we consider in some representative closed bounded domain; all we say holds without question as long as our sets are all sitting inside some closed bounded domain in $\mathbb{R}^n$.

THEOREM B.2.1. *$\mathcal{C}^n$ is a closed subset of $\mathcal{K}(B_{\ell_n^2})$.*

PROOF. Suppose $K \in \mathcal{K}(B_{\ell_n^2})$ is *not* convex. Then there are points $x, y \in K$ and numbers $\lambda, \varepsilon : 0 < \lambda, \varepsilon < 1$ so that if $z = \lambda x + (1-\lambda)y$, then $(z+\varepsilon B_{\ell_n^2}) \cap K = \emptyset$.

Suppose $\tilde{K} \in \mathcal{K}(B_{\ell_n^2})$ and $\delta(\tilde{K}, K) < \frac{\varepsilon}{3}$.

Then there exist $\tilde{x}, \tilde{y} \in \tilde{K}$ so that

$$\|x - \tilde{x}\| < \frac{\varepsilon}{3} \text{ and } \|y - \tilde{y}\| < \frac{\varepsilon}{3}.$$

If $\tilde{z} = \lambda\tilde{x} + (1-\lambda)\tilde{y}$, then $\|\tilde{z} - z\| < \frac{\varepsilon}{3}$, too. It follows that if $\tilde{z} \in \tilde{K}$ there would be a $w \in K$ so that $\|\tilde{z} - w\| < \frac{\varepsilon}{3}$; but then

$$\|z - w\| \leq \|z - \tilde{z}\| + \|\tilde{z} - w\| < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} < \varepsilon,$$

and this contradicts $(z + \epsilon B_{\ell_n^2}) \cap K = \emptyset$. The result is that $\tilde{z} \notin \tilde{K}$ and $\tilde{K}$ is *not* convex.

We've established that any $\tilde{K}$ within $\frac{\varepsilon}{3}$ of K in $(\mathcal{K}(B_{\ell_n^2}), \delta)$ is non-convex. Hence $\mathcal{K}\backslash\mathcal{C}^n$ is open, which means $\mathcal{C}^n$ is closed. $\square$

The compactness of $\mathcal{C}^n$ has added importance because of the following.

THEOREM B.2.2. *The function* $\text{vol} : \mathcal{C}^n \to [0, \infty)$ *given by*

$$\text{vol}(K) = \lambda_n(K)$$

is continuous on $(\mathcal{C}^n, \delta)$ where λ_n is Lebesgue measure on $\mathbb{R}^n$.

PROOF. Start with $K \in \mathcal{C}^n$.

Suppose $\text{vol}(K) = 0$. Then K must be contained in a hyperplane. Why? Because otherwise K contains vectors $x_1, \ldots, x_n$ that constitute a basis for $\mathbb{R}^n$; after translating, if necessary, we can assume K contains 0 as well as a basis $\{x_1, \ldots, x_n\}$ for $\mathbb{R}^n$. But now $K \supseteq \text{co}\{0, x_1, \ldots, x_n\}$, too; since

$$\text{co}\{0, x_1, \ldots, x_n\} = \{\sum_{i=1}^n \lambda_i x_i : 0 \leq \lambda_i \leq 1, \sum_{i=1}^n \lambda_i \leq 1\},$$

K contains the set $u(B^+_{\ell_n^1})$, where $u : \ell_n^1 \to \mathbb{R}^n$ takes e_n to x_n and $B^+_{\ell_n^1}$ is the set

$$\{(\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n : 0 \leq \lambda_i, \sum_{i=1}^n \lambda_i \leq 1\}.$$

It is plain that $\lambda_n(B^+_{\ell_n^1}) > 0$. Since u is invertible, it follows that $\lambda_n(u(B^+_{\ell_n^1})) > 0$. Hence

$$\text{vol}(K) \geq \text{vol}(u(B^+_{\ell_n^1})) > 0.$$

So if we assume $\text{vol}(K) = 0$, then it follows that K lies in some hyperplane $H \subseteq \mathbb{R}^n$. Suppose $\delta(K, \tilde{K}) < \alpha \leq 1$. Then $\tilde{K} \subseteq K + \alpha B_{\ell_n^2}$, so that

$$\text{vol}(\tilde{K}) \leq \text{vol}(K + \alpha B_{\ell_n^2}).$$

By rotating and translating, we may assume $H = \mathbb{R}^{n-1} \times \{0\}$. A picture now tells the story — draw your own.

$$\begin{aligned}
\mathrm{vol}(K + \alpha B_{\ell_n^2}) &= \lambda_n(K + \alpha B_{\ell_n^2}) \\
&\leq \int_{-\alpha}^{\alpha} \lambda_{n-1}(K + \alpha B_{\ell_{n-1}^2})dt \\
&= 2\alpha \cdot \lambda_{n-1}(K + \alpha B_{\ell_{n-1}^2}) \\
&= 2\alpha \cdot \lambda_{n-1}(K + B_{\ell_{n-1}^2}) \text{ (since } \alpha \leq 1) \\
&= c(K)\alpha,
\end{aligned}$$

where $c(K)$ is a constant depending on K.

So, if $\mathrm{vol}(K) = 0$, then all $\tilde{K}$'s in $\mathcal{C}^n$ that are close to K in the metric δ have volumes that are close to 0.

What happens when $\mathrm{vol}(K) > 0$? Well, K's interior, K°, must be non-empty; think about it: If $K^\circ = \emptyset$ then we've seen K *cannot* contain a basis for $\mathbb{R}^n$ and so it must lie inside a hyperplane. No harm is done if we translate our objects under consideration to allow 0 to be an interior point of K. So be it.

Let $\varepsilon > 0$. Choose $\gamma > 1$ so that

$$(\gamma^n - 1)\gamma^n \lambda_n(K) < \varepsilon.$$

Choose $\rho > 0$ so that

$$\rho B_{\ell_n^2} \subseteq K^\circ.$$

We need to take advantage of the relationship $\rho B_{\ell_n^2} \subseteq K^\circ$. To do this we recall that if $K \in \mathcal{C}^n$, then the support function of K is given by

$$S_K(u) = \sup\{(u, x) : x \in K\}$$

where $(\ ,\ )$ is the inner product in $\mathbb{R}^n$. S_K is a (continuous) convex function from $\mathbb{R}^n$ to $\mathbb{R}$ and satisfies

$$S_{K_1} \leq S_{K_2} \text{ if and only if } K_1 \subseteq K_2,$$

where $K_1, K_2 \in \mathcal{C}^n$. Now it is plain (or ought to be!) that $S_{K_1} \leq S_{K_2}$ if $K_1 \subseteq K_2$; if there were $x_0 \in K_1 \backslash K_2$, then there would be a $u \in \mathbb{R}^n$ so that

$$S_{K_2}(u) = \sup_{x \in K_2} (u, x) < (u, x_0) \leq \sup_{x \in K_1} (u, x) = S_{K_1}(u).$$

It is also easy to see that if $K_2 \subseteq K_1^\circ$, then not only is it true that $S_{K_1} \leq S_K$, but, in fact, $S_{K_2}(u) < S_{K_1}(u)$ for $u \neq 0$.

Now we're ready to take advantage of the relationship $\rho B_{\ell_n^2} \subseteq K^\circ$. Here's how: if $K_1, K_2 \in \mathcal{C}^n$ satisfy $K_2 \subseteq K_1^\circ$, then there is a number $\eta > 0$ so small that whenever $\tilde{K} \in \mathcal{C}^n$ and $\delta(K_1, \tilde{K}) < \eta : K_2 \subseteq K$.

Indeed, $K_2 \subseteq K_1^\circ$ ensures $S_{K_1} - S_{K_2}$ is a *positive* continuous function on $\mathbb{R}^n \backslash \{0\}$. As such, $S_{K_1} - S_{K_2}$ attains a *positive* minimum η on $S_{\ell_n^2}$. Let $\tilde{K} \in \mathcal{C}^n$ satisfy $\delta(K_1, \tilde{K}) < \eta$. Then

$$K_1 \subseteq \tilde{K} + \eta B_{\ell_n^2} \text{ and } \tilde{K} \subseteq K_1 + \eta B_{\ell_n^2}$$

so that

$$S_{K_1} \leq S_{\tilde{K}} + \eta \text{ and } S_{\tilde{K}} \leq S_{K_1} + \eta;$$

it follows that $|S_{K_1}(u) - S_{\tilde{K}}(u)| \leq \eta$ for $u \in S_{\ell_n^2}$. But for $u \in S_{\ell_n^2}$,

$$S_{K_1}(u) - S_{K_2}(u) \geq \eta,$$

so for $u \in S_{\ell_n^2}$

$$\begin{aligned} S_{K_2}(u) &\le S_{K_1}(u) - \eta \\ &\le S_{\tilde{K}}(u). \end{aligned}$$

Homogeneity soon says

$$S_{K_2} \le S_{\tilde{K}} \text{ and } K_2 \subseteq \tilde{K}.$$

Choose $\alpha < (\gamma - 1)\rho$ so that for $\tilde{K} \in \mathcal{C}^n$, it follows from $\delta(K, \tilde{K}) < \alpha$ that $\rho B_{\ell_n^2} \subseteq \tilde{K}$.

Suppose $\delta(K, \tilde{K}) < \alpha$. Then

$$\begin{aligned} K &\subseteq \tilde{K} + \alpha B_{\ell_n^2} \\ &\subseteq \tilde{K} + (\gamma - 1)\rho B_{\ell_n^2} \\ &\subseteq \tilde{K} + (\gamma - 1)\tilde{K} \\ &= \gamma \tilde{K}. \end{aligned}$$

Moreover,

$$\begin{aligned} \tilde{K} &\subseteq K + \alpha B_{\ell_n^2} \\ &\subseteq K + (\gamma - 1)\rho B_{\ell_n^2} \\ &\subseteq K + (\gamma - 1)K^{\circ} \\ &\subseteq K + (\gamma - 1)K \\ &= \gamma K. \end{aligned}$$

It follows that

$$\lambda_n(K) \le \lambda_n(\gamma \tilde{K}) = \gamma^n \lambda_n(\tilde{K}) \text{ and } \lambda_n(\tilde{K}) \le \lambda_n(\gamma K) = \gamma^n \lambda_n(K).$$

Ah ha!

$$\begin{aligned} \lambda_n(K) - \lambda_n(\tilde{K}) &\le \gamma^n \lambda_n(\tilde{K}) - \lambda_n(\tilde{K}) \\ &= (\gamma^n - 1)\lambda_n(\tilde{K}) \\ &\le (\gamma^n - 1)\gamma^n \lambda_n(K) \end{aligned}$$

and

$$\begin{aligned} \lambda_n(\tilde{K}) - \lambda_n(K) &\le \gamma^n \lambda_n(K) - \lambda_n(K) \\ &= (\gamma^n - 1)\lambda_n(K) \\ &\le (\gamma^n - 1)\gamma^n \lambda_n(K), \end{aligned}$$

because $\gamma^n \ge 1$. Regardless of the effects of the new world order, we have

$$|\lambda_n(\tilde{K}) - \lambda_n(K)| \le (\gamma^n - 1)\gamma^n \lambda_n(K) \le \varepsilon$$

and $\delta(K, \tilde{K}) \le \alpha$. □

B.3. Ellipsoids in finite dimensional Banach spaces

Let X be a Banach space and $u : \ell_n^2 \to X$ a bounded linear operator and let $K = \{v \in \ell_n^2 : u(v) = 0\}$ denote its kernel. Of course, K is a closed linear subspace of the Hilbert space ℓ_n^2 and, as such, has an orthogonal complement $K^\perp$. Take note that $u|_{K^\perp}$ is injective and $\|u|_{K^\perp}\| = \|u\|$. What's more, $u|_{K^\perp}(B_{K^\perp})$ is precisely $u(B_{\ell_n^2})$.

We say that the absolutely convex compact subset E of X is an *ellipsoid* if it is the image of $B_{\ell_n^2}$ under some bounded linear operator for some n. By what we've just noticed, if $E \subseteq \mathbb{R}^k$ is an ellipsoid, then there is an n and an injective linear operator u acting on ℓ_n^2 so that E is the image under the u of $B_{\ell_n^2}$.

Suppose F is a finite dimensional Banach space. Denote by $\mathcal{E}(F)$ the collection of all ellipsoids contained in the closed unit ball B_F of F. Here is what's true about $\mathcal{E}(F)$.

THEOREM B.3.1. *$\mathcal{E}(F)$ is a closed subset of the compact metric space $\mathcal{K}(B_F)$ of all non-empty compact convex subsets of B_F.*

PROOF. By our earlier remarks, if $n = \dim F$ and $E \in \mathcal{E}(F)$, then there is a linear operator $u_E : \ell_n^2 \to F$ so that $E = u_E(B_{\ell_n^2})$. It follows that if (E_m) is a sequence of ellipsoids inside B_F and $C = \lim_m E_m$ in $\mathcal{K}(B_F)$, then there is a sequence $u_m = u_{E_m} : \ell_n^2 \to F$ of linear operators such that $u_m(B_{\ell_n^2}) = E_m$; a fortiori, each u_m is in the closed unit ball of $\mathcal{L}(\ell_n^2; F)$, a compact space.

By passing to a subsequence, if necessary, we can assume $u = \lim_{m\to\infty} u_m$ exists. If we look at $E = u(B_{\ell_n^2})$, then E is an ellipsoid and is, in fact, C. Why is this so? Well, think what it means for two bounded linear operators $a, b : \ell_n^2 \to F$ to be close: If $\|a - b\| \leq \varepsilon$, then for any $x \in B_{\ell_n^2}$, $\|ax - bx\| \leq \varepsilon$; it is clear then that $aB_{\ell_n^2} \subseteq bB_{\ell_n^2} +$ small multiple of $B_{\ell_n^2}$ and $bB_{\ell_n^2} \subseteq aB_{\ell_n^2} +$ small multiple of $B_{\ell_n^2}$ and that just means that $aB_{\ell_n^2}$ and $bB_{\ell_n^2}$ are close in the Hausdorff metric in $\mathcal{K}(B_F)$. □

It is a stunning discovery of John (1948), that, in fact, the ellipsoid of maximum volume contained in the unit ball B_E of a finite dimensional Banach space E is unique. This result has played an important role in recent advances in the structure theory of Banach Spaces.

Note: The authors benefited a great deal from conversations with Artum Zvavitch about the material of this appendix, as well as frequent referrals to the book of Pisier (1989) as well as to the book of Schneider (1993).

APPENDIX C

A short introduction to Banach lattices

C.1. The facts, ma'm, just the facts

Banach lattices combine the best of Banach spaces and vector lattices thanks to demanding that the bigger the "absolute value" of an element the greater its length. To delve into the finer structure of a Banach lattice (as a Banach space) we must attend to certain order theoretic affairs.

For the present we will deal with *real* linear spaces.

Suppose X is a linear space and X comes equipped with a partial order $\leq$; if $x \pm z \leq y \pm z$ and $px \leq py$ for all x, y, z in X and all positive numbers p whenever $x \leq y$, then we call X an *ordered linear space.* If the order of the ordered linear space X is a lattice order, that is, for any $x, y \in X$, $x \vee y$, the least upper bound of x and y, as well as $x \wedge y$, the greatest lower bound of x and y, coexist, then X is called a *vector lattice.*

It is vector lattices that are the present object of our concern.

Some notation is called for: If x is an element of the vector lattice X, then

$$x^+ = x \vee 0,\ x^- = (-x) \vee 0,\ |x| = x^+ \vee x^-.$$

The positive cone X^+ of X is the set

$$X^+ = \{x \in X : x \geq 0\}.$$

As in any subject of attractive complexity a certain amount of grubby calculation is necessary; the relationships stated below require a bit of manipulative demonstration, but in light of their uses the tedium is worth our while.

THEOREM C.1.1. *Let X be a vector lattice. Then for any $x, y, z \in X$:*

(1) $x + y = (x \vee y) + (x \wedge y)$ *and* $x \vee y = -[(-x) \wedge (-y)]$;
(2) $(x \vee y) + z = (x + z) \vee (y + z)$ *and* $(x \wedge y) + z = (x + z) \wedge (y + z)$;
(3) $x = x^+ - x^-, |x| = x^+ + x^-$ *and* $x^+ \wedge x^- = 0$;
(4) *if* $x = uv$ *where* $u, v \in X^+$ *and* $u \wedge v = 0,$, *then* $x^+ = u$ *and* $x^- = v$.
(5) $|ax| = |a||x|$ *for all real numbers* a;
(6) $|x + y| \leq |x| + |y|$ *and, if* $|x| \wedge |y| = 0, |x + y| = |x| + |y| = |x| \vee |y|$;
(7) $(x \vee y) \wedge z = (x \wedge z) \vee (y \wedge z)$ *and* $(x \wedge y) \vee z = (x \vee z) \wedge (y \vee z)$;
(8) *(F. Riesz's decomposition property) if* $0 \leq z \leq x+y, x, y \geq 0$ *then there is a* $u \in X^+$ *and there is a* $v \in X^+$ *with* $u \leq x$ *and* $v \leq y$ *so that* $z = u + v$;
(9) $|x - y| = (x \vee y) - (x \wedge y) = |(x \vee z) - (y \vee z)| + |(x \wedge z) - (y \wedge z)|$.

PROOF. In so far as it is the only property positing the existence of external elements we start with the proof of F. Riesz's decomposition property (8).

Our setup is that $x, y, z \in X^+$ with $0 \leq z \leq x + y$. Let $u = x \wedge z$ and v has to be (what else but) $v = z - u$. Now it is plain and easy to see that $0 \leq u$ (x, z are both ≥ 0) and $u \leq x$ ($u = x \wedge z \leq x$) and $u \leq z$, too (same reason: $u = x \wedge z \leq z$). Since $u \leq z$, $v = z - u \geq 0$. Further, $v \leq y$:

$$\begin{aligned} y - v = y - (z - u) &= y - z + u = (y - z) + (x \wedge z) \\ &= y + (x - z) \wedge (z - z) = y + ((x - z) \wedge 0) \\ &= (y + x - z) \wedge y = (x + y - z) \wedge y \geq 0. \end{aligned}$$

Tra la! □

Such basic manipulations dominate the show that constitutes the proof of the above theorem.

A *Banach lattice* is a Banach space that is a vector lattice with $\|x\| \leq \|y\|$ whenever $0 \leq x \leq y$; alternatively, if $|x| \leq |y|$,, then $\|x\| \leq \|y\|$. Naturally $\|x\| = \||x|\|$.

THEOREM C.1.2. *Let X be a Banach lattice, then:*

(1) *The lattice operations are continuous.*
(2) *X^+ is closed.*
(3) *If (x_n) is an increasing convergent sequence, then $\sup_n x_n$ exists and is $= \lim_n x_n$.*

PROOF. (1) As is usual in bilinear type affairs, we proceed thusly,

$$|x_n \wedge y_n - x \wedge y| \leq |x_n \wedge y_n - x_n \wedge y| + |x_n \wedge y - x \wedge y|$$

which by Theorem C.1.1(9)

$$\leq |y_n - y| + |x_n - x|.$$

So

$$\begin{aligned} \|x_n \wedge y_n - x \wedge y\| &= \||(x_n \wedge y_n) - (x \wedge y)|\| \\ &\leq \||y_n - y| + |x_n - x|\| \\ &\leq \||y_n - y|\| + \||x_n - x|\| \\ &= \|y_n - y\| + \|x_n - x\| \end{aligned}$$

and (1) follows.

(2) Suppose $0 \leq x_n \to x$. Then $x = \lim_n x_n = \lim_n (x_n \vee 0) = (\lim_n x_n) \vee 0 = x \vee 0$. So $x = x \vee 0$ and $x = \lim_n x_n \in X^+$.

(3) Suppose $x_n \leq x_{n+1}$ for all $n \in \mathbb{N}$ and $x = \lim_n x_n$. If $m \geq n$, then $x_m - x_n \geq 0$. From (2) we see that $x - x_n = \lim_m (x_m - x_n) \geq 0$. So $x \geq x_n$ for each n; x is an upper bound for $\{x_n : n \in \mathbb{N}\}$.

Assume $u \in X$ and $x_n \leq u \leq x$ for all n. Then $u = x \wedge u = \lim_n x_n \wedge u = \lim_n x_n = x$. □

C.2. Some basics about duality in Banach lattices

Of course duality plays a crucial role in the study of Banach lattices and it is the behavior of "positive" linear functionals that opens the way.

