_K^c, \mathcal{N}_K)$ then follows from the fact that the point-wise supremum of any family of lower semi-continuous functions is again lower semi-continuous.

Theorem 3.5. *Let K be a compact Hausdorff space. Then $C(K)^\delta$ is isometrically lattice isomorphic to $L^\infty(\Sigma_K^c, \mathcal{N}_K)$.*

Proof. $C(K)$ is vector lattice subspace of $B(\Sigma_K^c)$. Furthermore, for $C(K) \cap N(\Sigma_K^c, \mathcal{N}_K) = \{\mathbf{0}\}$. Therefore the mapping $T : C(K) \ni u \mapsto \hat{u} \in L^\infty(\Sigma_K^c, \mathcal{N}_K)$ is a linear lattice isomorphism onto a vector lattice subspace of $L^\infty(\Sigma_K^c, \mathcal{N}_K)$. Clearly T is an isometry and $T[C(K)]$ is majorising in $L^\infty(\Sigma_K^c, \mathcal{N}_K)$.

Since $L^\infty(\Sigma_K^c, \mathcal{N}_K)$ is a Dedekind complete Banach lattice, it remains only to verify that $T[C(K)]$ is order dense in $L^\infty(\Sigma_K^c, \mathcal{N}_K)$. To this end, consider $\hat{\mathbf{0}} < \hat{u} \in L^\infty(\Sigma_K^c, \mathcal{N}_K)$. There exists a real number $\epsilon > 0$, an open set U in K and $N \in \mathcal{N}_K$ so that $\epsilon < u(x)$ for every $x \in U \Delta N$. According to (3.1) there exists $M \in \mathcal{N}_K$ so that $0 \leq u(x)$ for all $x \in K \setminus M$. Consider a function $v \in C(K)$ so that $\mathbf{0} < v \leq \epsilon \mathbf{1}_U$. Fix $x \in K \setminus (M \cup N)$. If $x \in U$ then $v(x) \leq \epsilon < u(x)$. If $x \in K \setminus U$ then $v(x) = 0 \leq u(x)$. Therefore $v(x) \leq u(x)$ for all $x \in K \setminus (M \cup N)$ so that $Tv = \hat{v} \leq \hat{u}$ by (3.1). □

4. THE DUAL OF $C(K)^\delta$

From the results discussed in Section 3, in particular Theorems 3.4 and 3.5, we obtain immediately two characterizations of the dual of $C(K)^\delta$. The first follows directly from the Riesz Representation Theorem for compact Hausdorff spaces.

Theorem 4.1. *Let K be a compact Hausdorff space. Then $(C(K)^\delta)^*$ is isometrically lattice isomorphic to $M(G_K)$.*

Theorem 3.5 yields a second characterization of $(C(K)^\delta)^*$. Denote by $M(\Sigma_K^C, \mathcal{N}_K)$ the set of bounded, finitely additive signed measures μ on Σ_K^C with the property that $\mu(N) = 0$ for every $N \in \mathcal{N}_K$. The space $M(\Sigma_K^C, \mathcal{N}_K)$ is a vector lattice [28, Section 1]. With respect to the total variation norm

$$\|\varphi\| = |\varphi|(K), \quad \varphi \in M(\Sigma_K^C, \mathcal{N}_K)$$

$M(\Sigma_K^C, \mathcal{N}_K)$ is a Banach lattice, see for instance [18]. We now have the following.

Theorem 4.2. *Let K be a compact Hausdorff space. Then $(C(K)^\delta)^*$ is isometrically lattice isomorphic to $M(\Sigma_K^C, \mathcal{N}_K)$. In particular, for every $\varphi \in (C(K)^\delta)^*$ there exists a unique $\mu_\varphi \in M(\Sigma_K^C, \mathcal{N}_K)$ so that*

$$\varphi(\hat{u}) = \int_K u d\mu_\varphi, \quad \hat{u} \in C(K)^\delta,$$

and the mapping $S : (C(K)^\delta)^* \ni \varphi \mapsto \mu_\varphi \in M(\Sigma_K^C, \mathcal{N}_K)$ is a linear lattice isometry onto $M(\Sigma_K^C, \mathcal{N}_K)$.