If X and Y are Banach lattices and $u : X \to Y$ is a linear mapping we say u is *positive* if $ux \geq 0$ whenever $x \geq 0$.

THEOREM C.2.1. *Positive linear mappings between Banach lattices are continuous.*

PROOF. Imagine, if you can, that $u : X \to Y$ is an unbounded positive linear mapping between the Banach lattices X and Y. Then u is unbounded on $B_X \subseteq B_{X^+} - B_{X^+}$, where $B_{X^+} = X^+ \cap B_X$ (here we notice that if $x \in B_X$, then $x = x^+ - x^-$ where $x^+, x^- \in B_{X^+}$ since $0 \le x^+, x^- \le |x| \in B_X$).

Hence u is unbounded on B_{X^+}. Thus we can find a sequence (x_n) in B_{X^+} such that for each n, $\|ux_n\| \ge n^3$.

Look at $z = \Sigma_n \frac{x_n}{n^2} \in X^+$. Plainly $z \ge \frac{x_n}{n^2}$ for each n, and so

$$uz \ge u\left(\frac{x_n}{n^2}\right) = \frac{u(x_n)}{n^2} \ge 0.$$

So

$$\|uz\| \ge \frac{\|u(x_n)\|}{n^2} \ge \frac{n^3}{n^2} = n.$$

OOPS! □

To get beneath the skin of the order/duality relations in a Banach lattice it is useful to introduce the space $X^\sharp$ of "order bounded" linear functionals on X. Recall that a set B in the vector lattice X is called *order bounded* if there is $x \in X^+$ so that $|b| \le x$ for all $b \in B$. A linear functional f on X is *order bounded*, or $f \in X^\sharp$, if f takes order bounded sets in X onto bounded sets of $\mathbb{R}$.

We can define an order structure on $X^\sharp$: If $f, g \in X^\sharp$, then $f \le g$ means $f(x) \le g(x)$ whenever $x \in X^+$.

THEOREM C.2.2. *The linear space $X^\sharp$ of all order bounded linear functionals on X is a Dedekind complete vector lattice. The lattice operations are given by their values on X^+ as follows: Supposing $x \in X^+$,*

$$\begin{aligned}(f \vee g)(x) &= \sup\{f(y) + g(z) : y, z \in X^+, x = y + z\},\\ (f \wedge g)(x) &= \inf\{f(y) + g(z) : y, z \in X^+, x = y + z\},\\ |f|(x) &= \sup\{|f(z)| : 0 \le z \le x\},\\ f^+(x) &= \sup\{f(y) : 0 \le y \le x\},\\ f^-(x) &= -\inf\{-f(y) : 0 \le y \le x\}.\end{aligned}$$

Should $\mathcal{F}$ be a non-empty directed subset of $X^\sharp$ which is bounded above, then $g = \sup \mathcal{F}$ is given on X^+ by $g(x) = \sup_{f \in \mathcal{F}} f(x)$.

Remark: In vector lattices functionals that are additive and positively homogeneous in the positive cone have unique *linear* extensions to the whole vector space.

PROOF. Suppose $f, g \in X^\sharp$ and $k \in X^\sharp$ with $f, g \le k$. Define $h : X^+ \to \mathbb{R}$ by

$$h(x) := \sup\{f(y) + g(x - y) : 0 \le y \le x\}.$$

It is plain that if $0 \le y \le x$, then

$$\begin{aligned}h(y) &\le f(y) + g(x - y)\\ &\le k(y) + k(x - y) = k(x).\end{aligned}$$

So, if h is linear (on E^+) and order-bounded, then it must be $f \vee g$. It is easy to see h is positively homogeneous on E^+; just let $\lambda \ge 0$ and look:

$$\begin{aligned} h(\lambda x) &= \sup\{f(y)+g(\lambda x-y) : 0\le y\le \lambda x\} \\ &= \sup\{f(\lambda z)+g(\lambda x-\lambda z) : \lambda z = y \text{ and } 0\le y\le \lambda x\} \\ &= \lambda \sup\{f(z)+g(x-z) : 0\le z\le x\} \\ &= \lambda h(x). \end{aligned}$$

Additivity calls in the Riesz decomposition property of vector lattices. Suppose $x, x_1, x_2 \in X^+$ with $x = x_1 + x_2$. If $0 \le y \le x$, then we can find $y_1 \in [0, x_1]$ and $y_2 \in [0, x_2]$ such that $y = y_1 + y_2$; as a consequence,

$$\begin{aligned} f(y)+g(x-y) &= f(y_1+y_2)+g((x_1+x_2)-(y_1+y_2)) \\ &= \underbrace{f(y_1)+g(x_1-y_1)}+\underbrace{f(y_2)+g(x_2-y_2)} \\ &\le \qquad h(x_1) \qquad + \qquad h(x_2). \end{aligned}$$

It follows that

$$h(x_1+x_2)\le h(x_1)+h(x_2).$$

Now should $0 \le y_1 \le x_1$ and $0 \le y_2 \le x_2$,, then $0 \le y_1 + y_2 \le x_1 + x_2 = x$. So

$$\begin{aligned} f(y_1)+g(x_1-y_1)+f(y_2)+g(x_2-y_2) &= f(y_1+y_2)+g((x_1+x_2)-(y_1+y_2)) \\ &= f(y_1+y_2)+g(x-(y_1+y_2)) \\ &\le \sup\{f(y)+g(x-y) : 0\le y\le x\} \\ &= h(x). \end{aligned}$$

Fixing y_2 for the moment (and remembering that x, x_1 and x_2 are going nowhere for the same moment) we see that whenever $0 \le y_1 \le x_1$,

$$f(y_1)+g(x_1-y_1)\le h(x)-[f(y_2)+g(x_2-y_2)].$$

So whenever $0 \le y_2 \le x_2$,

$$h(x_1)\le h(x)-[f(y_2)+g(x_2-y_2)]$$

or, whenever $0 \le y_2 \le x_2$,

$$f(y_2)-g(x_2-y_2)\le h(x)-h(x_1).$$

Now letting y_2 wander throughout its domain: $0 \le y_2 \le x_2$, we see

$$h(x_2)\le h(x)-h(x_1).$$

Tra la!

$$h(x_1)+h(x_2)\le h(x).$$

h is additive and positively homogeneous on X^+. It is clear that h is bounded by k on $[0, x]$ so, for all intents and purposes $h \in X^\sharp$.

Finally, suppose $\mathcal{F}$ is a non-empty directed set in $X^\sharp$ that is bounded above in $X^\sharp$. Then the functional

$$u : X^+ \to \mathbb{R}, u(x) := \sup\{f(x) : f\in\mathcal{F}\} \qquad (x\ge 0)$$

is positively homogeneous and additive on X^+ and dominated on X^+ by the upper bound of $\mathcal{F}$. Only additivity needs a word or two of explanation: Suppose $x, x_1, x_2 \in X^+$ and $x = x_1 + x_2$; on the one hand, if $f_1, f_2 \in \mathcal{F}$ and $f \in \mathcal{F}$ is $\ge f_1, f_2$, then, from

$$f_1(x_1)+f_2(x_2)\le f(x_1)+f(x_2)=f(x_1+x_2)=f(x)\le u(x)$$

it follows that $u(x_1) + u(x_2) \leq u(x)$; on the other hand,

$$\begin{aligned} u(x) &= \sup\{f(x) : f + \mathcal{F}\} \\ &= \sup\{f(x_1) + f(x_2) : f \in \mathcal{F}\} \\ &\leq \sup\{f(x_1) : f \in \mathcal{F}\} + \sup\{f(x_2) : f \in \mathcal{F}\} \\ &= u(x_1) + u(x_2). \end{aligned}$$

□

Since positive linear functionals are plainly in $X^\sharp$, and $X^\sharp$ is a vector lattice whose positive cone consists precisely of the positive linear functionals, $X^\sharp = X^{\sharp+} - X^{\sharp+}$. *$f \in X^\sharp$ precisely when f is the difference of positive linear functionals on X.*

COROLLARY C.2.3. *Let X be a Banach lattice. Then $X^* = X^\sharp$.*

PROOF. If X is a Banach lattice, then positive linear functionals are continuous on X (as are all positive linear operators to Banach lattices), so $X^\sharp \subseteq X^*$.

Members of X^* are bounded on all bounded sets, in particular, on order-bounded sets. □

A little bit of finagling with the order structure of X^* will come in handy. From the very fact that for $x^*, y^* \in X^*$, if $x \geq 0$,

$$(x^* \vee y^*)(x) = \sup\{x^*(u) + y^*(x - u) : 0 \leq u \leq x\}$$

and the easy observation that $0 \leq u \leq x$ precisely when $0 \leq x - u \leq x$, we see that if $x \geq 0$, then

$$\begin{aligned} |x^*|(x) &= [x^* \vee (-x^*)](x) \\ &= \sup\{x^*(u) + (-x^*)(x - u) : 0 \leq u, x - u \leq x\} \\ &= \sup\{x^*(u - (x - u)) : 0 \leq u, x - u \leq x\} \\ &= \sup\{x^*(y - z) : 0 \leq y, z \text{ and } y + z = x\}. \end{aligned}$$

It is plain that if $0 \leq y, z$ and $y+z = x$, then $y-z \leq y+z = x$ and $z-y \leq z+y = x$, so $|y - z| \leq x$. Hence, if $x \geq 0$,

$$|x^*|(x) \leq \sup\{x^*(w) : |w| \leq x\}.$$

On the other hand, if $w \leq x$, then

$$\begin{aligned} x^*(w) &= x^*(w^+ - w^-) \\ &= x^*(w^+) - x^*(w^-) \\ &\leq |x^*|(w^+) + |x^*|(w^-) \\ &= |x^*|(|w|) = |x^*|(x). \end{aligned}$$

It follows that if $x \geq 0$, then

$$|x^*|(x) = \sup\{x^*(w) : |w| \leq x\} = \sup\{|x^*(w)| : |w| = x\},$$

and so for general x,

$$|x^*(x)| \leq |x^*|(|x|).$$

COROLLARY C.2.4. *The dual X^* of a Banach lattice is a Banach lattice. If $x^* \in X^*$, then*

$$\|x^*\| = \sup\{|x^*(x)| : 0 \leq x \in B_X\}.$$

PROOF. Suppose $|x^*| \leq |y^*|$. Then

$$\begin{aligned}\|x^*\| &= \sup\{|x^*(x)| : x \in B_X\}\\ &\leq \sup\{|x^*|(|x|) : x \in B_X\}\\ &= \sup\{|x^*|(p) : 0 \leq p \in B_X\}\\ &\leq \sup\{|y^*|(p) : 0 \leq p \in B_X\}.\end{aligned}$$

But for $p \geq 0$,

$$\begin{aligned}|y^*|(p) &= \sup\{y^*(u) - y^*(p-u) : 0 \leq u \leq p\}\\ &= \sup\{y^*(p-2u) : 0 \leq u \leq p\},\end{aligned}$$

so,

$$\begin{aligned}\|x^*\| &\leq \sup\{y^*(p-2u) : 0 \leq u \leq p \in B_X\}\\ &= \sup\{\|y^*\|\|p-2u\| : 0 \leq u \leq p \in B_X\}.\end{aligned}$$

If $0 \leq u \leq p$, then $p - u$ and u are positive members of X whose sum is p so $|(p-u) - u| \leq p$ forcing us to conclude that

$$\|p - 2u\| = \||p-2u|\| = \||(p-u)-u|\| \leq \|p\| = 1.$$

Ah ha!

$$\|x^*\| \leq \|y^*\|.$$

$\square$

At this stage we can show that for any vector x in a Banach lattice X, $x \geq 0$ if and only if $\hat{x} \geq 0$ ($\hat{x}$ is just the image of x under the natural embedding of X into X^{**}). Indeed;

$$\begin{aligned}x \geq 0 &\Longrightarrow x^*(x) \geq 0 \text{ for each } x^* \in (X^*)^+\\ &\Longrightarrow \hat{x}(x^*) \geq 0 \text{ for each } x^* \in (X^*)^+\\ &\Longrightarrow \hat{x} \geq 0 \text{ as member of} (X^*)^*.\end{aligned}$$

Could it be that $\hat{x} \geq 0$ for some $x \notin X^+$? Well, if so, by X^+'s closedness, there would be an $x^* \in X^*$ so that $x^*(x) < 0$ yet $x^*(p) \geq 0$ for each $p \in X^+$. Plainly such an x^* is a positive linear functional. Just as plainly,

$$0 \leq \hat{x}(x^*) = x^*(x) < 0.$$

OOPS! So the answer to the earlier question is "no way, José".

It follows that in Banach lattices, $x \geq 0$ precisely when $x^*(x) \geq 0$ for each positive linear functional x^*.

In truth, more is so, namely, the natural embedding $X \to X^{**}$ of a Banach lattice X into its bidual is a lattice isomorphism. However, to show this a bit of experience with lattice homomorphisms is needed.

C.3. Lattice homomorphisms

A few words about linear operators that preserve the lattice structure of a Banach lattice are called for. We are particularly interested in lattice homomorphisms: A linear mapping $u : X \to Y$ between the vector lattices X and Y is called a *lattice homomorphism* if for any $x_1, x_2 \in X$, $u(x_1 \vee x_2) = u(x_1) \vee u(x_2)$ and $u(x_1 \wedge x_2) = u(x_1) \wedge u(x_2)$.

Because we've incorporated linearity within our definition of a lattice homomorphism, it is enough to check either preservation of $\vee$'s or $\wedge$'s to establish a linear map is a lattice homomorphism; this follows, by the way, because for linear maps $-u(-x) = u(x)$ and for any x_1 and $x_2 \in X$, $x_1 \vee x_2 = -(-x_1 \wedge -x_2)$ and $x_1 \wedge x_2 = -(-x_1 \vee -x_2)$.

PROPOSITION C.3.1. *Let $u : X \to Y$ be a linear operator between the vector lattices X and Y. Then the following are equivalent:*

(1) *u is a lattice homomorphism.*
(2) *$|u(x)| = u(|x|)$ for each $x \in X$.*
(3) *$u(x^+) \wedge u(x^-) = 0$ for each $x \in X$.*

PROOF. It is clear and easy that (1) implies (3) since $x^+ \wedge x^- = 0$.

Suppose (3) holds for u. Of course, u is a positive operator — after all, if $x \in X^+$, then $x = x^+$ and $x^- = 0$. So (3) says $u(x) \geq u(x^+) \wedge u(x^-) = 0$ so $u(x) \geq 0$ for each $x \geq 0$. But now $u(x) = u^+ - u^-$ with $u^+ \wedge u^- = 0$ is a unique decomposition of $u(x)$ into disjoint positive elements, so (3) just says $u(x^+) = u(x)^+$ and $u(x^-) = u(x)^-$; $|u(x)| = u(x)^+ + u(x)^- = u(x^+) + u(x^-) = u(x^+ + x^-) = u(|x|)$ follows.

Finally, suppose (2) holds. Again, it is plain that $u \geq 0$. It follows that $u(x)^+ = u(x) \vee 0 \leq u(x^+) \vee u(x^+) = u(x^+)$ and $u(x)^- \leq u(x^-)$, so from (2)

$$u(x)^+ + u(x)^- = |u(x)| = u(|x|) = u(x^+) + u(x^-).$$

This, in turn, says $u(x)^+ = u(x^+)$ and $u(x)^- = u(x^-)$ for all $x \in X$.

Suppose $x,\ y \in X$. Then

$$\begin{aligned} x \vee y &= x + y - x \wedge y \\ &= y + (x - x \wedge y) \\ &= y + [-[(x \wedge y) - x]] \\ &= y + [-(0 \wedge (y - x))] \\ &= y + (0 \vee (x - y)) = y + (x - y)^+. \end{aligned}$$

It follows that

$$\begin{aligned} u(x \vee y) &= u(y) + u((x - y)^+) \\ &= u(y) + [u(x - y)]^+ \\ &= u(y) + [u(x) - u(y)]^+ \\ &= u(x) \vee u(y). \end{aligned}$$

u is a lattice homomorphism. □

For functionals, the above proposition takes on particularly pleasing form.

COROLLARY C.3.2. *Let X be a Banach lattice and $x^* \in X^*$. Then the following statements regarding x^* are equivalent:*

(1) *x^* is a lattice homomorphism.*
(2) *For each $x \in X$, $\min\{x^*(x^+), x^*(x^-)\} = 0$.*
(3) *$x^* \in X^{*+}$ and $\ker(x^*)$ is an ideal; a subset I of X is an* ideal *if I is linear, and given $x \in X$ and $i \in I$ with $|x| \leq |i|$, then $x \in I$.*
(4) *$x^* \in X^{*+}$ and $X^*_{x^*}$ is one-dimensional.*

PROOF. We already know that (1) and (2) are equivalent statements. Naturally, $x^* \geq 0$ if x^* is a lattice homomorphism.

Suppose (1) holds. Let $x \in \ker(x^*)$ and suppose $|y| \leq |x|$. Then $|x^*(y)| = x^*(|y|) \leq x^*(|x|) = |x^*(x)| = 0$ so $y \in \ker(x^*)$, too; this is just (3). Now take $x \in X$ and suppose $x^*(x^+) \wedge x^*(x^-) > 0$. Choose $a > 0$ so that $x^*(x^+ - ax^-) = 0$, that is, $x^+ - ax^- \in \ker(x^*)$. Since x^+ and ax^- are each $\leq |x^+ - ax^-|$, (3) tells us $x^*(x^+)$ and $x^*(ax^-)$ are each $= 0$, i.e. $x^*(x^+), x^*(x^-) = 0$. OOPS!

It is easy to manipulate the above argument to show that denial of (2) leads to a denial of (3). So (2) follows from (3). (1), (2) and (3) are known equivalents! To get (4) "in the loop", notice that if $y^* \in X^*$ and $|y^*| \leq x^*$, then

$$|y^*(x)| \leq x^*(|x|)$$

for all $x \in X$. Hence $\ker(y^*) \subseteq \ker(x^*)$ in such a case, and so if (1) thru (3) are in effect, (4) follows; after all, either $\ker(y^*) = X$ or $\ker(y^*) = \ker(x^*)$; in the former case, well, $y^* = 0$ while in the latter, $y^* = \alpha x^*$ for some $\alpha : |\alpha| \leq 1$.

Alas, assume (4) is so. Fix $x \in X$ and let y^* be defined on X^+ by

$$y^*(y) := \lim_n x^*(y \wedge nx^+).$$

It is plain that y^* is positively homogeneous on X^+. Since for $p, q \in X^+$ we have

$$(p+q) \wedge (nx^+) \leq (p \wedge (nx^+)) + (q \wedge (nx^+)),$$

it is easy to see y^* is subadditive. If we fix $\varepsilon > 0$ and choose n so that

$$y^*(p) \leq x^*(p \wedge (nx^+)) + \varepsilon \text{ and } y^*(q) \leq x^*(q \wedge (nx^+)) + \varepsilon,$$

then

$$\begin{aligned} y^*(p) + y^*(q) &\leq x^*\left([p \wedge (nx^+)] + [q \wedge (nx^+)]\right) + 2\varepsilon \\ &\leq x^*\left((p+q) \wedge (2nx^+)\right) + 2\varepsilon \\ &\leq y^*(p+q) + 2\varepsilon. \end{aligned}$$

Additivity of y^* follows from this.

Of course, $0 \leq y^* \leq x^*$. By (4) $y^* = ax^*$ for some $a \geq 0$. If $x^*(x^+) > 0$, then $y^*(x^+) = x^*(x^+)$ and so $a = 1$; but, then $x^*(x^-) = y^*(x^-) = 0$ and (2) is the result. □

An extremely important example of a lattice homomorphism, indeed a lattice isomorphism, is the canonical embedding of a Banach lattice X into its bidual X^{**}. We saw earlier that this is a positive linear isometry. Now we show

THEOREM C.3.3. *The natural embedding $x \to \hat{x}$ of a Banach lattice X into its bidual is a vector lattice isomorphism and an isometry.*

PROOF. By our proposition, it is enough to show that $(\hat{x})^+$ and $\widehat{(x^+)}$ are the same in X^{**}, that is, both behave the same on X^*.

First, remember that $x \to \hat{x}$ *is* a positive linear isometry and $x \leq x^+ = x \vee 0$, so $\hat{x} \leq \widehat{(x^+)}$, and so $(\hat{x})^+ = \hat{x} \vee 0 \leq \widehat{(x^+)} \vee 0 = \widehat{(x^+)}$.

On the other hand, if we let x^* be a fixed positive linear functional and define χ on X^+ by

$$\chi(y) = \sup\{x^*(y') : 0 \leq y' \leq y, y' \in \bigcup_n n[0, x^+]\},$$

then proceeding as we did in the previous corollary, we soon realize that χ is (positively homogeneous and) additive.

So χ extends to a positive linear functional $y^* \in X^*$. It is plain that $0 \leq y^* \leq x^*$ and $y^*(x^-) = 0$. So

$$y^*(x) = y^*(x^+) = x^*(x^+),$$

and so

$$\widehat{(x^+)}(x^*) = x^*(x^+) = y^*(x) \leq \sup_{0 \leq z^* = x^*} z^*(x) = (\hat{x})^+(x^*).$$

□

C.4. AM-spaces and AL-spaces

A Banach lattice X is called an *AM-space* if whenever $x, y \in X^+$ we have $\|x \vee y\| = \|x\| \vee \|y\|$. If B_X has a biggest element u, then u is called the order unit of X.

A Banach lattice X is called an *AL-space* if whenever $x, y \in X^+$ we have $\|x + y\| = \|x\| + \|y\|$.

AM- and AL-spaces play a special role in the general theory of Banach lattices and in the general theory of Banach lattices and in the general theory of Banach spaces. Their role in each is defined by Kakutani's representation theorems and it is the pursuit of these that will dominate our efforts.

First, a remarkable duality result.

THEOREM C.4.1. *Let X be a Banach lattice.*

(1) *X is an AM-space if and only if X^* is an AL-space.*
(2) *X is an AL-space if and only if X^* is an AM-space with order unit.*

PROOF. Suppose X is an AM-space and let x^*, y^* be positive linear functionals on X. Then for any $x, y \geq 0$ in B_X we have $\|x \vee y\| = \|x\| \vee \|y\| \leq 1$, and so

$$\begin{aligned} x^*(x) + y^*(y) &\leq x^*(x \vee y) + y^*(x \vee y) \\ &= (x^* + y^*)(x \vee y) \\ &\leq \|x^* + y^*\| \|x \vee y\| \\ &\leq \|x^* + y^*\|. \end{aligned}$$

It follows that, if we fix y^* for the moment, whenever $x \geq 0$ is in B_X we have

$$x^*(x) \leq \|x^* + y^*\| - y^*(y).$$

So $\|x^*\| = \sup\{x^*(x) : 0 \leq x \in B_X\} \leq \|x^* + y^*\| - y^*(y)$.

Turnabout is fair play: For each $0 \leq y \in B_X$ we have

$$y^*(y) \leq \|x^* + y^*\| - \|x^*\|,$$

so that

$$\|y^*\| = \sup\{y^*(y) : 0 \leq y \in B_X\} \leq \|x^* + y^*\| - \|x^*\|.$$

We've shown that $\|x^*\| + \|y^*\| \leq \|x^* + y^*\|$; the mere mention of the "triangle inequality" should suffice to believe X^* is an AL-space.

Now suppose X is an AL-space. Define e^* on X^+ by $e^*(x) = \|x\|$. Then e^* is additive and positively homogeneous. (X is an AL-space!) e^* extends to a member of X^* which by its very definition is a positive linear functional on X. Take $x^* \in B_{X^*}$. Then for any $x \in B_X$ with $x \geq 0$ we have

$$x^*(x) \leq \|x^*\| = e^*(x).$$

It follows that $B_{X^*} \subseteq [-e^*, e^*]$. Since $\|e^*\|$ is plainly $= 1, B_{X^*} = [-e^*, e^*]$ and so $X^* = X^*_{e^*}$ is an AM-space with order unit e^*.

In tandem these two paragraphs show (1) and (2), since X is always a closed sublattice of X^{**}. □

Note: If X is an AL-space, then $S_{X^+} = \{x \geq 0 : \|x\| = 1\}$ is a convex set.

LEMMA C.4.2. *Let X be an AL-space. Then x is an extreme point of $S_{X^+} = \{x \in X^+ : \|x\| = 1\}$ if and only if X_x is one-dimensional.*

PROOF. Assume $x \geq 0$ is an extreme point of S_{X^+} and let $0 < y < x$. Then $x - y \geq 0$ and

$$x = y + (x - y)$$

so that

$$1 = \|x\| = \|y\| + \|x - y\| \ (= \lambda + (1 - \lambda), \text{ say}).$$

Therefore,

$$x = \|y\| \left(\frac{y}{\|y\|} \right) + \|x - y\| \left(\frac{x - y}{\|x - y\|} \right)$$

$$= \lambda x_1 + (1 - \lambda) x_2,$$

where $x_1 = \dfrac{y}{\|y\|}$ and $x_2 = \dfrac{x - y}{\|x - y\|}$ are both in S_{X^+}. But x is an extreme point so $x = x_1 = x_2$ and so $x = \dfrac{y}{\|y\|}$, that is, $\dim X_x = 1$.

Conversely, suppose $x \in S_{X^+}$ is such that X_x is one-dimensional. Suppose $0 < \lambda < 1$ and $x = \lambda y + (1 - \lambda) z$ where $y, z \in S_{X^+}$. Then, of course,

$$x = \lambda y + (1 - \lambda) z \geq \lambda y.$$

Since X_x is one-dimensional, there is a $\mu \geq 0$, so

$$\mu x = \lambda y.$$

But $\|x\| = 1 = \|y\|$ and $\lambda, \mu \geq 0$ so $x = y$ and, as a consequence, $x = z$, too. That is, x is an extreme point of S_{X^+}. □

We now apply this lemma to one of the most important concrete examples of an AL-space.

THEOREM C.4.3. *Let K be a compact Hausdorff space. For each $t \in K$, let $\delta_t \in C(K)^*$ be defined by*

$$\delta_t(f) = f(t).$$

(1) *A non-negative $\mu \in C(K)^*$ is an extreme point of $S_{(C(K)^*)^+}$ if and only if μ is a δ_t for some $t \in K$.*
(2) *$\mu \in B_{C(K)^*}$ is an extreme point if and only if $\mu = \pm\delta_t$ for some $t \in K$.*

PROOF. It is easy to see that each δ_t is a lattice homomorphism and so $C(K)^*_{\delta_t}$ is one-dimensional; since $C(K)^*$ is an AL-space, it follows that δ_t is an extreme point of $S_{(C(K)^*)^+}$.

On the other hand, if μ is an extreme point of $S_{(C(K)^*)^+}$, then, $C(K)^*$ being an AL-space, $C(K)^*_\mu$ is one-dimensional and so μ is a lattice homomorphism. We want to show $\mu = \delta_t$ for some $t \in K$ and this will follow from sharing that $\mu \wedge \delta_t \neq 0$ for some $t \in K$; consequently, we assume the contrary: $\mu \wedge \delta_t = 0$ for each $t \in K$. Then testing $\mu \wedge \delta_t$ at 1 given (for each $t \in K$),

$$\begin{aligned} 0 &= (\mu \wedge \delta_t)(1) \\ &= \inf\{\mu(1-g) + \delta_t(g) : 0 \leq g \leq 1\} \\ &= \inf\{1 - \mu(g) + g(t) : 0 \leq g \leq 1\}. \end{aligned}$$

So for each $t \in K$ there is a $g_t \in C$ so $0 \leq g_t \leq 1$ and $0 \leq 1 - \mu(g_t) + g_t(t) < \frac{1}{2}$. It is a quick and easy calculation to see that $\mu(g_t) > \frac{1}{2}$; after all, $1 - \mu(g_t) \leq 1 - \mu(g_t) + g_t(t) < \frac{1}{2}$. It is also easy to see that $g_t(t) < \frac{1}{2}$; indeed, $g_t(t) < \mu(g_t) - \frac{1}{2} \leq 1 - \frac{1}{2} = \frac{1}{2}$.

Define the open sets

$$U(t) = \left[g_t < \frac{1}{2}\right]$$

and notice $t \in U(t)$; hence, $\{U(t) : t \in K\}$ constitutes an open cover of K. K's compactness ensures the existence of a finite subfamily $\{U(t_1), \ldots, U(t_m)\}$ which still covers K. If we let $g = g_{t_1} \wedge \ldots \wedge g_{t_m}$, then for any $x \in K$ we know $x \in U(t_i)$ for some $1 \leq i \leq m$ and so $g(x) \leq g_{t_i}(x) < \frac{1}{2}$. It follows that applying the probability μ to g gives $\mu(g) < \frac{1}{2}$. BUT $\mu(g) = \mu(g_{t_1} \wedge \ldots \wedge g_{t_m})$ is, thanks to μ being a lattice homomorphism, $= \mu(g_{t_1}) \wedge \ldots \wedge \mu(g_{t_m}) > \frac{1}{2}$. OOPS!

Our supposition that $\mu \wedge \delta_t \neq 0$ for each $t \in K$ leads us astray. Deny $\mu \wedge \delta_t \neq 0$ for each $t \in K$. Don't worry. Be happy. ☐

C.5. Kakutani's vector lattice version of the Stone-Weierstrass theorem

Here is an approximation theorem that will play a crucial role in Kakutani's representation of AM-spaces. It is of considerable interest in its own right.

THEOREM C.5.1 (Kakutani (1941b)). *Let V be a vector sublattice of $C(K)$, K a compact Hausdorff space. Suppose V separates the points of K and contains the constants. Then $\overline{V} = C(K)$.*

PROOF. We proceed in three stages.

I. *For every $s, t \in K$ such that $s \neq t$ and $a, b \in \mathbb{R}$ there is an $f \in V$ so that $f(s) = a$ and $f(t) = b$.*

Since V separates the points of K, there is a $g \in V$ such that $g(s) \neq g(t)$. Define f by

$$f(x) = \frac{[ag(x) - ag(t) - bg(x) + bg(s)]}{g(s) - g(t)}.$$

Then $f \in V$, $f(s) = a$ and $f(t) = b$.

II. *If $h \in C(K)$, $\varepsilon > 0$ and $s \in K$, then there is $g_s \in V$ such that $g_s(s) = h(s)$ and for each $t \in K$ we have $g_s(t) > h(t) - \varepsilon$.*

Indeed, I allows us to choose for each $t \in K$ an $f_t \in V$ so that

$$f_t(s) = h(s) \text{ and } f_t(t) = h(t).$$

Let $U(t)$ be the open set $U(t) := [f_t > h - \varepsilon]$; plainly, $t \in U(t)$ for each $t \in K$, so the collection $\{U(t) : t \in K\}$ constitutes an open cover of the compact space K. It follows that there is a finite subfamily $\{U(t_1), \ldots, U(t_m)\}$ which also covers K. If we let $g_s \in V$ be

$$g_s = f_{t_1} \vee \ldots \vee f_{t_m},$$

then not only is $g_s \in V$ but $g_s(s) = h(s)$ and regardless of $t \in K$, as soon as $t \in U(t_i)$,

$$g_s(t) \geq f_{t_i}(t) > h(t) - \varepsilon.$$

III. *Fini*

Let $h \in C(K)$, $\varepsilon > 0$. For each $s \in K$ choose $g_s \in V$ ala II, so

$$g_s(s) = h(s) \text{ and } g_s(t) > h(t) - \varepsilon \text{ for all } t \in K.$$

Now define the open set $W(s)$ by $W(s) = [g_s < h + \varepsilon]$. It is clear that $s \in W(s)$ and so the collection $\{W(s) : s \in K\}$ constitutes an open cover of the compact space K. There is a finite subfamily $\{W(s_1), \ldots, W(s_n)\}$ which still covers K. If we let

$$g = g_{s_1} \wedge \ldots \wedge g_{s_n},$$

then $g \in V$ (each g_{s_j} is), and regardless of $t \in K$,

$$h(t) - \varepsilon < g(t) < h(t) + \varepsilon$$

the left-hand inequality coming through the still-present graces of II and the right-hand by the work of III. The end result: $\|g - h\|_\infty < \varepsilon$, where $g \in V$ is the issue. □

C.6. Kakutani's characterization of AM-spaces with unit

Recall that a Banach lattice X is called an AM-space if for $x, y \in X^+$, $\|x \vee y\| = \|x\| \vee \|y\|$. If B_X has a biggest element u, then u is called an order unit.

We've seen that AM-spaces with unit arise frequently within the theory of Banach lattices: Indeed, whenever X is a Banach lattice and x is a non-zero member of X^+, then X_x is an AM-space with unit x if equipped with $[-x, x]$ as a unit ball. Here's the final word about AM-spaces, practically speaking.

THEOREM C.6.1 (Kakutani (1941b)). *Let X be an AM-space with order unit e. Then the positive face $(S_X^*)^+$ of S_{X^*} is weak* compact and convex, the set K of extreme points of $(S^*)^+$ is weak* compact and X is vector lattice isomorphic and isometric to $C((K,$ weak*$))$ via the operator*

$$X \longrightarrow C((K, \text{weak}^*)) : x \mapsto f_x(\cdot)$$

where $f_x(x^*) = x^*(x)$ for $x^* \in K$.

PROOF. Since X^* is an AL-space, $S^{*+} = S_{X^*} \cap (X^*)^+$ is a convex set; it is also bounded and weak* closed because

$$S^{*+} = (X^*)^+ \cap \{x^* \geq 0 : x^*(e) = 1\}.$$

So S^{*+} is weak* compact convex (and non-empty). It follows from the Krein-Milman Theorem that S^{*+} has extreme points. Again X^*'s being an AL-space, the results of section C.4 tell us that the set K of extreme points of S^{*+} consists precisely of those positive linear functionals x^* of norm-one on X for which $X^*_{x^*}$ is one-dimensional; and so K consists entirely of the norm-one lattice homomorphism of X into $\mathbb{R}$. But this provides us with the means to recognize K as being weak* closed; after all, the pointwise limit of lattice homomorphisms is a lattice homomorphism.

Define $f_x \in C((K, \text{weak}^*))$ as we did above: $f_x(x^*) = x^*(x)$. If $x \in X$, then

$$\begin{aligned}\|x\| &= \sup\{|x^*(x)| : 0 \le x^* \in S_{X^*}\}\\ &= \sup\{|x^*(x)| : x^* \in \text{ext}S^{*+}\}\\ &= \sup\{|f_x(x^*)| : x^* \in K\}\\ &= \|f_x\|_\infty.\end{aligned}$$

The map $x \to f_x$ is a linear isometry.

Now K is not just the set of extreme points of S^{*+}: K consists of lattice homomorphisms! So if $x, y \in X$ and $x^* \in K$, then

$$f_{x\vee y}(x^*) = x^*(x \vee y) = x^*(x) \vee x^*(y) = f_x(x^*) \vee f_y(x^*)$$

and the map $x \to f_x$ *is* as advertised, a vector lattice, isometric isomorphism of X into $C((K, \text{weak}^*))$.

Oh, yes! We also promised $\{f_x : x \in X\}$ would be *all* of $C((K, \text{weak}^*))$, and for this we note that $\{f_x : x \in X\}$ is a closed vector sublattice of C that contains the constant functions ($f_e(x^*) = 1$ all $x^* \in K$) and separates the points of K. So Kakutani's vector lattice form of the Stone-Weierstrass theorem finishes this proof. □

C.7. AL-spaces: The Freudenthal-Kakutani theorem

In this section we make serious headway in the study of AL-spaces; Kakutani's characterization of $L^1(\mu)$'s looms large as a result.

In AL-spaces we have a *monotone convergence theorem*:

PROPOSITION C.7.1. *Let (x_n) be a sequence in an AL-space such that*

$$0 \le x_1 \le x_2 \le \cdots \le x_n \le \cdots \le y$$

for some $y \in X$. Then $x = \lim_n x_n$ exists (in norm) and $0 \le x \le y$.

PROOF. Note that $x_{k+1} - x_k \ge 0$, so by the AL-nature of the beast,

$$\sum_{k=1}^{n-1} \|x_{k+1} - x_k\| = \|\sum_{k=1}^{n-1} x_{k+1} - x_k\| = \|x_n - x_1\| \le \|y - x_1\|.$$

Hence $\sum_{k=1}^{\infty} \|x_{k+1} - x_k\| < \infty$ and so $\lim x_n = x$ exists and it is plain to see that $0 \le x \le y$. □

We say that the norm of a Banach lattice X is *order continuous* if whenever $(x_d)_{d\in D}$ is a monotone non-increasing net with $\wedge_D x_d = 0$, then $\lim_D \|x_d\| = 0$.

PROPOSITION C.7.2. *The norm of any AL-space X is order continuous.*

PROOF. If $(x_d)_{d\in D}$ is a monotone non-increasing net for which $\wedge_D x_d = 0$, but $\lim_D \|x_d\| > 0$, then $(x_d)_{d\in D}$ cannot be a Cauchy net. It then follows that $(x_d)_{d\in D}$ contains a non-increasing sequence which is not convergent, which is contradictory to the Monotone Convergence Theorem. □

A Banach lattice X is *Dedekind complete* if, whenever a nonempty set $A \subseteq X$ is bounded below (in the lattice X), then $\inf A$ exists.

PROPOSITION C.7.3. *Any AL-space X is Dedekind complete.*

PROOF. We may assume, by enlarging A if necessary, that A is directed downward and sits inside X^+. Let L be the subset of lower bounds of A (lying inside X^+) and let $C = A - L = \{a - l : a \in A, l \in L\}$. Then, C too, is directed downward. Further, $\inf C = 0$. Since X has order continuous norm, we can, for each $n \in \mathbb{N}$ find $x_n \in A$, $l_n \in L$, so that $\|x_n - l_n\| < \frac{1}{n}$, $x_{n+1} \leq l_n$, $l_{n+1} \leq l_n$.

Naturally, for any $k, n \in \mathbb{N}$ we have

$$0 \leq x_n - x_{n+k} \leq x_n - l_{n+k} \leq x_n - l_n;$$

(x_n) is a Cauchy sequence, with limit x $(= \lim_n y_n)$, so $x \in L$ and $x = \inf A$. □

An *ideal* in X is a closed linear subspace Y of X such that if $y \in Y$ and $|x| \leq |y|$, then $x \in Y$. An ideal Y is a *band* if whenever $\{y_i : i \in I\} \subseteq Y$ and $\vee_I y_i$ exists in X, then $\vee_I y_i \in Y$.

A positive element e in a Banach lattice is called a *weak order unit* if $e \wedge x = 0$ implies $x = 0$. The natural model is an everywhere positive integrable function in $L^1(\mu)$, where μ is a σ-finite measure.

Much of our early analysis centers around some natural projections. If $x \geq 0$, then we define P_x as follows: Suppose $z \geq 0$ and put

$$P_x(z) = \vee_{\mathbb{N}}(nx \wedge z);$$

to be sure, we assume that X is at the very least Dedekind σ-complete. Since $nx \wedge z \leq z$ for all n, (in case of Dedekind σ-completeness) the definition makes sense; for $y = y^+ - y^- \in X$ we put $P_x(y) = P_x(y^+) - P_x(y^-)$ and linearity follows. Of course, P_x is a positive linear map and, in fact, P_x is a projection: For $z \geq 0$,

$$\begin{aligned} P_x P_x z &= P_x\left(\vee_{\mathbb{N}}(nx \wedge z)\right) \\ &= \vee_m(mx \wedge (\vee_n(nx \wedge z))) \\ &= \vee_m \vee_n (mx \wedge nx) \wedge z \\ &= P_x z. \end{aligned}$$

So P_x *is a positive linear projection of* X *onto* PX. Further, for $x, y \geq 0$, $x \wedge (y - P_x y) = 0$.

Notice that if $e \geq 0$ is a weak order unit in X, then $P_e = id_X$.

A fundamental building block of our analysis is the following:

THEOREM C.7.4 (Kakutani (1941a)). *Any Banach lattice with order continuous norm (in particular, any AL-space) can be decomposed into an unconditional direct sum of a (possibly uncountable) family of mutually disjoint ideals* X_α, *each having a weak order unit* $x_\alpha > 0$.