Proof. It follows immediately from [28, Theorem 2.3] that S is a linear isometry onto $M(\Sigma_K^C, \mathcal{N}_K)$. That both S and its inverse are positive operators follows immediately from the construction of μ_φ from $\varphi \in (C(K)^\delta)^*$ and the definition of the integral. Indeed, for $\varphi \in (C(K)^\delta)_+^*$ and $\mu \in M(\Sigma_K^C, \mathcal{N}_K)_+$

$$\mu_\varphi(B) = \varphi(\hat{\mathbf{1}}_B), \quad B \in \Sigma_K^C$$

and

$$\int_K f d\mu = \sup_{s \in S_f} \int_K s d\mu, \quad f \in B(\Sigma_K^C)_+$$

where S_f consists of all simple, positive functions dominated by f , see [18] and [28] for the details. \square

Recall that Σ_K^C/\mathcal{N}_K is a complete Boolean algebra, isomorphic to \mathcal{R}_K . For $B \in \Sigma_K^C$ let \hat{B} denote the equivalence class in Σ_K^C/\mathcal{N}_K containing B . For $B_0, B_1 \in \Sigma_K^C$, $\hat{B}_0 = \hat{B}_1$ if and only if $B_0 \Delta B_1 \in \mathcal{N}_K$. Denote by $M(\mathcal{R}_K)$ the space of bounded finitely additive measures on \mathcal{R}_K . This space is a Banach lattice with respect to the pointwise order and variation norm, see for instance [12, 326Y (j)].

Let $\varphi \in M(\Sigma_K^C, \mathcal{N}_K)$. Since $\varphi[\mathcal{N}_K] = \{0\}$, φ induces a (signed) finitely additive measure μ_φ on \mathcal{R}_K ,

$$\mu_\varphi(\hat{B}) = \varphi(B), \quad B \in \Sigma_K^C.$$

Conversely, every finitely additive measure μ on \mathcal{R}_K induces a measure $\varphi^\mu \in M(\Sigma_K^C, \mathcal{N}_K)$,

$$\varphi^\mu(B) = \mu(\hat{B}), \quad B \in \Sigma_K^C.$$

The maps $M(\Sigma_K^C, \mathcal{N}_K) \ni \varphi \mapsto \mu_\varphi \in M(\mathcal{R}_K)$ and $M(\mathcal{R}_K) \ni \mu \mapsto \varphi^\mu \in M(\Sigma_K^C, \mathcal{N}_K)$ are positive, linear isometries, and each is the inverse of the other. Therefore we have the following.

Corollary 4.3. *Let K be a compact Hausdorff space. The following statements are true.*

- (i) *The following spaces are pairwise isometrically lattice isomorphic: $(C(K)^\delta)^*$, $M(\mathcal{R}_K)$, $M(\Sigma_K^C, \mathcal{N}_K)$ and $M(G_K)$.*

(iv) *If K is Stonean then $C(K)^*$, $M(\mathcal{R}_K)$ and $M(K)$ pairwise isometrically lattice isomorphic.*

All this is well (but perhaps not widely) known, see for instance [12, Chapter 32]. That $M(\Sigma_K^c, \mathcal{N}_K)$ and $M(G_K)$ are isometrically lattice isomorphic can also be derived from [28, Paragraph 4.5], and [12, 362A] may be used to show that $M(\mathcal{R}_K)$ is isometrically lattice isomorphic to $M(G_K)$.

For the remainder of the paper we will be primarily concerned with the representation of $(C(K)^\delta)^*$ given in Theorem 4.2.

5. ORDER CONTINUOUS ELEMENTS OF $(C(K)^\delta)^*$ AND HYPER-STONEAN SPACES

The first result of this section is a characterisation of σ -order continuous functionals on $C(K)^\delta$. In contrast with the classical representation theorem for functionals on $C(K)$ in terms of Borel measures, there is an exact correspondence between countably additivity of a measure and σ -order continuity of the corresponding functional.