The decomposition is straightforward: After a bit of harmless "zornication" find a maximal family of mutually disjoint positive elements $\{x_\alpha : \alpha \in A\}$ in X. Let

$$X_\alpha = \{x \in X : \text{ if } x_\alpha \wedge y = 0, \text{ then } |x| \wedge y = 0\};$$

A bit of play soon reveals that X_α is precisely $P_{x_\alpha} X$. Indeed, testing $x \geq 0$, if $x_\alpha \wedge y = 0$, then

$$\begin{aligned} P_{x_\alpha} x \wedge y &= (\vee_n n x_\alpha \wedge x) \wedge y \\ &= \vee_n (n x_\alpha \wedge y \wedge x) \\ &= \vee_n \left(n(x_\alpha \wedge \frac{y}{n})\right) \wedge x \end{aligned}$$

$$= 0 \wedge x$$
$$= 0$$

so $P_{x_\alpha} x \in X_\alpha$. On the other hand, if $x \in X_\alpha^+$, then $x_\alpha \wedge (x - P_{x_\alpha} x) = 0$ so $0 = x \wedge (x - P_{x_\alpha} x)$, which since $x - P_{x_\alpha} x \leq x$, says that $x = P_{x_\alpha} x \in P_{X_\alpha} X$.

Naturally, we've set things up so that x_α is a weak unit for X_α.

Fix $y \geq 0$ and look at $\{P_{x_\alpha} y : \alpha \in A\}$. For any sequence (α_n) in A, the series $\sum_n P_{x_{\alpha_n}}(y)$ converges to $\vee_n P_{x_{\alpha_n}}(y)$ and so only countably many of the $P_{x_\alpha}(y)$'s are non-zero and $\sum_{\alpha \in A} P_{x_{\alpha_n}}(y)$ converges unconditionally to a member y_0 of X. Since $\sum_{\alpha \in A} P_{x_\alpha}(y)$ is just $\vee_{\alpha \in A} P_{x_\alpha}(y)$ and each term is $\leq y$, $y_0 \leq y$. But should $y - y_0 > 0$, then the maximality of the family $\{x_\alpha : \alpha \in A\}$ would produce an $\hat{\alpha} \in A$ so that $x_{\hat{\alpha}} \wedge (y - y_0) \neq 0$. But then

$$0 < x_{\hat{\alpha}} \wedge (y - y_0) \leq x_{\hat{\alpha}} \wedge (y - P_{x_{\hat{\alpha}}} y) = 0.$$

It follows that $y = y_0$ and the unconditional decomposition has been established.

The X_α's in the above theorem are even bands in X. Moreover, if $X_\alpha^\perp = \{x \in X : |x| \perp |y| \text{ for each } y \in X_\alpha\}$, then $X = X_\alpha \oplus X_\alpha^\perp$, so each of the X_α's is a "projection band" and, of course, inside X_α, x_α is a weak order unit.

Henceforth, we'll concentrate on the pieces $\{X_\alpha : \alpha \in A\}$ of X. We need some notation and notions. A family $\mathcal{B}$ of commuting bounded linear projections on a Banach space X is called a *Boolean algebra of projections* if $P, Q \in \mathcal{B}$ ensures PQ and $P + Q - PQ$ are each in $\mathcal{B}$ as well. If $\mathcal{B}$ is a Boolean algebra of projections, we can make $\mathcal{B}$ a lattice if we set $P \wedge Q = PQ$ and $P \vee Q = P + Q - PQ$. The Boolean algebra $\mathcal{B}$ of projections is *σ-complete* if $\vee_n P_n, \wedge_n P_n$ both exist in $\mathcal{B}$ for any $(P_n) \subseteq \mathcal{B}$ and

$$(\vee_n P_n)(X) = \overline{\text{span}(\vee_n P_n X)}, \qquad \wedge_n P_n X = (\wedge_n P_n)(X).$$

For a Boolean algebra $\mathcal{B}$ of projections on X, if $x \in X$, then

$$\mathcal{N}(x) = \overline{\text{span}\{Px : P \in \mathcal{B}\}}$$

is called the cyclic space generated by x.

The main result of this section is the following:

THEOREM C.7.5 (Freudenthal (1936), [Kakutani (1941a)]). *Let X be a Dedekind complete Banach lattice with order continuous norm and weak order unit e. Then the family $\{P_x : x \geq 0\}$ is a σ-complete Boolean algebra of projections and $X = \mathcal{N}(e)$ is a cyclic space with respect to this Boolean algebra.*

One can easily verify that

(a) $P_e = id_X$;
(b) if $x, y \geq 0$, then $P_x P_y = P_{x \wedge y}$;
(c) if $x \geq y \geq 0$, then $P_x - P_y = P_{x - P_y(x)}$;
(d) the σ-completeness of this Boolean algebra follows from X's order completeness and the order-continuity of its norm.

To prove $X = \mathcal{N}(e)$, fix $x \geq 0$ and for real $\lambda \geq 0$ put

$$x(\lambda) = P_{(\lambda e - x)^+}(e).$$

If $0 \leq \lambda \leq \eta$, then $(\lambda e - x)^+ \leq (\eta e - x)^+$, and so

$$\begin{aligned} x(\lambda) = P_{(\lambda e - x)^+}(e) &= \vee_n (n(\lambda e - x)^+ \wedge e) \\ &\leq \vee_n (n(\eta e - x)^+ \wedge e) = P_{(\eta e - x)^+}(e) = x(\eta). \end{aligned}$$

As a result $x : [0, \infty) \to X^+$ is a monotone non-decreasing function. We can mimic the Riemann-Stieltjes construction and talk about

$$\int_0^\infty \lambda dx(\lambda).$$

This integral is the limit in norm — if such exists — of sums of the form

$$\sum_{i \leq n} \lambda_i (x(\lambda_i) - x(\lambda_{i-1})).$$

We propose to show that

$$x = \int_0^\infty \lambda dx(\lambda),$$

which will show that $x \in \mathcal{N}(e)$ and so X^+, and with it all of $X \subseteq \mathcal{N}(e)$ as we wanted.

First a small aid.

LEMMA C.7.6. *Suppose $0 \leq z \in X$ satisfies $z = P_z(e)$; so $z \wedge (e - z) = 0$. Then whenever $z \leq x(\lambda)$, we have $P_z(x) \leq \lambda z$ and whenever $z \leq e - x(\lambda)$, we have $\lambda z \leq P_z(x)$.*

PROOF. $z \leq x(\lambda) = P_{(\lambda e - x)^+}(e)$ says that

$$\begin{aligned} 0 \leq (x - \lambda e)^+ \wedge z &\leq (x - \lambda e)^+ \wedge x(\lambda) \\ &= (x - \lambda e)^+ \wedge P_{(\lambda e - x)^+}(e) \\ &= (x - \lambda e)^+ \wedge (\vee_n ((n(\lambda e - x)^+ \wedge e))) \\ &= \vee_n n(\lambda e - x)^+ \wedge e \wedge (x - \lambda e)^+ \\ &= \vee_n n(\lambda e - x)^+ \wedge (x - \lambda e)^+ \wedge e = 0, \end{aligned}$$

since $(\lambda e - x)^+ \wedge (x - \lambda e)^+ = 0$. But then

$$(x - \lambda e)^+ \wedge z = 0,$$

and so

$$(x - \lambda e)^+ \wedge 2z = \left(\frac{x - \lambda e}{2}\right)^+ \wedge z + \frac{(x - \lambda e)^+}{2} \wedge z = 0 + 0 = 0$$

and generally

$$(x - \lambda e)^+ \wedge nz = 0,$$

so that

$$P_z((x - \lambda e)^+) = 0.$$

Since $x \leq (x - \lambda e)^+ + \lambda e$ (after all, $(x - \lambda e)^+ = (x - \lambda e) \vee 0$, so $(x - \lambda e)^+ + \lambda e = x \vee \lambda e \geq x$), we see that

$$\begin{aligned} P_z(x) &\leq P_z\big((x - \lambda e)^+ + \lambda e\big) \\ &= P_z((x - \lambda e)^+) + \lambda P_z(e) \\ &= 0 + \lambda P_z(e) \\ &= \lambda z. \end{aligned}$$

The other fact mentioned in the lemma is proved similarly. □

Now on to the proof that $x = \int_0^\infty \lambda dx(\lambda)$.

Let $\varepsilon > 0$ be given.

Let $0 < \Lambda < \infty$ and partition $[0, \Lambda]$ into a partition $\pi : 0 = \lambda_0 < \lambda_1 < \cdots < \lambda_n = \Lambda$ where the mesh of π, $\|\pi\| = \max_{0 \le i \le n-1}(\lambda_{i+1} - \lambda_i)$ is small, say $< \varepsilon$. Let

$$z_i = x(\lambda_i) - x(\lambda_{i-1}) \text{ for } i = 1, 2, \ldots, n.$$

Notice that

$$z_i = P_{z_i}(e)$$

and

$$z_i \le e - x(\lambda_{i-1}).$$

Since $z_i \wedge z_j = 0$ if $i \ne j$ we have the lower sum

$$\begin{aligned} s(\pi) &= \sum_{i=1}^{n} \lambda_{i-1}(x(\lambda_i) - x(\lambda_{i-1})) \\ &= \sum_{i=1}^{n} \lambda_{i-1} z_i \le \sum_{i=1}^{n} P_{z_i}(x) = P_{\sum_{i=1}^n z_i}(x) = P_{x(\Lambda)}(x) \end{aligned}$$

by the second part of our lemma. On the other hand, $z_i \le x(\lambda_i)$, so the first part of our lemma tells us that

$$P_{x(\Lambda)}(x) = \sum_{i=1}^{n} P_{z_i}(x) \le \sum_{i=1}^{n} \lambda_i z_i = S(\pi),$$

the upper sum. Hence

$$\|S(\pi) - s(\pi)\| \le \varepsilon \|e\|$$

which, if we let $\varepsilon \to 0$, gives us

$$\int_0^\Lambda \lambda dx(\lambda) = P_{x(\Lambda)}(x).$$

But $x(\Lambda) = P_{(\Lambda e - x)^+}(e) \le e$ and X is Dedekind complete with order continuous norm and so, since $x(\Lambda) = P_{(\Lambda e - x)^+}(e) \le e$,

$$x(\infty) = \lim_{\Lambda \to \infty} x(\Lambda) = \sup_{\lambda} x(\lambda)$$

exists in X.

Now

$$e - x(\infty) = P_{e - x(\infty)}(e) \text{ and } e - x(\infty) \le e - x(\Lambda)$$

for any Λ. By the second part of our lemma we have

$$\Lambda(e - x(\infty)) \le P_{e - x(\infty)}(x).$$

Hence

$$\Lambda \|e - x(\infty)\| \le \|P_{e - x(\infty)}(x)\|$$

and

$$\|e - x(\infty)\| \le \frac{\|P_{e - x(\infty)}(x)\|}{\Lambda} \to 0 \text{ as } \Lambda \to \infty.$$

$x(\infty) = e$ follows. But now the σ-completeness of the Boolean algebra $\{P_x : x \ge 0\}$ of projections comes into play to tell us that

$$\lim_{\Lambda \to \infty} P_{x(\Lambda)}(x) = P_e(x) = x$$

and from this

$$\int_0^\infty \lambda dx(\lambda) = x$$

follows.

What we've shown is the following:

THEOREM C.7.7 (Spectral theorem of Freudenthal (1936) and Kakutani (1941a)). *Let X be a Dedekind complete Banach lattice with an order continuous norm and a weak order unit e. Then the Boolean algebra of commuting projections $\{P_x : x \geq 0\}$ is σ-complete and if $x \in X$ with $x \geq 0$, then defining $x(\lambda) = P_{(\lambda e - x)^+}(e)$ we have that $x = \int_0^\infty \lambda dx(\lambda)$, where this integral is interpreted in the classical Riemann-Stieltjes way.*

C.8. Kakutani's characterization of AL-spaces

THEOREM C.8.1 (Kakutani). *Let X be an AL-space with weak unit. Then X is Banach lattice isometric to an $L^1(\mu)$-space for some finite measure μ.*

X is order complete and its norm is order continuous. We'll assume that X's weak unit $e > 0$ has $\|e\| = 1$. Consider the Boolean algebra $B = \{P_x : x \geq 0\}$ of projections on X and realize that $X = \mathcal{N}(e)$. We'll find an isometry of X to an $L^1(\mu)$ so that e is carried onto the constant function 1.

By Stone's Representation Theorem, the Boolean algebra B (by the way, complementation in B is given by $P'_x = P_{e-(x\wedge e)})$ can be identified with the Boolean algebra A of all clopen sets of some compact Hausdorff, totally disconnected space S. If we now set $\mu(P_x) = \|P_x(e)\|$ for $x \geq 0$, we obtain a finitely additive measure on (S, A). But such a measure is plainly countable additive thereupon! After all, any member of A that is a countable union of a pairwise disjoint sequence (A_n) of members of A is surprised to find that but for finitely many n's, $A_n = \phi$; this follows from the compact/open nature of members of A. Caratheodory lets us extend μ to the σ-field Σ of subsets of S generated by A in a countably additive (real-valued) fashion. Here's how we take X to $L^1(\mu)$: Let $x_1, \ldots, x_n \in X^+$ be disjoint ($x_i \wedge x_j = 0$ if $i \neq j$) and $a_1, \ldots, a_n \in \mathbb{R}$, define $T(\sum_{i\leq n} a_i P_{x_i}(e)) = \sum a_i \chi_{P_{x_i}}$. Then

$$\|\sum a_i P_{x_i}(e)\| = \sum |a_i| \|P_{x_i}(e)\| = \sum |a_i| \mu(P_{x_i})$$

and T is an isometry from

$$\{\sum_{i\leq n} a_i P_{x_i}(e) : a_1, \ldots, a_n \in \mathbb{R},\ x_1, \ldots, x_n \in X^*,\ x_i \wedge x_j = 0 \text{ if } i \neq j\}$$

into $L^1(\mu)$. But $X = \mathcal{N}(e)$ so the domain of T is dense in X. On the other hand, the Caratheodory procedure ensures that span $\{\chi_A : A \in \mathcal{N}\}$ is dense in $L^1(\mu)$, too. DONE DEAL.

In tandem with Kakutani's Theorem C.7.4, Theorem C.8.1 has the following crisp version.

COROLLARY C.8.2. *Let X be an AL-space. Then there is a measure μ such that X is Banach lattice isometrically isomorphic to $L^1(\mu)$.*

PROOF. By Kakutani's Theorem C.7.4, X is an unconditional sum of a family $\{X_i : i \in I\}$ of bands each of which has an element that serves as a weak order unit. On the one hand, Theorem C.8.1 tells us that each X_i must be an $L^1(\mu_i)$, for some measure μ_i; on the other hand, since X is an AL-space, the unconditionality is just an ℓ^1-sum. So X is Banach lattice isometrically isomorphic to the space $(\bigoplus X_i)_{\ell^1} = (\bigoplus L^1(\mu_i))_{\ell^1} = L^1((\bigoplus \mu_i)_{\ell^1})$, where $\mu = (\bigoplus \mu_i)_{\ell^1}$ is the measure defined on the disjoint union of the domains of μ_i in the most natural manner imaginable. □

Note: The μ of Corollary 2 enjoys the added property that $L^1(\mu)^* = L^\infty(\mu)$.

C.9. Grothendieck's inequality for Banach lattices

Kakutani's representation theorems allow us to develop a beautiful functional calculus in Banach lattices, a calculus that will let us make sense of quantities like

$$\Big(\sum_{k\le n} |x_n|^p\Big)^{\frac{1}{p}}$$

where p is a real number > 1. This in hand we can formulate, and prove, a natural generalization of Grothendieck's inequality in Banach lattices.

Let L be a Banach lattice.

For any $n \in \mathbb{N}$, we'll denote by $\mathcal{C}_n$, the family of all real-valued functions on $\mathbb{R}^n$ which are obtained from the coordinate functions

$$(t_1, \dots, t_n) \to t_k,\ 1 \le k \le n,$$

by finitely many operations of addition, multiplication by a real number and taking suprema and infima of finitely many real functions.

Note that if $h, h' \in \mathcal{C}_n$ and

$$h(t_1, \dots, t_n) = h'(t_1, \dots, t_n), \text{ for all } (t_1, \dots, t_n) \in \mathbb{R}_n,$$

then

$$h(x_1, \dots, x_n) = h'(x_1, \dots, x_n), \text{ for all } x_1, \dots, x_n \in L.$$

Indeed, temporarily fix $x_1, \dots, x_n \in L$ and look at $x = |x_1| \vee \cdots \vee |x_n| \in L$. Since both $h, h' \in \mathcal{C}_n$ it follows that $h(x_1, \dots, x_n)$ and $h'(x_1, \dots, x_n)$ belong to $\mathcal{I}(x)$, the ideal generated by x. Of course $\mathcal{I}(x)$ is an M-space with unit x and so thanks to Kakutani, we have that $\mathcal{I}(x)$ is isomorphic (as a Banach lattice) to $C(K)$ for some compact K. But the algebraic and order relations in $C(K)$ are defined pointwise! So having

$$h(t_1, \dots, t_n) = h'(t_1, \dots, t_n), \text{ for all } (t_1, \dots, t_n) \in \mathbb{R}_n,$$

ensures that for all $x_1, \dots, x_n (\in L)$ we have

$$h(x_1, \dots, x_n) = h'(x_1, \dots, x_n)$$

in $\mathcal{I}(x)$ (aka $C(K)$) and also in L.

Hence we have a well-defined map

$$j : \mathcal{C}_n \longrightarrow L$$

that takes $h \in \mathcal{C}_n$ to $h(x_1, \dots, x_n)$. Since $\mathcal{C}_n$ is plainly a vector lattice, it makes sense (and is so) that j is linear and preserves order.

Now we bootstrap and approximate functions like

$$(t_1, \dots, t_n) \to \Big(\sum_{k\le n} |t_k|^p\Big)^{\frac{1}{p}}$$

by members of $\mathcal{C}_n$. Here a call will be made on Kakutani's vector version of the Stone-Weierstrass theorem (Theorem C.5.1).

Let $\mathcal{H}_n$ denote the space of all continuous real-valued functions f on $\mathbb{R}^n$ that are positively homogeneous of order 1, that is, that satisfy

$$f(\lambda t_1, \ldots, \lambda t_n) = \lambda f(t_1, \ldots, t_n) \text{ for all } (t_1, \ldots, t_n) \in \mathbb{R}_n, \text{ and all } \lambda \geq 0.$$

It is plain that $\mathcal{H}_n$ is a vector lattice; in fact, $\mathcal{H}_n$ is a different model of $C(S_{\ell_n^\infty})$. Now $\mathcal{C}_n$ is a sublattice of $\mathcal{H}_n$. Moreover, $\mathcal{C}_n$ separates the points of $S_{\ell_n^\infty}$ and contains the constants.

Hence $\mathcal{C}_n$ is dense in $\big(C(S_{\ell_n^\infty}), \|\cdot\|_\infty\big)$.

Now things fall into place. The map $j : \mathcal{C}_n \to L$ is continuous and linear on $\mathcal{C}_n$ (viewed as sitting in $C(S_{\ell_n^\infty})$) and so extends uniquely to a continuous linear operator, that we'll still call j, from $C(S_{\ell_n^\infty})$ $(= \mathcal{H}_n)$ to L which is order preserving.

It is natural to denote $j(h)$ by $h(x_1, \ldots, x_n)$ and realize that the procedure just described takes $h \in \mathcal{H}_n$ and, with $x_1, \ldots, x_n$ in hand, assigns the element $h(x_1, \ldots, x_n) \in L$ in an unambiguous manner.

This is what we refer to as the *functional calculus in L*.

We make quick mention of a few more or less standard facts that indicate the power inherent in this functional calculus. The proofs are quite direct and painless. For details we refer to [Diestel, Jarchow, and Tonge (1995), pp. 328-329].

PROPOSITION C.9.1. *Let $x_1, \ldots, x_n$ be members of the Banach lattice L.*

(a) *if $1 \leq p < \infty$, then*

$$\Big(\sum_{k\leq n} |x_n|^p\Big)^{\frac{1}{p}} = \sup\Big\{\sum_{k\leq n} a_k x_k : a = (a_1, \ldots, a_n) \in B_{\ell_n^{p'}}\Big\}$$

where p' is the index conjugate to p;

(b) $\sup_{k\leq n} |x_k| = \sup\Big\{\sum_{k\leq n} a_k x_k : a = (a_1, \ldots, a_n) \in B_{\ell_n^1}\Big\}$;

(c) *for any $1 \leq p < \infty$, there are constants $A_p, B_p > 0$ so that*

$$A_p\Big(\sum_{k\leq n} |x_k|^2\Big)^{\frac{1}{2}} \leq \Big(\int_0^1 \Big|\sum_{k\leq n} r_k(t) x_k\Big|^p\Big)^{\frac{1}{p}} \leq B_p\Big(\sum_{k\leq n} |x_k|^2\Big)^{\frac{1}{2}},$$

where $r_k(\cdot)$ denotes the k-th Rademacher function.

This version of the functional calculus is closest to the development given by Krivine (1974); a slightly different version that ends up with the same result was developed by Carne (1980). Either approach allows us to formulate and prove a generalization of Grothendieck's inequality.

Here then is Grothendieck's inequality as it appears in a Banach lattice setting.

THEOREM C.9.2 (Krivine (1978), Krivine (1979), Carne (1980)). *Let X and Y be Banach lattices and $T : X \to Y$ be a bounded linear operator. Then for any $x_1, \ldots x_n, \in X$ we have*

$$\Big\|\Big(\sum_{k\leq n} |Tx_k|^2\Big)^{\frac{1}{2}}\Big\| \leq K_G \|T\| \Big\|\Big(\sum_{k\leq n} |x_k|^2\Big)^{\frac{1}{2}}\Big\|.$$

PROOF. We'll take three giant steps.

Step 1. Suppose $T : \ell_m^\infty \to \ell_m^1$. Let $x_1, \ldots, x_n \in \ell_m^\infty$ with $x_i = (a_{i1}, \ldots, a_{im})$ for $i = 1, 2, \ldots, n$. Let $(\alpha_{kj})_{1\leq k,j\leq m}$ be the matrix representing T. We'll want to estimate quantities of the form $\left\|\left(\sum_{i\le n}|z_i|^2\right)^{\frac12}\right\|_p$ where $z_i\in\mathbb{R}_m$. Of course, the vector $\left(\sum_{i\le n}|z_i|^2\right)^{\frac12}$ is easy to recognize: If $z_i=(z_{i1},z_{i2},\dots,z_{im})$, then

$$((\sum_{i\le n}|z_{i1}|^2)^{\frac12},(\sum_{i\le n}|z_{i2}|^2)^{\frac12},\dots,(\sum_{i\le n}|z_{im}|^2)^{\frac12})=(\sum_{i\le n}|z_i|^2)^{\frac12}.$$

So,

$$\left\|\left(\sum|z_i|^2\right)^{\frac12}\right\|_p=\begin{cases}\left(\sum_{j\le m}(\sum_{i\le n}|z_{ij}|^2)^{\frac p2}\right)^{\frac1p} & \text{if } 1\le p<\infty,\\ \sup_{j\le m}\left(\sum_{i\le n}|z_{ij}|^2\right)^{\frac12} & \text{if } p=\infty.\end{cases}$$

In other words, $\|(\sum_{i\le n}|z_i|^2)^{\frac12}\|_p$ is just the norm of the vector

$$((z_{11},\dots,z_{n1},0,0\dots),(z_{12},\dots,z_{n2},0,0,\dots),\dots,(z_{1m},\dots,z_{nm},0,0,\dots))$$

in $\ell^p_m(\ell^2)$. This in mind we can appeal to the duality between $\ell^1_m(\ell^2)$ and $\ell^\infty_m(\ell^2)$ and choose vectors $y_1,\dots,y_n\in\ell^\infty_m=(\ell^1_m)^*$ so that

$$\left\|\left(\sum_{i\le n}|y_i|^2\right)^{\frac12}\right\|_\infty=1 \text{ and } \left\|\left(\sum_{i\le n}|Tx_i|^2\right)^{\frac12}\right\|_1=\sum_{i\le n}y_i(Tx_i).$$

Suppose $y_i=(b_{i1},\dots,b_{im})$.

Let's look at the vectors $u_k=(a_{1k},\dots,a_{nk})$ and $v_k=(b_{1k},\dots,b_{nk})$ in ℓ^2_n. Direct computation shows

$$\begin{aligned}\left\|\left(\sum_{i\le n}|Tx_i|^2\right)^{\frac12}\right\|_1&=\sum_{i\le n}y_i(Tx_i)\\&=\sum_{i\le n}\sum_{1\le k,j\le m}\alpha_{kj}a_{ik}b_{ij}\\&=\sum_{1\le k,j\le m}\alpha_{kj}(u_k,v_j).\end{aligned}$$

Of course, computations similar to those above show

$$\max_{1\le k\le m}\|u_k\|_2=\left\|\left(\sum_{i\le n}|x_i|^2\right)^{\frac12}\right\|_\infty$$

and

$$\begin{aligned}\max_{1\le k\le m}\|v_k\|_2&=\left\|\left(\sum_{i\le n}|y_i|^2\right)^{\frac12}\right\|_\infty\\&=1.\end{aligned}$$

If we now just apply Grothendieck's inequality to the matrix $\left(\frac{\alpha_{kj}}{\|T\|}\right)$ we see that

$$\begin{aligned}\frac{1}{\|T\|}\left\|\left(\sum_{i\le n}|Tx_i|^2\right)^{\frac12}\right\|_1&=\sum_{k,j}\frac{\alpha_{kj}}{\|T\|}(u_k,v_j)\\&\le K_G\max_{1\le k\le m}\|u_k\|\max_{1\le k\le m}\|v_k\|\\&\le K_G\left\|\left(\sum_{i\le n}|x_i|^2\right)^{\frac12}\right\|_\infty.\end{aligned}$$

Step 2. Now let $T : C(K) \to L_1(\mu)$ be a bounded linear operator and $f_1, \dots, f_n \in C(K)$ be given.

For a fixed $\varepsilon > 0$ there is a partition of unity $\phi_1, \dots, \phi_m$ and $\tilde{f}_1(\varepsilon), \dots, \tilde{f}_n(\varepsilon)$ in $[\phi_j : j \le m]$ such that for $i = 1, \dots, n$,

$$\|f_i - \tilde{f}_i(\varepsilon)\| \le \varepsilon \|f_i\|.$$

Pick $\psi_1, \dots, \psi_m$ simple functions in $L_1(\mu)$ so that for $j = 1, \dots, m$,

$$\|T\phi_j - \psi_j\|_1 \le \frac{\varepsilon}{m}.$$

The ψ_j's lie in an isometric and lattice isomorphic copy of ℓ_1^k inside $L_1(\mu)$ and the operator $T_\varepsilon : [\phi_j] \to [\psi_j]$ defined by $T_\varepsilon \phi_j = \psi_j$ enjoys

$$\left\|T_\varepsilon - T\mid_{[\phi_j : j \le m]}\right\| \le \varepsilon$$

by our choice of ψ_j and the fact that $\phi_1, \dots, \phi_m$ is the unit vector basis of

$$[\phi_j : j \le m]$$

which is isometric and lattice isomorphic to ℓ_∞^m *inside* $C(K)$.

$$T_\varepsilon : [\phi_j : j \le m] \to [\psi_j : j \le m] \subseteq \ell_1^k.$$

Hence, by Step 1,

$$\Big\|\Big(\sum_{i \le n} |T_\varepsilon(\tilde{f}_i(\varepsilon))|^2\Big)^{\frac{1}{2}}\Big\| \le K_G \|T_\varepsilon\| \Big\|\Big(\sum_{i \le n} |\tilde{f}_i(\varepsilon)|^2\Big)^{\frac{1}{2}}\Big\|.$$

Now the idea is, as almost always, to let $\varepsilon \to 0$ and see what remains when the dust clears.

To start,

$$\begin{aligned} \|T_\varepsilon\| &\le \left\|T\mid_{[\phi_j : j \le m]}\right\| + \left\|T_\varepsilon - T\mid_{[\phi_j : j \le m]}\right\| \\ &\le \|T\| + \varepsilon, \end{aligned}$$

and

$$\left|f_i - \tilde{f}_i(\varepsilon)\right| \le \varepsilon \|f_i\|_\infty,$$

so

$$\begin{aligned} \Big(\sum_{i \le n} \left|\tilde{f}_i(\varepsilon)\right|^2\Big)^{\frac{1}{2}} &\le \Big(\sum_{i \le n} \left|\tilde{f}_i(\varepsilon) - f_i\right|^2\Big)^{\frac{1}{2}} + \Big(\sum_{i \le n} |f_i|^2\Big)^{\frac{1}{2}} \\ &\le \varepsilon \Big(\sum_{i \le n} \|f_i\|^2\Big)^{\frac{1}{2}} + \Big(\sum_{i \le n} |f_i|^2\Big)^{\frac{1}{2}}. \end{aligned}$$

So

$$\Big\|\Big(\sum_{i \le n} \left|T_\varepsilon \tilde{f}_i(\varepsilon)\right|^2\Big)^{\frac{1}{2}}\Big\| \le K_G (\|T\| + \varepsilon)\Big(\Big\|\Big(\Big(\sum_{i \le n} |f_i|^2\Big)^{\frac{1}{2}}\Big)\Big\| + \varepsilon \Big(\sum_{i \le n} \|f_i\|^2\Big)^{\frac{1}{2}}\Big).$$

To get control of the left-hand side, notice that for $i = 1, \dots, n$

$$\lim_{\varepsilon \to 0} T_\varepsilon \tilde{f}_i(\varepsilon) = T f_i \text{ (in } L_1(\mu)).$$

So by F. Riesz's good graces we can find a sequence $\varepsilon_k \searrow 0$ so that for $i = 1, \dots, n$,

$$\lim_k T_{\varepsilon_k} \tilde{f}_i(\varepsilon_k) = T f_i \ \mu - \text{a.e.}$$

But now — and here's where it's important that we're in $L_1(\mu)$ —

$$\lim_k \Big(\sum_{i\le n} \big|T_{\varepsilon_k}\tilde{f}_i(\varepsilon_k)\big|^2\Big)^{\frac12} = \Big(\sum_{i\le n} |Tf_i|^2\Big)^{\frac12} \ \mu - \text{a.e.}$$

Fatou's lemma tells us

$$\begin{aligned}\Big\|\Big(\sum_{i\le n}|Tf_i|^2\Big)^{\frac12}\Big\|_1 &\le \liminf_{k\to\infty}\Big\|\Big(\sum_{i\le n}\big|T_{\varepsilon_k}\tilde{f}_i(\varepsilon_k)\big|^2\Big)^{\frac12}\Big\|_1\\ &\le \liminf_{k\to\infty} K_G\big(\|T\|+\varepsilon_k\big)\Big(\Big\|\Big(\sum_{i\le n}|f_i|^2\Big)^{\frac12}\Big\| + \varepsilon_k\Big(\sum_{i\le n}\|f_i\|^2\Big)^{\frac12}\Big)\\ &= K_G\|T\|\Big\|\Big(\sum_{i\le n}|f_i|^2\Big)^{\frac12}\Big\|.\end{aligned}$$

Note: Above we use Riesz's theorem that if f is the L^1-limit of (f_n), then there is a subsequence (g_n) of (f_n) that converges to f almost everywhere. This is plainly so in finite measure spaces, hence, in σ-finite measure spaces and, since sequences are involved, in any measure space.

Step 3. Let T be any bounded linear operator from X to Y. Take $x_1,\dots,x_n \in X$. Put $x_0 = \big(\sum_{i\le n}|x_i|^2\big)^{\frac12}$ and $y_0 = \big(\sum_{i\le n}|Tx_i|^2\big)^{\frac12}$. Let $\mathcal{I}(x_0)$ be the ideal generated by x_0; so $x \in \mathcal{I}(x_0)$ means there is a $\lambda > 0$ so that $|x| \le \lambda x_0/\|x_o\|$. Look at the order interval $\big[-\frac{x_0}{\|x_0\|}, \frac{x_o}{\|x_o\|}\big]$ and let $|||\cdot|||$ be the gauge functional of $\big[-\frac{x_0}{\|x_0\|}, \frac{x_0}{\|x_o\|}\big]$. $(\mathcal{I}(x_0), |||\cdot|||)$ is a Banach lattice with an order unit $\frac{x_o}{\|x_o\|}$; in fact, $(\mathcal{I}(x_0), |||\cdot|||)$ is an abstract M-space. Kakutani warns us that $\big(\mathcal{I}(x_0), |||\cdot|||\big)$ is isometrically isomorphic as a Banach lattice to a space $C(K)$.

Next, let $y_o^* \in Y^{*+}$ satisfy $y_0^* \in S_{Y^*}$ and

$$y_0^*(y_0) = \|y_0\|.$$

For $y \in Y$, define $\|y\|_1$ by

$$\|y\|_1 = y_0^*(|y|).$$

Notice that $\|\cdot\|_1$ is a seminorm on Y. If we factor out those $z \in Y$ such that $\|z\|_1 = 0$ and, then complete the resulting quotient normed lattice, then the completion Y_1 is an abstract L-space and so, with another call on Kakutani for his own particular brand of magic, we find Y_1 to be isometrically isomorphic, as a Banach lattice, to an $L^1(\mu)$-space for some measure μ.

Let $J_\infty : \big(\mathcal{I}(x_0), |||\cdot|||\big) \hookrightarrow X$ and $J_1 : Y \to Y_1$ be the natural maps defined implicitly by the above machinations. $J_1TJ_\infty : \mathcal{I}(x_0) \to Y_1$ is a bounded linear operator from the $C(K)$-space $\big(\mathcal{I}(x_0), |||\cdot|||\big)$ into the L^1-space Y_1 with $\|J_1TJ_\infty\| \le \|T\|$. By Step 2,

$$\Big\|\Big(\sum_{i\le n}|J_1TJ_\infty(x_i)|^2\Big)^{\frac12}\Big\|_1 \le K_G\|T\|\Big\|\Big(\sum_{i\le n}|x_i|^2\Big)^{\frac12}\Big\|_\infty.$$

But

$$\Big\|\Big(\sum_{i\le n}|x_i|^2\Big)^{\frac12}\Big\|_\infty = \|x_0\|_\infty = \|x_0\|$$

and

$$\Big\|\Big(\sum_{i\le n}|J_1TJ_\infty(x_i)|^2\Big)^{\frac12}\Big\|_1 = \|y_0\|_1 = y_0^*y_0 = \|y_0\|;$$

hence

$$\Big\|\Big(\sum_{i\le n}|Tx_i|^2\Big)^{\frac{1}{2}}\Big\|_Y = \|y_0\| \le K_G\|T\|\|x_0\| \le K_G\|T\|\Big\|\Big(\sum_{i\le n}|x_i|^2\Big)^{\frac{1}{2}}\Big\|.$$

□

Notes and remarks. There are many outstanding references to be consulted when studying Banach lattices. We mention but a few: *Riesz spaces, I* by Luxemburg and Zaanen (1971), *Riesz spaces, II* by Zaanen (1983), *Banach lattices* by Meyer-Nieberg (1991), *Banach lattices and positive operators* by Schaefer (1974), *Classical Banach Spaces I* by Lindenstrauss and Tzafriri (1977), *Classical Banach Spaces II* by Lindenstrauss and Tzafriri (1979), as well as the survey paper by Buhvalov, Veksler, and Lozanovskiĭ (1979). We also often found ourselves referring to the original papers of Kakutani: [Kakutani (1941a)] and [Kakutani (1941b)].

APPENDIX D

Stonean spaces and injectivity

D.1. The Nakano Stone Theorem

A compact Hausdorff space S is called *Stonean* (or *extremally disconnected*) if the closure $\bar{U}$ of any open set U in S is open. Alternatively, S is *Stonean* if given disjoint open sets U and V in S, $\bar{U}$ and $\bar{V}$ are disjoint, too.

These spaces, isolated independently by Nakano (1941) and Stone (1949), are central to the study of Banach spaces and enjoy many spectacular properties.

A hint at the importance of Stonean spaces in Banach space theory is contained in the following now-classical result.

THEOREM D.1.1 (Nakano (1941), Stone (1949)). *Let S be a compact Hausdorff space. S is Stonean if and only if $C(S)$, the space of continuous real-valued functions defined on S, is a Dedekind complete lattice.*

PROOF. Suppose $C(S)$ is a Dedekind complete lattice and let V be an open subset of S. By Urysohn's lemma we can find a family $\{f_\alpha\}$ of continuous real-valued functions on S with $0 \leq f_\alpha \leq 1$, each f_α vanishing outside of V and such that for each $s \in S$,

$$\sup_\alpha f_\alpha(s) = \chi_V(s).$$

Let f_0 be the supremum of f_α's in $C(S)$:

$$f_0 = \vee_\alpha f_\alpha.$$

Then $f_0(s) = 1$ if $s \in V$ since some $f_\alpha(s) = 1$ for such s. It follows that $f_0(s) = 1$ if $s \in \bar{V}$. However, if $s_0 \notin \bar{V}$, then there is a $g \in C(S)$ such that $0 \leq g \leq 1, g(s_0) = 0$ and $g(u) = 1$ for all $u \in \bar{V}$. It follows that g is an upper bound for $\{f_\alpha\}$ and so $f_0 \leq g$. From this we see that $f_0(s_0) = 0$ and so $f_0 = \chi_{\bar{V}}$!! $\bar{V}$ must be open and S Stonean.

Now for the more delicate part of the proof, we suppose S is Stonean and consider a bounded family $\{f_\alpha\}$ in $C(S)$. Define $h_0 : S \to \mathbb{R}$ by $h_0(s) = \sup_\alpha f_\alpha(s)$, $s \in S$.

Of course, $h_0 \in \ell^\infty(S)$ and if $\lambda \in \mathbb{R}$, then $[h_0 > \lambda] = \bigcup_\alpha [f_\alpha > s]$ so $[h_0 > \lambda]$ is open. Suppose that for each $s \in S$, $-M_0 < h_0(s) < M_0$ and partition the interval $[-M_0, M_0]$ into (small) pieces

$$\pi : -M_0 = \lambda_0 < \lambda_1 < \cdots < \lambda_n = M_0.$$

Each of the sets $C_k = [h_0 > \lambda_k]$, $k = 0, 1, \ldots, n$, is open, $C_0 = S, C_n = \emptyset$ and $C_k \supseteq C_{k+1}$. Moreover, the sets $\bar{C}_k$ are open and closed, thanks to S's Stonean nature. So $S = \bigcup_{k=0}^{n-1} (\bar{C}_k \backslash \bar{C}_{k+1})$ decomposes S into the union of pairwise disjoint clopen sets.

Let g_π be given by $g_\pi(s) = \sum_{k=0}^{n} \lambda_k \chi_{\bar{C}_k \backslash \bar{C}_{k+1}}(s)$. Then $g_\pi \in C(S)$.

Let G_π be the nowhere dense set $G_\pi = \bigcup_{k=1}^{n}(\bar{C}_k \setminus C_k)$. Notice that the symmetric difference of $\bar{C}_k \setminus \bar{C}_{k+1}$ and $C_k \setminus C_{k+1}$ satisfies

$$(\bar{C}_k \setminus \bar{C}_{k+1}) \triangle (C_k \setminus C_{k+1}) \subseteq (\bar{C}_k \setminus C_k) \bigcup (\bar{C}_{k+1} \setminus C_{k+1}) \subseteq G_\pi$$

for $k = 0, 1, \ldots, n-1$. Also notice that because $\lambda_k < h_0(s) \leq \lambda_{k+1}$ for $s \in C_k \setminus C_{k+1}$ we have $|h_0(s) - g_\pi(s)| \leq \text{mesh}(\pi)$ for $s \in S \setminus G_\pi$. Okay?

Now choose a sequence (π_n) of partitions of $[-M_0, M_0]$, each a refinement of its predecessor, so that $\lim_n \text{mesh}(\pi_n) = 0$. The result: (g_{π_n}) is a Cauchy, hence convergent, sequence in $C(S)$ with limit $g_0 \in C(S)$. Of course, $g_0(s) = h_0(s)$ for $s \in S \setminus \bigcup_n G_{\pi_n}$. But $S \setminus \bigcup_n G_{\pi_n}$ is dense in S!! Why is this?

Imagine not. Then $\left(\overline{S \setminus \bigcup_n G_{\pi_n}}\right)^c$ would be a non-empty open subset of the compact Hausdorff space S. But

$$\overline{S \setminus (\bigcup_n G_{\pi_n})}^c = \bigcup_n \left(\overline{(S \setminus \bigcup_n G_{\pi_n})}^c \bigcap G_{\pi_n} \right),$$

a set of the first category, something emphatically forbidden by Baire's theorem.