Theorem 5.1. *A measure $\varphi \in (C(K)^\delta)^*$ is σ -order continuous if and only if it is countably additive on Σ_K^c .*

Proof. Both the countably additive elements of $M(\Sigma_K^c, \mathcal{N}_K)$ and the σ -order continuous elements in $(C(K)^\delta)^*$ are ideals in the respective ambient spaces. Therefore we may assume that $\varphi \geq 0$.

Assume that φ is σ -order continuous. Let (C_n) be a decreasing (with respect to inclusion) sequence of sets in Σ_C so that $\bigcap \{C_n : n \in \mathbb{N}\} = \emptyset$. For each $n \in \mathbb{N}$ let $f_n = \mathbf{1}_{C_n}$. Then $\hat{f}_n \downarrow \hat{\mathbf{0}}$ in $L^\infty(\Sigma_K^c, \mathcal{N}_K)$. Therefore

$$\varphi(C_n) = \int_K \hat{f}_n d\varphi \longrightarrow 0$$

so that φ is countably additive.

Conversely, assume that φ is countably additive. Consider a sequence $\hat{f}_n \downarrow \hat{\mathbf{0}}$ in $L^\infty(\Sigma_K^c, \mathcal{N}_K)$. Replacing each f_n with $(f_1 \wedge \dots \wedge f_n) \vee \mathbf{0}$ if necessary and using (3.1) we may assume that the sequence (f_n) in $B(\Sigma_K^c)$ is pointwise decreasing and bounded below by 0. Let $f : K \ni x \mapsto \inf_{n \in \mathbb{N}} f_n(x) \in \mathbb{R}$. Then f is Σ_K^c -measurable and bounded on K ; that is, $f \in B(\Sigma_K^c)$. But $\hat{\mathbf{0}} \leq \hat{f} \leq \hat{f}_n$ for all $n \in \mathbb{N}$. Therefore $\hat{f} = \hat{\mathbf{0}}$ so that $f^{-1}[\mathbb{R}_+] \in \mathcal{N}$. By the Lebesgue Dominated Convergence Theorem,

$$\varphi(\hat{f}_n) = \int_K f_n d\varphi \longrightarrow \int_K f d\varphi = 0.$$

so that φ is σ -order continuous. \square

For a measure $\varphi \in M(\Sigma_K^c, \mathcal{N}_K)_+$ we define regularity in the same way as for Borel measures. That is, φ is *regular* if for every $B \in \Sigma_K^c$,

$$\sup\{\mu(C) : C \subseteq B \text{ is compact}\} = \mu(B) = \inf\{\mu(U) : U \supseteq B \text{ is open}\}.$$

We note that each of the identities above implies the other. A general measure $\varphi \in M(\Sigma_K^c, \mathcal{N}_K)$ is regular if $|\varphi|$ is regular.

Theorem 5.2. *A measure $\varphi \in (C(K)^\delta)^*$ is order continuous if and only if it is countably additive and regular on Σ_K^c .*

Proof. Let $\varphi \geq 0$. Assume that φ is order continuous. By Theorem 5.1, φ is countably additive. Let $A \in \Sigma_K^c$. Then $\hat{\mathbf{1}}_A \in L^\infty(\Sigma_K^c, \mathcal{N}_K)$. Fix $\epsilon > 0$ and let $D_\epsilon = \{\hat{f} \in L^\infty(\Sigma_K^c, \mathcal{N}_K) : f \in C(K), (1 + \epsilon/2)\mathbf{1}_A \leq f\}$. According to Theorem 3.5, $D_\epsilon \downarrow (1 + \epsilon/2)\hat{\mathbf{1}}_A$ in $L^\infty(\Sigma_K^c, \mathcal{N}_K)$. By order continuity of φ ,

$$(1 + \epsilon/2)\varphi(A) = \int (1 + \epsilon/2)\mathbf{1}_A d\phi = \inf_{\hat{f} \in D_\epsilon} \int_K f d\varphi.$$