Hence, $g_0(s) \geq f_\alpha(s)$ for each s in the dense set $S \setminus (\bigcup_n G_{\pi_n})$ and all α. It follows that $g_0(s) \geq f_\alpha(s)$ for each $s \in S$ and all α. Hence g_0 is an upper bound of $\{f_\alpha\}$ in $C(S)$.

But if $h \in C(S)$ satisfies $h \geq f_\alpha$ for all α, then $h(s) \geq h_0(s)$ for all $s \in S$. Since $h_0(s) = g_0(s)$ for $s \in S \setminus (\bigcup_n G_{\pi_n})$ we see $h(s) \geq g_0(s)$ for all $s \in S \setminus (\bigcup_n G_{\pi_n})$, a dense subset of S. Hence $h \geq g_0$ and g_0 is the least upper bound of $\{f_\alpha\}$ in $C(S)$. □

D.2. Injective Banach spaces

A Banach space Z is called an *injective Banach space* if whenever X is a Banach space (over the same scalar field as Z) and $u : X \to Z$ is a bounded linear operator, then regardless of the Banach space Y that contains X as a closed linear subspace, there is a bounded linear operator $U : Y \to Z$ such that $U|_X = u$ and $\|u\| = \|U\|$.

THEOREM D.2.1. *If S is a Stonean compact Hausdorff space, then the Banach space $C(S)$ is injective.*

We'll need the following lemma which has a familiar look to it.

LEMMA D.2.2. *Let Y be a real linear space and X be a linear subspace of Y. Suppose $p : Y \to C(S)$ satisfies the following two conditions for all $\lambda \geq 0$ and $y, y_1, y_2 \in Y$:*

$$p(y_1 + y_2) \leq p(y_1) + p(y_2),$$
$$p(\lambda y) = \lambda p(y).$$

Assume $u : X \to C(S)$ is a linear operator such that $u(x) \leq p(x)$ for each $x \in X$. Then there is a linear operator $U : Y \to C(S)$ such that

$$U|_X = u \text{ and } Uy \leq p(y) \text{ for all } y \in Y.$$

Of course, this is a modified version of the Hahn-Banach extension procedure for functionals. Its proof is also a small modification.

PROOF. To be sure, the key step is to extend u from X to the linear span of $X \cup \{y_0\}$ for some $y_0 \in Y \backslash X$.

Now a typical vector y in this linear span is of the form

$$y = x + \alpha y_0$$

for some $x \in X$ and $\alpha \in \mathbb{R}$. The problem is to choose $U(y)$ in such a manner that $U(y') \leq p(y')$ for all y' in the linear span of $X \cup \{y_0\}$. Regardless of the choice we know $Uy = ux + \alpha U y_0$ will be the only possible choice of values for the desired linear extension.

Let's test the waters: Let x, x' be arbitrary members of X. Then

$$\begin{aligned} u(x) - u(x') &= u(x - x') \\ &\leq p(x - x') \\ &= p(x + y_0 - y_0 - x') \\ &\leq p(x + y_0) + p(-y_0 - x'). \end{aligned} \tag{1}$$

Consequently, for any $x, x' \in X$,

$$-p(-y_0 - x') - u(x') \leq p(x + y_0) - u(x). \tag{2}$$

But $C(S)$ is a Dedekind complete lattice, so a long look at (2) reveals that

$$\sup_{x' \in X} \{-p(-y_0 - x') - u(u')\} \leq \inf_{x \in X} \{p(x + y_0) - u(x)\},$$

where the sup and the inf are taken in $C(S)$. Choose f_0 so that

$$-p(-y_0 - x') - u(x') \leq f_0 \leq p(x + y_0) - u(x) \tag{3}$$

for all $x, x' \in X$. Define $U(y_0) = f_0$. The gods are smiling with satisfaction at our choice! U's linearity is clear. Test p's continued domination.

Look at

$$y = x + \alpha y_0.$$

If $\alpha > 0$: $f_0 \leq p(x + y_0) - u(x)$ for all $x \in X$; replace x by $\frac{x}{\alpha}$ and notice that $\alpha f_0 \leq p(x + \alpha y_0) - u(x)$ results, or, likewise,

$$\begin{aligned} U(y) = u(x) + \alpha U(y_0) &= u(x) + \alpha f_0 \\ &\leq p(x + \alpha y_0) = p(y). \end{aligned}$$

If $\alpha < 0$, then we turn to the other side of (3) for help: For all $x' \in X$,

$$-p(-y_0 - x') - u(x') \leq f_0,$$

so replacing x' by $\frac{x}{\alpha}$ gives

$$-p(-y_0 - \frac{x}{\alpha}) - u(\frac{x}{\alpha}) \leq f_0,$$

which, if we multiply all in sight by α, is the same as

$$-\alpha p(-y_0 - \tfrac{x}{\alpha}) - u(x) \geq \alpha f_0 \tag{$\dagger$}$$

(remember $\alpha < 0$); but $-\alpha > 0$, so

$$-\alpha p(-y_0 - \frac{x}{\alpha}) = p(\alpha y_0 + x).$$

Piecing things together we get

$$\begin{aligned} U(y) &= U(x+\alpha y_0) = u(x) + \alpha U(y_0) \\ &= u(x) + \alpha f_0 \le -\alpha p(-y_0 - \frac{x}{\alpha}) \qquad \text{(by (†))} \\ &= p(\alpha y_0 + x) = p(y). \end{aligned}$$

Of course, the proof of the lemma is now completed through standard use of Zorn's lemma. □

PROOF OF THEOREM D.2.1. We first consider the real case. If $p(x) = \|x\| \cdot 1$, then a linear operator $u : X \to C(S)$ has norm ≤ 1 precisely when $u(x) \le p(x)$ for all $x \in X$. The injectivity of $C(S)$ for S Stonean is now a simple consequence of the lemma (and the Dedekind completeness of $C(S)$).

Now to the case of complex scalars: Let X be a linear subspace of the complex Banach space Y and let $u : X \to C(S)$ be a bounded linear operator into the Banach space $C(S)$ of all continuous complex-valued functions on the Stonean space S. For $x \in X$, look at

$$u(x) = \mathrm{Re}\big(u(x)\big) + i\mathrm{Im}\big(u(x)\big).$$

Notice (with Bohnenblust and Sobczyck) that if α and β are real and $x, x' \in X$, then

$$\mathrm{Re}\big(\alpha u(x) + \beta u(x')\big) = \alpha\mathrm{Re}\big(u(x)\big) + \beta\mathrm{Re}\big(u(x')\big)$$

and

$$|\mathrm{Re}\big(u(x)\big)| \le \|x\| \cdot 1,$$

where all equalities and inequalities are comparing members of $C(S)$.

Viewing Y as a *real* Banach space and keeping $p(y) = \|y\| \cdot 1$ clearly in our field-of-vision, use what the gods have given us earlier to extend $\mathrm{Re} \circ u$ to all of Y in a fashion that ensures the extension $RU : Y \to C(S)$ is *real-linear* and enjoys the domination

$$|RU(y)| \le \|y\| \cdot 1$$

for all $y \in Y$. Define $U : Y \to C(S)$ by

$$Uy = RU(y) - iRU(iy).$$

U is linear. U extends u. Each of these is so as they were when dealing with functionals and for the same reasons. We see that $\|U\| = \|u\|$. It is enough to do this in case $\|u\| = 1$. So suppose $y \in Y$ and $\|y\| = 1$. For any $s \in S$, there is a $\phi_s \in [-\pi, \pi]$ so that

$$Uy(s) = |Uy(s)|e^{i\phi_s}.$$

Hence,

$$\begin{aligned} |Uy(s)| = e^{-i\phi_s}Uy(s) &= U(e^{-i\phi_s}y)(s) \\ &= RU(e^{-i\phi_s}y)(s) \end{aligned}$$

never exceeds 1; after all, $\|e^{-i\phi_s}y\| = 1$. Hence $\|Uy\| \le 1$ and $\|U\| \le 1$ is the consequence. □

We now set out to establish the converse, that is, that if Z is an injective Banach space, then there is a Stonean compact Hausdorff space S such that Z is isometrically isomorphic to $C(S)$. This remarkable result is due in the real case to Nachbin (1950), Goodner (1950) and Kelley (1952), while the complex case was established by Hasumi (1958).

We've chosen to take a different path, one blazed by Gleason (1958). Along the way we encounter some beautiful scenery, much of a homological nature, with Stonean spaces coloring the background.

We start with the notion of an irreducible map. Let S, T be compact Hausdorff spaces and $\pi : S \twoheadrightarrow T$ a continuous surjection; we say that π is *irreducible* if whenever S_0 is a closed subset of S such that $\pi(S_0) = T$, $S_0 = S$.

Irreducibility is always around when we have a continuous surjection, indeed, Zorn's Lemma whispers: *If $\pi : S \twoheadrightarrow T$ is a continuous surjection from the compact Haussdorff space S onto the compact Hausdorff space T, then there is a closed subset S_0 of S so that $\pi|_{S_0} : S_0 \twoheadrightarrow T$ is an irreducible continuous surjection.*

A hint of the role played by Stonean spaces is found in the following:

THEOREM D.2.3 (Gleason (1958)). *Let S and T be compact Hausdorff spaces and $\pi : S \twoheadrightarrow T$ an irreducible continuous surjection. Suppose that T is Stonean. Then π is a homeomorphism.*

PROOF. First, we'll show that if G is an open subset of S, then $\pi(G) \subseteq \overline{[\pi(G^c)]^c}$. Let $s \in G$ and let U be an open set containing $\pi(s)$. $G \cap \pi^{\leftarrow}(U)$ is a non-empty open subset of S with a complement that is a proper closed subset of S; from this and π's irreducibility it follows that $\pi([G \cap \pi^{\leftarrow}(U)]^c)$ is a proper closed subset of T. Let t be a point in T that is *not* in $\pi([G \cap \pi^{\leftarrow}(U)]^c)$. Necessarily, $t \in \pi(G^c)^c$. But π is a surjection so $t = \pi(s_0)$ where plainly $s_0 \in G \cap \pi^{\leftarrow}(U)$. So $t = \pi(s_0) \in \pi(\pi^{\leftarrow}(U)) = U$ and so $t \in U$ *and* $\pi(G^c)^c$; together these tell us that $U \cap \pi(G^c)^c \neq \phi$. Any open set U containing a point $\pi(s)$, where $s \in G$, intersects $\pi(G^c)^c$. It follows that $\pi(G)$ is contained in $\overline{\pi(G^c)^c}$, as claimed.

Now for the main course. Suppose s_1 and s_2 are distinct points of S for which $\pi(s_1) = \pi(s_2)$. Let G_1 and G_2 be disjoint open subsets of S with $s_1 \in G_1$ and $s_2 \in G_2$. G_1^c and G_2^c are compact so $\pi(G_1^c)^c$ and $\pi(G_2^c)^c$ are open. Further, they are disjoint since $G_1^c \cup G_2^c = S$, ensuring $\pi(G_1^c) \cup \pi(G_2^c) = \pi(G_1^c \cup G_2^c) = \pi(S) = T$. Now $\pi(G_1^c)^c$ and $\pi(G_2^c)^c$ are disjoint open sets in the Stonean space T and so they have disjoint closures, $\overline{\pi(G_1^c)^c}$ and $\overline{\pi(G_2^c)^c}$. But $s_1 \in G_1$ so $\pi(s_1) \in \pi(G_1) \subseteq \overline{\pi(G_2^c)^c}$ and $s_2 \in G_2$, so $\pi(s_2) \in \pi(G_2) \subseteq \overline{[\pi(G_2^c)]^c}$ making $\pi(s_1) = \pi(s_2)$ a common point of disjoint sets. OOPS! ☐

To bring the above notions into play in our study of Stonean spaces and the nature of injective Banach spaces, we take an interesting and informative side trip and present more of Gleason-style magic. This is another case of "abstract nonsense" that makes perfect sense.

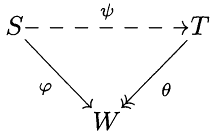
Suppose S is a Stonean compact Hausdorff space and denote by $|S|$, S with the discrete topology. If $i : |S| \to S$ is the formal identity, then i has a unique continuous extension $\varphi : \beta|S| \twoheadrightarrow S$ to the Čech-Stone compactification $\beta|S|$; φ is surjective. We know now that there must be a closed subset Q of $\beta|S|$ so that $\varphi|_Q : Q \twoheadrightarrow S$ is an irreducible continuous surjection. But S is Stonean and so $\varphi|_Q$ is a homeomorphism. So what? Well, this tells us (if we just stop to listen) that

$$(\varphi|_Q)^{-1} \circ \varphi : \beta|S| \to Q$$

is a retraction of $\beta|S|$ onto a homeomorphic image of S. So *Stonean compact Hausdorff spaces are (homeomorphic to) retracts of $\beta(D)'s$ for discrete spaces D.*

A compact Hausdorff space S is called *projective* if given any compact Hausdorff spaces T and W and any continuous functions $\theta : T \twoheadrightarrow W$ and $\varphi : S \to W$ with θ

being surjective, there is a continuous $\psi : S \to T$ so that $\theta \circ \psi = \varphi$, that is, we can fill in the dotted arrow



For any discrete space $D, \beta D$ is projective. After all if T, W, φ and θ are given in the above set-up and $d \in D$, then $\theta^{\leftarrow}(\{\varphi(d)\})$ is a non-empty subset of T, thanks to θ's surjectivity. Choose a point $\psi(d)$ from $\theta^{\leftarrow}(\{\varphi(d)\})$, and be happy with the map $\psi : D \to T$: It is continuous! So the very nature of ψ ensures that the formula

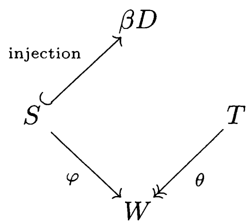
$$\theta \circ \psi = \varphi$$

holds on D. Now ψ has a unique continuous extension (still called) ψ to a continuous map $\psi : \beta D \to T$ which a fortiori satisfies the same equation it obeyed on D.

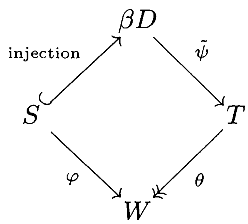
Now suppose S is a closed subset of βD, where D is still a discrete space, and that S is in fact a retract of βD. This means there is a continuous surjection $v : \beta D \twoheadrightarrow S$ so that

$$v|_S = id_S.$$

Let's look at the test diagram for projectivity:



Here T and W are compact Hausdorff spaces, θ and φ are continuous and $\theta : T \twoheadrightarrow W$ is surjective. Extend φ from S to βD by $\tilde{\varphi} = \varphi \circ v$. βD *is* projective so we can find $\tilde{\psi} : \beta D \to T$ so that this diamond-in-the-rough shines



Define $\psi : S \to T$ by $\psi = \tilde{\psi}|_S$. Then

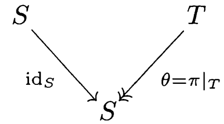
$$\varphi = \theta \circ \psi$$

and S is projective, too. We've shown that *retracts of βD's are projective for discrete D's.*

Now suppose we let S be a projective compact Hausdorff space and let G be an open subset of S. Take the two point Hausdorff space $\{p, q\}$ and look at T:

$$T = [G^c \times \{p\}] \cup [\bar{G} \times \{q\}] \subseteq W = S \times \{p, q\}.$$

Both T and W are compact Hausdorff spaces. Let π be the natural projection of W onto $S, \pi(s,p) = \pi(s,q) = s$ for $s \in S$; let $\theta : T \to W$ be $\pi\mid_T$. Here is what we have



Now θ *is* surjective and so we can apply S's projectivity to find a continuous $\psi : S \to T$ so that

$$\theta \circ \psi = id_S.$$

But θ is injective from $G \times \{q\}$ to G so for $x \in G$,

$$\psi(x) = (x, q).$$

Hence, $\bar{G} = \psi^{\leftarrow}(\bar{G} \times \{q\})$. But $\bar{G} \times \{q\}$ is open in T!! It follows that

$$\bar{G} = \psi^{\leftarrow}(\bar{G} \times \{q\})$$

is open in S!! We took an arbitrary open subset G of our projective compact Hausdorff space S and showed that $\bar{G}$ is open as well. *All projective compact Hausdorff spaces are Stonean.*

We've proved the following:

THEOREM D.2.4 (Gleason (1958)). *Let S be a compact Hausdorff space. Then the following statements are equivalent:*

(1) *S is Stonean;*
(2) *S is (homeomorphic to) a retract of βD, for some discrete space D;*
(3) *S is projective.*

LEMMA D.2.5. *Let S be a compact Hausdorff space and $\varphi : S \to S$ a continuous map. Suppose that φ is* not *id_S. Then there is a proper closed subset Q of S such that $S = Q \cup \varphi^{\leftarrow}(Q)$.*

PROOF. Let $s \in S$ be chosen so that $\varphi(s) \neq s$. Let U and V be disjoint open subsets of S with $s \in U$ and $\varphi(s) \in V$. Look at

$$Q = [U \cap \varphi^{\leftarrow}(V)]^c = U^c \cup [\varphi^{\leftarrow}(V)]^c.$$

Clearly $s \in Q^c = U \cap \varphi^{\leftarrow}(V)$ so that Q is a *proper* closed subset of S. Further,

$$Q^c = U \cap \varphi^{\leftarrow}(V) \subseteq \varphi^{\leftarrow}(V) \subseteq \varphi^{\leftarrow}(U^c) \subseteq \varphi^{\leftarrow}(Q),$$

and so

$$S = Q \cup Q^c \subseteq Q \cup \varphi^{\leftarrow}(Q) \subseteq S$$

and the lemma's proof is complete. □

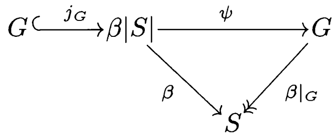
THEOREM D.2.6 (Gleason (1958)). *For every compact Hausdorff space S there is a pair (G, φ), where G is a Stonean space and φ is an irreducible continuous surjection of G onto S.*

The space G is called the *Gleason space* of S.

PROOF. Let $|S|$ be S with the discrete topology. Of course, the formal identity map $i : |S| \to S$ is a continuous surjection and has a unique extension $\beta : \beta|S| \twoheadrightarrow S$ that is continuous and surjective. Let G be a closed subset of $\beta|S|$ such that $\beta|_G$ is irreducible. Here is the picture:

$$G \xhookrightarrow{j_G} \beta|S| \xrightarrow[\beta]{} S \overset{}{\underset{\beta|_G}{\twoheadleftarrow}} G$$

where $j_G : G \hookrightarrow \beta|S|$ is the natural inclusion of G into $\beta|S|$. But $\beta|S|$ *is* projective so we can find a continuous map $\psi : \beta|S| \to G$ such that the following diagram commutes:



Claim: $\psi \circ \beta_G = id_G$. Once established ψ will be seen to be a retraction of $\beta|S|$ onto G and so G must be Stonean. Let us establish the claim.

Because $\beta|_G$ is just what it is,

$$\beta \circ j_G = \beta|_G,$$

and so

$$(*) \qquad \beta|_G \circ \psi \circ j_G = \psi \circ j_G = \beta|_G.$$

Now *if* $\psi \circ j_G$ is *NOT* id_G, then our lemma assures us that there is a *proper* closed subset Q of G such that

$$G = Q \cup (\psi \circ j_G)^{\leftarrow}(Q).$$

But this means that

$$\begin{aligned}\beta|_G(G) &= \beta|_G(Q \cup [(\psi \circ j_G)^{\leftarrow}(Q)])\\ &= \beta|_G(Q) \cup \beta|_G((\psi \circ j_G)^{\leftarrow}(Q))\\ &= \beta|_G(Q) \cup (\beta|_G \circ \psi \circ j_G)((\psi \circ j_G)^{\leftarrow}(Q))\\ &= \beta|_G(Q)\end{aligned}$$

contradicting the irreducibility of the continuous surjection $\beta|_G : G \twoheadrightarrow S$.

It follows that $\psi \circ j_G$ *is* id_G and so G is a retract of the Stonean space $\beta|S|$. □

Now we're heading into the home stretch.

To help us find our way to the finish we'll need to know about the dual ball of a Banach space and, more particularly, about the structure of the set of extreme parts thereof. The role played by the extreme points in this situation was first recognized by Kelley (1952) and it is his ideas that were the driving force behind finishing the real case of this characterization of injective spaces. We, however, will follow a slightly different tact, one aimed at dealing with both real and complex spaces, more or less in the same manner. In detail we follow Cohen (1964).

LEMMA D.2.7. *Let X be a* real *Banach space. Then there is a subset U of* $\mathrm{ext}B_{X^*}$ *such that*

(1) $\overline{U \cup (-U)}^{\text{weak}^*} = \overline{\mathrm{ext}B_{X^*}}^{\text{weak}^*}$ *and*

(2) $(-U) \cap \overline{U}^{\text{weak}^*} = \emptyset$.

PROOF. We open with a *CLAIM*: If O is a weak* open symmetric (non-empty) subset of $\overline{\text{ext}B_{X^*}}^{\text{weak}^*}$, then O contains a non-empty open set V for which $V \cap (-V) = \emptyset$. In fact, if we choose $x_0^* \in O$ with $\|x_0^*\| = 1$, then there is an $x_0 \in X$ so $x_0^* x_0 = 1$. Look at $\{x^* \in O : x^*(x_0) > \frac{1}{2}\} = V$ and be done with it.

Consider the collection of all non-empty weak*-open O's inside $\overline{\text{ext}B_{X^*}}^{\text{weak}^*}$ such that $O \cap (-O) = \emptyset$. Order by inclusion and consider a chain, $(O_\alpha : \alpha \in A)$ of such; $\bigcup_\alpha O_\alpha$ is such a set, too. So there is an open subset W of $\overline{\text{ext}B_{X^*}}^{\text{weak}^*}$ which is (non-empty and) *maximal* with respect to all open subsets V of $\overline{\text{ext}B_{X^*}}^{\text{weak}^*}$ that are disjoint from $-V$.

For such a W we have

(†) $[W \cap \text{ext}B_{X^*}] \cup [(-W) \cap \text{ext}B_{X^*}]$ is weak* dense in $\overline{\text{ext}B_{X^*}}^{\text{weak}^*}$.

But what would the alternative be? The set

$$\overline{\text{ext}B_{X^*}}^{\text{weak}^*} \setminus \overline{W \cup (-W)}^{\text{weak}^*}$$

would be non-empty; it is certainly open and symmetric. In tandem we see a contradiction to W's maximality by tacking the set

$$\overline{\text{ext}B_{X^*}}^{\text{weak}^*} \setminus \overline{W \cup (-W)}^{\text{weak}^*}$$

onto W to get a bigger set than W of the same ilk. OKAY?

Look at $U = W \cap \text{ext}B_{X^*}$: (1) is just (†).

To see (2) imagine an $x^* \in (-U) \cap \overline{U}^{\text{weak}^*}$: $x^* \in (-U)$, hence $x^* \in (-W)$ and $(-W)$ is a weak* open subset of $\overline{\text{ext}B_{X^*}}^{\text{weak}^*}$; it follows that $(-W)$ intersects $\overline{U}^{\text{weak}^*}$ (x^* is in each) so $(-W) \cap U \neq \emptyset$ forcing $(-W) \cap W (\supseteq (-W) \cap U)$ to be non-empty, too. OOPS! (2) follows. □

If $A \subseteq X$, then the circled hull $\text{cir}(A)$ is the set $\{\lambda a : |\lambda| = 1, a \in A\}$; A is *circled* if $A = \text{cir}(A)$.

LEMMA D.2.8. *Let X be a* complex *Banach space. Then there exists a subset U of* $\text{ext}B_{X^*}$ *such that*

(1) $\overline{\text{cir}(U)}^{\text{weak}^*} = \overline{\text{ext}B_{X^*}}^{\text{weak}^*}$,

(2) *for any* $u^* \in U$, $\text{cir}\{u^*\} \cap \overline{U}^{\text{weak}^*} = \{u^*\}$.

PROOF. Just as in the proof of Lemma D.2.7 we start by making a CLAIM! It is just this: Every non-empty circled open subset Q of $\overline{\text{ext}B_{X^*}}^{\text{weak}^*}$ contains a non-empty open subset V for which, if $v^* \in V$, then $\lambda v^* \in V$ for all $|\lambda| = 1$ except one such λ. In fact, let D be the open unit disk in $\mathbb{C}$ with $[0,1)$ removed, choose $x_0^* \in Q$, $x_0^* \neq 0$ (Q does intersect $\text{ext}B_{X^*}$) and then choose $x_0 \in X$ so that $x_0^*(x_0) \in D$. Let V be the set $\{x^* \in Q : x^*(x_0) \in D\}$; V is just $x_0^{\leftarrow}(D) \cap W$ and $x_0^* \in V$. If $x^* \in V$, then $x^*(x_0) \in D$ and so with precisely one exception for all $\lambda, |\lambda| = 1, \lambda x^*(x_0) \in D$, too, leaving all λx^*'s in V but for one. Zorn speaks: There is a maximal non-empty open "deleted" subset W in $\overline{\text{ext}B_{X^*}}^{\text{weak}^*}$.

For such a W we have

(††) $\text{ext}B_{X^*} \cap \text{cir}(W)$ is weak* dense in $\overline{\text{ext}B_{X^*}}^{\text{weak}^*}$,

or, likewise, $\text{cir}(W)$ is weak* dense in $\overline{\text{ext}B_{X^*}}^{\text{weak}^*}$. As a matter of record let it be known that "otherwise" would entail $\overline{\text{ext}B_{X^*}}^{\text{weak}^*} \setminus \overline{\text{cir}(W)}^{\text{weak}^*}$ being non-empty as well as open and circled. But then our CLAIM would jump in to provide an open subset $V \neq \emptyset$ inside $\overline{\mathrm{ext}B_{X^*}}^{\mathrm{weak}^*} \backslash \overline{\mathrm{cir}(W)}^{\mathrm{weak}^*}$ for which given $v^* \in V$ for all λv^*'s, $|y| = 1$, belong to V save for exactly one λ. Tacking V onto W contradicts W's maximality. (††) is a "fait accompli".

Let $U = \{x^* \in \mathrm{ext}B_{X^*} : x^* \notin W$ but $\lambda x^* \in W$ for all *other* $\lambda, |\lambda| = 1\}$. (1) follows from (††) since $\mathrm{cir}(U) = \mathrm{ext}B_{X^*} \cap \mathrm{cir}(W)$.

To see (2) notice that W is open in $\overline{\mathrm{ext}B_{X^*}}^{\mathrm{weak}^*}$ and $U \subseteq W^c$, a weak* closed set. Hence, $\overline{U}^{\mathrm{weak}^*} \subseteq W^c$, too. But $\overline{U}^{\mathrm{weak}^*} \cap W = \emptyset$ follows. Now should $u^* \in U$ and $\lambda \neq 1$ with $|\lambda| = 1$, then $\lambda u^* \in W$ so $\lambda u^* \notin \overline{U}^{\mathrm{weak}^*}$. Hence, the only point $\lambda u^*, |\lambda| = 1$, that is in $\overline{U}^{\mathrm{weak}^*}$ is u^* itself. □

So inside $\mathrm{ext}B_{X^*}$ we can always find a subset U such that

- if X is *real*, then
  (a) $\overline{U \cup (-U)}^{\mathrm{weak}^*} = \overline{\mathrm{ext}B_{X^*}}^{\mathrm{weak}^*}$
  and
  (b) $(-U) \cap \overline{U}^{\mathrm{weak}^*} = \emptyset$
  and
- if X is *complex*, then
  (a) $\overline{\mathrm{cir}U}^{\mathrm{weak}^*} = \overline{\mathrm{ext}B_{X^*}}^{\mathrm{weak}^*}$
  and
  (b) $\mathrm{cir}\{u^*\} \cap \overline{U}^{\mathrm{weak}^*} = \{u^*\}$ for each $u^* \in U$.

Let $K = \overline{U}^{\mathrm{weak}^*}$. K is a compact subset of $(B_{X^*}, \mathrm{weak}^*)$. Let S be the Gleason space of K and $\beta : S \twoheadrightarrow K$ the associated Gleason map; S is Stonean and β is an irreducible continuous surjection.

Define $J : X \to C(S)$ by

$$Jx(s) = \beta(s)(x), \qquad x \in X, s \in S.$$

For any $s \in S, \beta(s) \in K \subseteq B_{X^*}$ so $\beta(s)(x)$ makes sense. What is more, β is continuous from S to K, which *is* equipped with the weak* topology making evaluation at x weak* continuous.

LEMMA D.2.9. *J is a linear isometry.*

PROOF. It is plain that for any $x \in X$,

$$\|Jx\| = \sup_{s \in S} |\beta(s)(x)| \leq \sup_{B_{X^*}} |x^*(x)| = \|x\|.$$

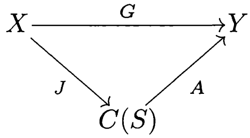
Take an $x \in X$ and let $\varepsilon > 0$ be our margin of error. Then there is an $x^* \in \mathrm{ext}B_{X^*}$ so that $x^*(x) = \|x\|$; after all, the support functionals at x are an extremal subset of B_{X^*} and so contain a point of $\mathrm{ext}B_{X^*}$. Look to U as described above; there is a $u^* \in U$ so that $|u^*(x)| > \|x\| - \varepsilon$, since U satisfies (a). Since $u^* \in U, u^* \in K$, and so there is an $s_0 \in S$ so that $\beta(s_0) = u^*$. Hence,

$$|Jx(s_0)| = |\beta(s_0)(x)| = |u^*(x)| > \|x\| - \varepsilon.$$

J is an isometry. □

Here is a property of the pair $(J, C(S))$ that is key to our characterization of 1-injectivity. We continue with the accumulated notations in hand.

LEMMA D.2.10. *The pair* $(J, C(S))$ *enjoys the following phenomenon: If* $G : X \to Y$ *is a linear isometry and* $A : C(S) \to Y$ *is a bounded linear operator with* $\|A\| \leq 1$ *such that the diagram*



commutes, then A *is an isometry.*

PROOF. Suppose $Y, G : X \to Y, A : C(S) \to Y$ have all answered the stage call and are ready to play their parts.

Let $f \in C(S)$ and $\varepsilon > 0$.

Our goal is to estimate $\|Af\|$; this in mind we'll suppose $\|f\| = f(s_0)$ for some $s_0 \in S$ and look at the open subset

$$V_\varepsilon = [|f - \|f\|| < \varepsilon].$$

Since $\beta : S \to K$ is irreducible and V_ε is open in S,

$$\beta(V_\varepsilon) \subseteq K \backslash \overline{\beta(V_\varepsilon^c)},$$

and so the open set $K \backslash \overline{\beta(V_\varepsilon^c)}$ in K has a preimage under β that lies inside V_ε and is, in fact, dense therein. Now K is just U's weak* closure so we can find $u^* \in U \subseteq \text{ext} B_{X^*}$ so that $\beta^{\leftarrow}(\{u^*\}) \subseteq V_\varepsilon$, again by β's irreducibility.

A change of view. Let $\hat{S}$ be S viewed as point changes $\{\delta_s : s \in S\}$ in $(C(S)^*, \text{weak}^*)$. If $\mu \in \overline{\text{co}}(\beta^{\leftarrow}(\{u*\}))$, then $|\mu(f)| \geq \|f\| - 2\varepsilon$! Why is this so? Well, $\beta^{\leftarrow}(\{u^*\}) \subseteq V_\varepsilon$, so any $s \in \beta^{\leftarrow}(\{u^*\})$ belongs to V_ε; that is,

$$|f(s) - \|f\|| = |\delta_s(f) - \|f\|| < \varepsilon.$$

Alternatively, for any $s \in \beta^{\leftarrow}(\{u^*\})$,

$$\|f\| - \varepsilon < \delta_s(f) < \|f\| + \varepsilon.$$

Now this inequality plainly is enjoyed by any $\mu \in co(\{\delta_s : s \in \beta^{\leftarrow}(\{u^*\})\})$. So for any such μ we have

$$\|f\| - \varepsilon < |\mu(f)|.$$

We can approximate any $\mu \in \overline{\text{co}}^{\text{weak}^*}(\{\delta_s : s \in \beta^{\leftarrow}(\{u^*\})\})$ at f within ε by μ's in $\text{co}(\{\delta_s : s \in \beta^{\leftarrow}(\{u^*\})\})$ so, indeed, for all μ's in $\overline{\text{co}}^{\text{weak}^*} \widehat{\beta^{\leftarrow}(\{u^*\})}$, $|\mu(f)| > \|f\| - 2\varepsilon$.

But $\widehat{\beta^{\leftarrow}(\{u^*\})} = J^{*\leftarrow}(\{u^*\}) \cap \hat{S}$ and we know from our study of U that

$$J^{*\leftarrow}(\{u^*\}) \cap \hat{S} = J^{*\leftarrow}(\{u^*\}) \cap \text{cir}(\hat{S});$$

after all, if $|\lambda| = 1$ and $J^* \lambda \delta_s = \lambda \beta(s) = u^*$, then $u^* \in U$ and $\frac{u^*}{\lambda} = \beta(s) \in K = \overline{U}^{\text{weak}^*}$ so $\frac{1}{\lambda} = 1 = \lambda$.

Here is where we find ourselves: $u^* \in \text{ext} B_{X^*}$ and J^* is linear and weak*-weak* continuous (as well as norm bounded), so the weak* compact convex set

$$J^{*\leftarrow}(\{u^*\}) \cap B_{C(S)^*}$$

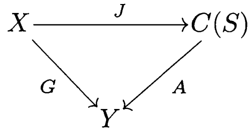
is extremal, and so

$$\begin{aligned} \overline{\text{co}}^{\text{weak}^*} \widehat{\beta^{\leftarrow}(\{u^*\})} &= \overline{\text{co}}^{\text{weak}^*} (J^{*\leftarrow}(\{u^*\}) \cap \hat{S}) \\ &= \overline{\text{co}}^{\text{weak}^*} (J^{*\leftarrow}(\{u^*\}) \cap \text{cir}\hat{S}) \end{aligned}$$

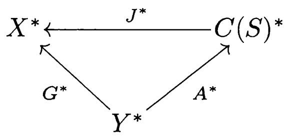
$$= \overline{\text{co}}^{\text{weak}^*}(J^{*\leftarrow}(\{u^*\}) \cap \text{ext} B_{C(S)^*})$$
$$= \overline{\text{co}}^{\text{weak}^*}(\text{ext}(J^{*\leftarrow}(\{u^*\}) \cap B_{C(S)^*})$$
$$= J^{*\leftarrow}(u^*) \cap B_{C(S)^*}.$$

Now we're ready to compute!

Start with



Dualize



The Hahn-Banach theorem assures us there is a $y^* \in Y^*$ of norm one with $G^*y^* = u^*$.

So

$$J^*A^*(y^*) = (J^*A^*)(y^*) = (AJ)^*(y^*) = G^*y^* = u^*.$$

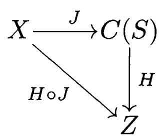
$A^*y^* \in J^{*\leftarrow}(\{u^*\}) \cap B_{C(S)^*} = \overline{\text{co}}^{\text{weak}^*}\widehat{\beta^{\leftarrow}(\{u^*\})}$, and so

$$|y^*(Af)| = |A^*y^*(f)| > \|f\| - 2\varepsilon.$$

$\varepsilon > 0$ is arbitrary and $\|A\| \leq 1$, so this can only happen if $\|Af\| = \|f\|$. Done. □

LEMMA D.2.11. *Let X be a Banach space. Then there is a pair (J, Y) where Y is an injective Banach space and $J : X \to Y$ is a linear isometry such that if Z is any injective subspace of Y such that $J(X) \subseteq Z \subseteq Y$, then $Z = Y$.*

PROOF. Naturally, $(J, C(S))$, where S is the Stonean compact Hausdorff space constructed earlier and $J : X \to C(S)$ is the map discussed at length above, is the pair we have in mind. If Z is an injective subspace of $C(S)$ that contains $J(X)$, then Z is complemented in $C(S)$ via a norm one projection $H : C(S) \to C(S), H(C) = Z$. Look at the diagram



Since J is an isometry and $JX \subseteq Z$, $H \circ T$ is an isometry. But now we can appeal to our previous theorem to conclude that H must itself be an isometry, so $Z = C(S)$. □

Remark: The pair (J, Y) is actually unique and called the *injective envelope* of X.

COROLLARY D.2.12. *If X is an injective Banach space, then X is isometrically isomorphic to a space $C(S)$, where S is a Stonean compact Hausdorff space.*

PROOF. If $(J, C(S))$ is the injective envelope of X, then J is an isometry and as JX is an injective subspace of $C(S)$. By golly we can apply Lemma D.2.11 to conclude that $JX = Z = C(S)$! □

Notes and remarks. In this section we've characterized the "injective" Banach Spaces in the **isometric** *category* of Banach spaces. Our exposition owes a great deal to the beautiful lecture notes of Bade (1971) and Lindenstrauss (1964).

An incisive analysis of lifting and extension properties of compact (and weakly compact) linear operators is to be found in the thesis of Lindenstrauss (1964). Among the many beautiful conclusions drawn in this important work one finds connections drawn between extending compact linear operators and isometrically injective spaces. For instance:

THEOREM. *For a Banach space X, the following statements are equivalent:*

(1) *X^{**} is isometrically injective;*
(2) *if Y is a closed linear subspace of the Banach space Z and $J : Y \to X$ is a compact linear operator, then for each $\varepsilon > 0$ there is a compact linear operator $\tilde{T}_\varepsilon : Z \to X$ such that $\tilde{T}_\varepsilon|_Y = T$ and $\|\tilde{T}_\varepsilon\| = (1+\varepsilon)\|T\|$;*
(3) *whenever $X \subseteq W$ and $T : X \to Y$ is a compact linear operator, there is a compact linear operator $\tilde{T} : W \to Y$ such that $\tilde{T}|_X = T$ and $\|\tilde{T}\| = \|T\|$;*
(4) *whenever $X \subseteq W$ and $T : X \to Y$ is a weakly compact linear operator, there is a weakly compact linear operator $\tilde{T} : W \to Y$ such that $\tilde{T}|_X = T$ and $\|\tilde{T}\| = \|T\|$.*

Note that (3) and (4) are extension theorems for operators *with X as a domain.*

Lindenstrauss's analysis exposes the profound relationship between extension properties of operators and intersection properties of balls, a relationship first broached in the work of Nachbin (1950).

There is a related notion in the **isomorphic** *category*: A Banach space X is *isomorphically injective* if given any bounded linear operator $u : Y \to X$ where Y is a closed linear subspace of the Banach space Z, there is a bounded linear operator $U : Z \to X$ so that $U|_Y = u$. Any isomorphically injective Banach space is a $\mathcal{P}_\lambda$-space for some $\lambda \geq 1$: X is a $\mathcal{P}_\lambda$-space if, whenever X is a closed linear subspace of the Banach space Y there is a bounded projection $P : Y \to Y$ with $P(Y) = X$ and $\|P\| \leq \lambda$. Alternatively, X is a $\mathcal{P}_\lambda$-space if, whenever $u : Z \to X$ is a bounded linear operator and Z is a closed linear subspace of the Banach space W there is a bounded linear $U : W \to X$ with $\|U\| \leq \lambda\|u\|$ and $U|_Z = u$.

The classification of the isomorphically injective Banach spaces remains an open question. In fact it is a famous open problem whether or not an isomorphically injective Banach space is isomorphic to an isometrically injective Banach space.

Several striking passes have been made at the problem. Here are a few results.

From [Lindenstrauss and Pełczyński (1968)] and [Lindenstrauss and Rosenthal (1969)] we know:

- *X is a $\mathcal{L}^1$-space if and only if X^* is isomorphically injective.*
- *X is a $\mathcal{L}^\infty$-space if and only if X^{**} is isomorphically injective.*

[Rosenthal (1970)] contains a wealth of information and directions to the erstwhile student of this topic.

A striking result of Haydon (1978) must be mentioned: *If X is an isomorphically injective bidual space, then X is isomorphic to $\ell^\infty(\Gamma)$ for some set Γ.*

Earlier Isbell and Semadeni (1963) and Amir (1962) had shown that if a $C(K)$-space is a $\mathcal{P}_\lambda$-space with $\lambda < 2$, then K is extremally disconnected; Wolfe (1978) contributed to the understanding of $C(K)$-spaces that are $\mathcal{P}_\lambda$ for $\lambda \leq 3$.