Therefore there exists $\hat{f} \in D_\epsilon$ so that

$$\int_K f d\varphi < (1 + \epsilon)\varphi(A).$$

Let $U = f^{-1}[(1, \infty)]$. Then $U \supseteq A$ is open and

$$\varphi(U) = \int_K \mathbf{1}_U d\varphi \leq \int_K f d\varphi < (1 + \epsilon)\varphi(A).$$

Therefore $\varphi(A) \leq \inf\{\varphi(U) : U \supseteq A \text{ open}\} < (1 + \epsilon)\varphi(A)$. This holds for all $\epsilon > 0$ so that $\varphi(A) = \inf\{\varphi(U) : U \supseteq A \text{ open}\}$. Therefore φ is outer regular, hence regular.

Conversely, assume that φ is countably additive and regular on Σ_K^c . Then the restriction φ_0 of φ to Σ_K^B is countably additive and regular, and $\varphi_0(N) = 0$ for every meagre Borel set N . Therefore φ_0 is a normal Borel measure on K so that the restriction of the functional φ to $C(K)$ is order continuous. By Lemma 2.1 (i), φ is order continuous on $C(K)^\delta$. \square

Proposition 5.3. *Let $\varphi \in (C(K)^\delta)_+^*$. Then φ is strictly positive if and only if, for every every $A \in \Sigma_K^c$, $\varphi(A) = 0$ if and only if $A \in \mathcal{N}_K$.*

Proof. Assume that φ is strictly positive. Let $A \in \Sigma_K^c \setminus \mathcal{N}_K$. Then $\hat{\mathbf{0}} < \hat{\mathbf{1}}_A$ so that $\varphi(A) = \varphi(\hat{\mathbf{1}}_A) > 0$.

Assume that for every $A \in \Sigma_K^c$, $\varphi(A) = 0$ if and only if $A \in \mathcal{N}_K$. Consider any $\hat{\mathbf{0}} < \hat{f} \in L^\infty(\Sigma_K^c, \mathcal{N}_K)$. Then there exists an $\epsilon > 0$ so that $A = f^{-1}[[\epsilon, \infty]] \in \Sigma_K^c \setminus \mathcal{N}_K$. Consequently,

$$\varphi(\hat{f}) \geq \int_K \mathbf{1}_A d\varphi = \varphi(A) > 0.$$

Therefore φ is strictly positive. \square

Theorems 2.5, 5.1 and 5.2, and Proposition 5.3 now yields the following result, establishing the relationship between order continuous functionals and category measures.

Corollary 5.4. *Let K be a compact Hausdorff space. Then the following statements are equivalent.*

- (i) K satisfies property (***)
- (ii) K admits a category measure.
- (iii) $C(K)^\delta$ admits a strictly positive σ -order continuous linear functional.
- (iv) $C(K)$ admits a strictly positive σ -order continuous linear functional.
- (v) $C(K)$ admits a strictly positive order continuous linear functional.
- (vi) $C(K)^\delta$ admits a strictly positive order continuous linear functional.

Proof. That (i) and (ii) are equivalent is Theorem 2.5. The equivalence of (ii) and (iii) follows immediately from the definition of a category measure, Theorem 5.1 and Proposition 5.3. That (iii) implies (iv) follows from Lemma 2.1 (ii).

To see that (iv) implies (v), assume that $C(K)$ admits a strictly positive σ -order continuous linear functional φ . Then there exists a fully supported regular Borel measure on K , hence K satisfies cc, see for instance [7, Proposition 4.1.6]. Then $C(K)$ is order separable so that φ is order continuous.

The equivalence of (v) and (vi) follows from Lemma 2.1 (i) and (ii). Lastly, that (vi) implies (iii) is obvious, which complete the proof. \square

Remark 5.5. For the equivalence of (iii) to (vi) in Corollary 5.4 the assumption of strict positivity is essential. Indeed, there exists a compact Hausdorff space K and a σ -order continuous functional on $C(K)$ which is not order continuous [7, Example 4.7.16]. Furthermore, the statement ‘there exists a Stonean space K and a σ -order continuous functional on $C(K)$ which is not order continuous’ is equivalent to the existence of a measurable cardinal, see [17].

As an application of Corollary 5.4 we obtain the following topological characterization of hyper-Stonean spaces.

Theorem 5.6. *Let K be a Stonean space. Then K is hyper-Stonean if and only if there exists a collection \mathcal{U} of clopen subsets of K so that $\bigcup \mathcal{U}$ is dense in K and every $U \in \mathcal{U}$ satisfies property (***)*.

Proof. Assume that K is hyper-Stonean. For each normal measure μ on K , let S_μ denote the support of μ . Let $\mathcal{U} = \{S_\mu : \mu \in \mathcal{N}(K)\}$. By definition of a hyper-Stonean space $\bigcup \mathcal{U}$ is dense in K . Each $S_\mu \in \mathcal{U}$ is clopen, hence itself Stonean, and μ defines a strictly positive order continuous functional on $C(S_\mu)$. Corollary 5.4 implies that each S_μ satisfies property (***)

Assume that there exists a collection \mathcal{U} of clopen subsets of K so that $\bigcup \mathcal{U}$ is dense in K and every $U \in \mathcal{U}$ satisfies property (***)

. We claim that each $U \in \mathcal{U}$ is the support of a normal measure on K . Fix $U \in \mathcal{U}$. By Corollary 5.4 there exists a strictly positive order continuous functional φ on $C(U)$. Consider the linear functional $\psi : C(K) \ni f \mapsto \varphi(f|U) \in \mathbb{R}$. Because U is clopen, hence regular closed in K , ψ is order continuous by [15, Theorem 3.4]. Therefore there exists a unique normal measure μ on K so that

$$\psi(f) = \int_K f d\mu, \quad f \in C(K).$$

For every $f \in C(K)_+$, $f|S_\mu = \mathbf{0}$ if and only if $\psi(f) = 0$, if and only if $f|U = \mathbf{0}$. Therefore $U = S_\mu$, which verifies our claim.

Since $\bigcup \mathcal{U}$ is dense in K , the union of the supports of the normal measures on K is dense in K ; that is, K is hyper-Stonean. \square

Remark 5.7. It is possible to obtain our main results, namely Theorem 4.2, from which the results in Section 5 follow in a less direct manner. We briefly recall how this may be achieved.

In [8], de Jonge and van Rooij give a construction of $C(K)^\delta$ in terms of Borel measurable functions which is very similar to that given in Theorem 3.5. If \mathcal{B} denotes the vector lattice of bounded Borel measurable functions on K and \mathcal{N} the subspace of \mathcal{B} consisting of these functions which vanish on a meagre Borel set, then $C(K)^\delta$ can be identified with $\mathcal{D}(K) = \mathcal{B}/\mathcal{N}$. Dales et al. [7] call the space $\mathcal{D}(K)$

the Dixmier algebra of K . Using this construction our results can be obtained via the machinery of measures on Boolean algebras as set out, for instance, in [12]. We have opted for a direct and more transparent approach.

REFERENCES

1. C. D. Aliprantis and O. Burkinshaw, *Locally solid Riesz spaces*, Academic Press, New York-London, 1978.
2. C.D. Aliprantis and O. Burkinshaw, *Positive operators*, Springer, Dordrecht, 2006, reprint of the 1985 original.
3. R. Anguelov, *Dedekind order completion of $C(X)$ by Hausdorff continuous functions*, Quaest. Math. **27** (2004), no. 2, 153–169.
4. S. A. Argyros, *On compact spaces without strictly positive measure*, Pacific J. Math. **105** (1983), no. 2, 257–272.
5. E. Behrends, R. Danckwerts, R. Evans, S. Göbel, P. Greim, K. Meyfarth, and W. Müller, *L^p -structure in real Banach spaces*, Lecture Notes in Mathematics, Vol. 613, Springer-Verlag, Berlin-New York, 1977.
6. M. Cambern and p. Greim, *Uniqueness of preduals for spaces of continuous vector functions*, Canad. Math. Bull. **32** (1989), no. 1, 98–104.
7. H. G. Dales, F. K. Dashiell, Jr., T.-M. Lau, and D. Strauss, *Banach spaces of continuous functions as dual spaces*, CMS Books in Mathematics/Ouvrages de Mathématiques de la SMC, Springer, Cham, 2016.
8. E. de Jonge and A. C. M. van Rooij, *Introduction to Riesz spaces*, Mathematisch Centrum, Amsterdam, 1977.
9. B. de Pagter and C. B. Huijsmans, *On z -ideals and d -ideals in Riesz spaces. II*, Nederl. Akad. Wetensch. Indag. Math. **42** (1980), no. 4, 391–408.
10. R. P. Dilworth, *The normal completion of the lattice of continuous functions*, Trans. Amer. Math. Soc. **68** (1950), 427–438.
11. J. Dixmier, *Sur certains espaces considérés par M. H. Stone*, Summa Brasil. Math. **2** (1951), 151–182.
12. D. H. Fremlin, *Measure theory. volume 3*, Torres Fremlin, Colchester, 2002.
13. A. M. Gleason, *Projective topological spaces*, Illinois J. Math. **2** (1958), 482–489.
14. A. Horn, *The normal completion of a subset of a complete lattice and lattices of continuous functions*, Pacific J. Math. **3** (1953), 137–152.
15. M. Kandić and A. Vavpetič, *Topological aspects of order in $C(X)$* , Positivity **23** (2019), no. 3, 617–635.
16. J. L. Kelley, *Measures on Boolean algebras*, Pacific J. Math. **9** (1959), 1165–1177.
17. W. A. J. Luxemburg, *Is every integral normal?*, Bull. Amer. Math. Soc. **73** (1967), 685–688.
18. ———, *Integration with respect to finitely additive measures*, Positive operators, Riesz spaces, and economics (Pasadena, CA, 1990), Springer, Berlin, 1991, pp. 109–150.
19. W. A. J. Luxemburg and A. C. Zaanen, *Riesz spaces. Vol. I*, North-Holland Publishing Co., Amsterdam-London; American Elsevier Publishing Co., New York, 1971.
20. F. Maeda and T. Ogasawara, *Representation of vector lattices*, J. Sci. Hiroshima Univ. Ser. A. **12** (1942), 17–35.
21. D. Maharam, *Category, Boolean algebras and measure*, Proceedings of the Fourth Prague Topological Symposium, held in Prague, August 1976 (Josef Novák, ed.), Society of Czechoslovak Mathematicians and Physicists, Prague, 1977, Part B: Contributed papers, pp. 124–135.
22. W. Maxey, *The dedekind completion of $c(x)$ and its second dual*, Ph.D. thesis, Purdue University, 1973.
23. J. C. Oxtoby, *Spaces that admit a category measure*, J. Reine Angew. Math. **205** (1960/1961), 156–170.
24. ———, *Measure and category*, second ed., Graduate Texts in Mathematics, vol. 2, Springer-Verlag, New York-Berlin, 1980.
25. H. H. Schaefer, *Banach lattices and positive operators*, Springer-Verlag, New York-Heidelberg, 1974, Die Grundlehren der mathematischen Wissenschaften, Band 215.
26. J. M. Ayerbe Toledano, *Category measures on Baire spaces*, Publ. Mat. **34** (1990), no. 2, 299–305.

27. J. H. van der Walt, *Representations of the Dedekind completions of spaces of continuous functions*, Positivity and noncommutative analysis, Trends Math., Birkhäuser/Springer, Cham, [2019] ©2019, pp. 563–581.
28. K. Yosida and E. Hewitt, *Finitely additive measures*, Trans. Amer. Math. Soc. **72** (1952), 46–66.

DEPARTMENT OF MATHEMATICS AND APPLIED MATHEMATICS, UNIVERSITY OF PRETORIA, CORNER OF LYNNWOOD ROAD AND ROPER STREET, HATFIELD 0083, PRETORIA, SOUTH AFRICA