One late addition to the classification of isomorphically injective spaces is the following result of Argyros, Castillo, Granero, Jiménez, and Moreno (2002): *Let X be an isomorphically injective Banach space. Suppose B_{X^*} has weak* density character τ and X contains a weakly compact set of density character τ. Then X is isomorphic to $\ell^\infty(\tau)$.*

Incidentally, it is not always plain that certain viable candidates for isomorphically injectivity pass or fail the test; the space of bounded Lebesgue measurable functions defined on $[0,1]$ and the space of bounded Borel functions defined on $[0,1]$, fails to be isomorphically injective. This was shown by Argyros (1983). It is crucial to note that each of these spaces is a function space; no equivalence classes modulo null sets are involved.

Epilogue

The past twenty years have seen the development of a third important category of Banach spaces: operator spaces. Described by Gilles Pisier as noncommutative Banach space theory, operator spaces incorporate the Banach space structure of a (complex) space with its implicit topological-algebraic structure when embedded into a C^*-algebra. Invented by Effros, Ruan, Blecher and Paulsen, operator spaces have enjoyed considerable success in solving old problems in operator theory and have breathed new life into Banach space theory, allowing new, finer gradations of Banach spaces.

Relevant to these deliberations is the fact that operator spaces have been a fertile ground for the Grothendieck programme. Tensor products live!

What, then, is an *operator space*?

The short answer is any closed linear subspace of a space $\mathcal{B}(H)$ of all bounded linear operators on some Hilbert space H, equipped with the operator norm. Now this includes all Banach spaces, thanks to the Gelfand-Naimark-Segal theory. So what's difficult? Well, the manner of embedding matters as does our manner of comparison.

At the heart of the matter is the notion of a completely bounded linear operator. Denoting by $M_{m,n}(E)$ the space of all $m \times n$ matrices with entries in the operator space E (and making the abbreviation, $M_n(E)$, for all $n \times n$ matrices with entries in E), a linear mapping T between two operator spaces $X \subseteq \mathcal{B}(H)$ and $Y \subseteq \mathcal{B}(K)$ is *completely bounded* if the induced operators $T_n : M_n(X) \to M_n(Y)$ have uniformly bounded norm. Here $M_n(X)$ inherits its norm from $\mathcal{B}(\ell_n^2(H))$ and $T_n((x_{ij})) = (Tx_{ij})$. So the completely bounded norm of T, $\|T\|_{\text{cb}}$, is just

$$\|T\|_{\text{cb}} = \sup_n \|T_n\|_{M_n(X) \to M_n(Y)}.$$

When we speak of the category of operator spaces, our objects are Banach spaces with the embeddings into $\mathcal{B}(H)$'s in hand; the morphisms are the completely bounded linear maps.

Naturally, two operator spaces X and Y are *completely isomorphic* if there is a completely bounded invertible $T : X \to Y$ whose inverse is also completely bounded.

Beware! This new category is really different from our old friends, the isometric and the isomorphic categories of Banach spaces. In fact, this new category is chock full of delicious surprises.

To wit: In the classical categories of Banach spaces, the density character of a Hilbert space of infinite dimensions completely characterizes the equivalence class the space belongs to. If you've seen one infinite dimensional separable Hilbert space, you've seen them all.

Not so in the operator space category. Hilbert space comes in many guises. One is *column Hilbert space* C: Starting with $\mathcal{B}(\ell^2)$, look at C, the closed linear span in $\mathcal{B}(\ell^2)$ of $\{e_{n1} : n \in \mathbb{N}\}$; another is *row Hilbert space* R: Starting with $\mathcal{B}(\ell^2)$, look at R, the closed linear span of $\{e_{1n} : n \in \mathbb{N}\}$. Of course, e_{mn} is the operator in $\mathcal{B}(\ell^2)$ whose matrix representation has a 1 in the m^{th} row and n^{th} column and 0's elsewhere. It is plain and easy-to-see that C and R are isometrically isomorphic (to ℓ^2, even) in the isometric and isomorphic categories. However, and this is a startling fact (at least for the novice) discovered by Mathes (1994), if $T : C \to R$ is completely bounded, then T is a Hilbert-Schmidt operator. In the category of operator spaces, C and R are completely different, pardon the irresistible pun.

The above definition of operator spaces, while attractive for its simplicity, lacks flexibility. A remarkable characterization of operator spaces, due to Ruan (1988), was discovered about twenty years ago. It was the spark that set off an amazing rush of activity, activity that continues to this very day. Here is Ruan's fundamental result.

Let E be a complex vector space given together with a sequence (α_n) of norms on the spaces $M_n(E)$, that is, for each n we have a norm α_n on $M_n(E)$. We assume these norms are compatible in the sense that by embedding $M_n(E)$ into $M_{n+1}(E)$ (by adding zeros in the last row and column), α_n coincides with α_{n+1}'s restriction to $M_n(E)$. Rather than work with this sequence of norms, we might choose to work with one norm, the natural norm defined on the "union" $K_0(E)$ of all the spaces $M_n(E)$. We can view $K_0(E)$ as the union of an ascending sequence

$$M_1(E) \subseteq M_2(E) \subseteq \ldots \subseteq M_n(E) \subseteq M_{n+1}(E) \subseteq \ldots$$

and equip $K_0(E)$ with the norm α induced by the spaces $M_n(E)$.

It is easy to see that the sequence of norms that come from an operator space structure on E (assuming such already exists) satisfy the following:

(R_1) for all $n \in \mathbb{N}$, if $x \in M_n(E)$ and $a, b \in M_n(\mathbb{C})$, then

$$\alpha_n(a \cdot x \cdot b) \leq \|a\|_{M_n} \alpha_n(x) \|b\|_{M_n};$$

and

(R_2) for all $m, n \in \mathbb{N}$, if $x \in M_n(E), y \in M_m(E)$, then

$$\alpha_{n+m}(x \oplus y) = \max \{\alpha_n(x), \alpha_m(y)\},$$

where $x \oplus y$ is the $(n+m)$ by $(n+m)$ matrix $\begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}$.

Further, and this reflects the C^*-algebraic character implicit in operator spaces, we have

(R) for any finite sequences (a_i) and (b_i) in $K_0(\mathbb{C})$ and any finite sequence (x_i) in $K_0(E)$, we have

$$\alpha\Big(\sum_i a_i \cdot x_i \cdot b_i\Big) \leq \Big\|\sum_i a_i a_i^*\Big\|^{\frac{1}{2}} \sup_i \alpha\big((x_i)\big) \Big\|\sum_i b_i^* b_i\Big\|^{\frac{1}{2}}.$$

Here, then, is the fundamental (and extremely elegant) result of Ruan.

THEOREM (Ruan (1988)). *Let E be a complex vector space. Let (α_n) be as above, a sequence of compatible norms on the spaces $M_n(E)$ and let α be the corresponding norm on $K_0(E)$. The following statements are equivalent:*

(1) *(R_1) and (R_2) are satisfied.*
(2) *(R) is satisfied.*

(3) *For a judiciously chosen Hilbert space H, there is a linear embedding $J : E \to \mathcal{B}(H)$ such that for any $n \in \mathbb{N}$, $\mathrm{id}_{M_n} \otimes J$ is an isometry between $(M_n(E), \alpha_n)$ and $M_n(J(E)) \subseteq M_n(\mathcal{B}(H))$, where $M_n(J(E))$ is equipped with the C^*-algebra norm of $M_n(\mathcal{B}(H))$ restricted to $M_n(J(E))$.*

Since $M_n(\mathcal{B}(H))$ is easily identified with $\mathcal{B}(\ell_n^2(H))$, (3) just says E is an operator space.

Armed with Ruan's theorem, the practitioners of operator space theory presented a unified approach to such fundamental constructs as "dual space", "quotient space", "bidual", "direct sum", "complex interpolation" and "ultra product"; in short all the usual abstract accompaniments we want within a category dealing with Banach spaces regardless of view point.

We rush to interject here an important bit of information. Although the investigation into operator spaces per se started in earnest only in the late 1980's, already there are three excellent monographs by leaders in the field. Each introduces the ambitious, able student to the subject from a particular vantage point. We refer the reader to the references at the end of this Epilogue for complete details.

[Effros and Ruan (2000)] offers a beautifully detailed exposition by two of the creators of "quantized functional analysis'!

[Paulsen (2002)] offers a look into how operator spaces shed light on operator algebras, seen through the eyes of one of the originators of the subject, one who is responsible for many of the successes. Again, written with the apt student in mind.

[Pisier (2003)] may be the "Résumé" of operator spaces, setting forth the basics in early chapters and positing a list of possible building blocks for the future development of the subject.

With texts like these and the expanding number of truly outstanding practitioners, operator space theory is experiencing remarkable progress. It is *not* our intention to chronicle such — it would be far beyond our capabilities anyway. Rather, in the next few paragraphs we want to speak of how the Grothendieck programme looks in this new setting.

As in the classical setting, the operator space projective and operator space injective tensor products play a central role in the theory.

First, to reflect the added structure of an operator space, the norms we consider are *subcross matrix norms*: For operator spaces V and W, an operator space norm μ on $V \otimes W$ is a *subcross matrix norm* if $\|v \otimes w\|_\mu \leq \|v\| \, \|w\|$ for all $v \in M_p(V)$ and $w \in M_q(W)$; if $\|v \otimes w\|_\mu = \|v\| \, \|w\|$ for all v's, w's, then μ is called a *cross matrix norm.*

Subcross matrix norms play a role in operator space theory similar to reasonable crossnorms in classical Banach space theory.

We are now ready to define the *operator space projective tensor norm* $\| \cdot \|_\wedge$ on $V \otimes W$, where V and W are any pair of operator spaces. We know, by Ruan's theorem that we need to specify $\|u\|_\wedge$ for $u \in M_n(V \otimes W)$, and all $n \in \mathbb{N}$ in such a way that (R_1) and (R_2) are satisfied: Given an element $u \in M_n(V \otimes W)$, we define

$$\|u\|_\wedge = \inf\{\|\alpha\|\|v\|\|w\|\|\beta\| : u = \alpha \cdot v \otimes w \cdot \beta\},$$

where the infimum is taken over arbitrary decompositions of u with $v \in M_p(V)$, $w \in M_q(W)$, $\alpha \in M_{n,p\times q}$, $\beta \in M_{p\times q,n}$, $p, q \in \mathbb{N}$. This *is* an operator space norm, the *largest* subcross matrix norm on $V \otimes W$.

If V and W are operator spaces, we will denote the completion (as in the Banach space case) of $(V \otimes W, \|\cdot\|_\wedge)$ by $V \overset{\wedge}{\otimes} W$ and we call this space the *operator space projective tensor product.*

The norm $\|\cdot\|_\wedge$ enjoys the following Universal Mapping Property: *If V, W, X are operator spaces, then there is a natural completely isometric identification of $\mathcal{CB}(V \overset{\wedge}{\otimes} W; X)$ and $\mathcal{CB}(V \times W; X)$.* Here $\mathcal{CB}(V \times W; X)$ denotes the space of completely bounded bilinear maps $\varphi : V \times W \to X$, that is, those bilinear φ's such that there is a $K > 0$ so that for any $p, q \in \mathbb{N}$, the mappings

$$\varphi_{p,q} : M_p(V) \times M_q(W) \to M_{p+q}(X)$$

given by

$$\varphi_{p,q}(v, w) = (\varphi(v_{i,j}, w_{k,l}))$$

satisfy $\|\varphi_{p,q}\| \le K$. Naturally $\|\varphi\|_{\mathrm{cb}} = \sup_{p,q}\|\varphi_{p,q}\|$. This basic feature of the operator space projective tensor product was discovered by Blecher, Paulsen, Effros and Ruan. We refer to the bibliography at the end of this epilogue for further references.

At the other end of tensorial considerations, we have the *operator space injective tensor norm*, $\|\cdot\|_\vee$. For operator spaces U and V and $u \in M_n(V \otimes W)$ we define

$$\|u\|_\vee = \sup\left\{\|(f \otimes g)_n(u)\| : f \in M_p(V^*), g \in M_q(W^*), \|f\| \le 1, \|g\| \le 1\right\}.$$

The completion of $(V \otimes W, \|\cdot\|_\vee)$ is called the *operator space injective tensor product* and denoted by $V \overset{\vee}{\otimes} W$.

Just as the operator space projective tensor product mimics the Banach space projective tensor product via a Universal Mapping Property, the operator space injective tensor product mimics its Banach space counterpart in a myriad of mannerisms. To state but two: *The natural embedding of $V^* \overset{\vee}{\otimes} W$ of the operator space injective tensor product of the operator spaces V^* and W into the operator space $\mathcal{CB}(V, W)$ of completely bounded bilinear functionals on $V \times W$ is completely isometric*; again, *if $\varphi : V \to V_1$, $\psi : W \to W_1$ are complete isometries, then $\varphi \otimes \psi : V \overset{\vee}{\otimes} W \to V_1 \overset{\vee}{\otimes} W_1$ is a complete isometry*

There are, of course, many operator space tensor norms, just as there are many tensor norms in the Banach space setting. In the Banach space setting the 14 natural tensor norms of Grothendieck rightfully occupy center stage in the development of the theory.

In the operator space setting there is one operator space tensor norm, the *Haagerup norm* that plays a key role in the theory and *there is no corresponding tensor norm in the Banach space setting.* Since we plan only to talk *about* the Haagerup norm, we refer the reader to the texts listed earlier for lively and informative discussions regarding this norm. (Also see [Blecher and Smith (1992)].) Suffice it to say that within the category of operator spaces *the Haagerup norm is projective, injective and self dual!* Of course, as we saw in our discussion of the Gordon-Lewis Theorem A.2.2 no tensor norm that is both projective and injective exists in the Banach space setting.

Of course Grothendieck's programme entailed much more than just tensor products in various tensor norms. He generated classes of α-integral and α-nuclear operators with striking factorizations, characteristic of injective and/or projective properties of the generating tensor norms. So, too, operator space theory has its classes of integral operators and nuclear operators with corresponding operator space factorizations. (See [Effros and Ruan (1994b)], [Effros and Ruan (1994a)], [Effros and Ruan (1997)] and [Effros, Junge, and Ruan (2000)].)

Notions that arise in the Banach space categories have analogous notions in the operator space category. Injectivity [Effros, Ozawa, and Ruan (2001)], approximation property [Effros and Ruan (1990)], local reflexivity [Effros, Junge, and Ruan (2000)], all have special meanings in the new enriched category and precise delineation is a quest well worth pursuing. We make special mention here of the point of departure of these categories, separable injectivity. A space is separable injective if it is separable and complemented by a projection of suitable nature in any separable super space. It is a still wonderful theorem of Zippin (1977) that says c_0 *is the only* separable isomorphically injective infinite dimensional Banach space; that c_0 is separably injective (in the isomorphic theory) is a now classical result of Sobczyk (1941). In the category of operator spaces, the proper setting for this problem is to ask which (separable) operator spaces U have the property that given a separable operator space V and a completely bounded linear operator $a : V \to U$ such that if V is a subspace of the separable operator space W, then there is a completely bounded operator $A : W \to U$ such that $A|_V = a$? This question has striking results already in evidence, results different from those in the isomorphic category. Rather than tell tales, we refer those interested, to the paper of Rosenthal (2000) for an informative read.

We close with one more hint of how the subject of operator spaces has progressed in a manner parallel to the understanding of the Résumé: The operator space version of Grothendieck's inequality.

Conjectured by Effros and Ruan (1991a): If A and B are C^*-algebras and $\varphi : A \times B \to \mathbb{C}$ is a jointly completely bounded bilinear functional, then there exists states f_1, f_2 on A and states g_1, g_2 on B, such that for all $a \in A$ and $b \in B$,

$$|\varphi(a,b)| \le K\|\varphi\|_{\mathrm{cb}}\big(f_1(aa^*)^{\frac{1}{2}} g_1(b^*b)^{\frac{1}{2}} + f_2(a^*a)^{\frac{1}{2}} g_1(bb^*)^{\frac{1}{2}}\big)$$

where K is a universal constant.

Pisier and Shlyakhtenko (2002) verified the Effros-Ruan conjecture (and much, much more) with $K = 2^{\frac{3}{2}}$ provided that at least one of the C^*-algebras A and B is "exact". Recently, Haagerup and Musat (2007) have verified the Effros-Ruan conjecture for any C^*-algebras with $K = 1$. Consequently, any completely bounded linear map $T : A \to B^*$ from a C^*-algebra A to the dual B^* of a C^*-algebra B admits a factorization $T = vw$ through $R \oplus C$ (R is row Hilbert space and C is column Hilbert space) where $v : R \oplus C \to B^*$ and $w : A \to R \oplus C$ are completely bounded linear operators such that

$$\|v\|_{\mathrm{cb}}\|w\|_{\mathrm{cb}} \le 2\|T\|_{\mathrm{cb}}.$$

Enough said here about operator spaces. No better advice can be given than to go to the sources cited herein. Enjoy!

References to operator spaces

Monographs.

[A] Effros, E. G. and Ruan, Z.-J. (2000). *Operator spaces*, volume 23 of *London Mathematical Society Monographs. New Series*. The Clarendon Press, Oxford University Press, New York.

[B] Paulsen, V. (2002). *Completely bounded maps and operator algebras*, volume 78 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge.

[C] Pisier, G. (2003). *Introduction to operator space theory*, volume 294 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge.

Papers.

[1] Blecher, D. P. (1992). Tensor products of operator spaces. II. *Canad. J. Math.*, **44**(1) 75–90.

[2] Blecher, D. P. and Paulsen, V. I. (1991). Tensor products of operator spaces. *J. Funct. Anal.*, **99**(2) 262–292.

[3] Blecher, D. P. and Smith, R. R. (1992). The dual of the Haagerup tensor product. *J. London Math. Soc. (2)*, **45**(1) 126–144.

[4] Effros, E. and Ruan, Z.-J. (1994a). The Grothendieck-Pietsch and Dvoretzky-Rogers theorems for operator spaces. *J. Funct. Anal.*, **122**(2) 428–450.

[5] Effros, E. G., Junge, M., and Ruan, Z.-J. (2000). Integral mappings and the principle of local reflexivity for noncommutative L^1-spaces. *Ann. of Math. (2)*, **151**(1) 59–92.

[6] Effros, E. G., Ozawa, N., and Ruan, Z.-J. (2001). On injectivity and nuclearity for operator spaces. *Duke Math. J.*, **110**(3) 489–521.

[7] Effros, E. G. and Ruan, Z.-J. (1990). On approximation properties for operator spaces. *Internat. J. Math.*, **1**(2) 163–187.

[8] Effros, E. G. and Ruan, Z.-J. (1991a). A new approach to operator spaces. *Canad. Math. Bull.*, **34**(3) 329–337.

[9] Effros, E. G. and Ruan, Z.-J. (1991b). Self-duality for the Haagerup tensor product and Hilbert space factorizations. *J. Funct. Anal.*, **100**(2) 257–284.

[10] Effros, E. G. and Ruan, Z.-J. (1994b). Mapping spaces and liftings for operator spaces. *Proc. London Math. Soc. (3)*, **69**(1) 171–197.

[11] Effros, E. G. and Ruan, Z.-J. (1997). On the analogues of integral mappings and local reflexivity for operator spaces. *Indiana Univ. Math. J.*, **46**(4) 1289–1310.

[12] Haagerup, U. and Musat, M. (2007). The Effros-Ruan conjecture for bilinear forms on C*-algebras.

[13] Mathes, B. (1994). Characterizations of row and column Hilbert space. *J. London Math. Soc. (2)*, **50**(1) 199–208.

[14] Pisier, G. and Shlyakhtenko, D. (2002). Grothendieck's theorem for operator spaces. *Invent. Math.*, **150**(1) 185–217.

[15] Rosenthal, H. (2000). The complete separable extension property. *J. Operator Theory*, **43**(2) 329–374.

[16] Ruan, Z.-J. (1988). Subspaces of C^*-algebras. *J. Funct. Anal.*, **76**(1) 217–230.

[17] Sobczyk, A. (1941). Projection of the space (m) on its subspace (c_0). *Bull. Amer. Math. Soc.*, **47** 938–947.

[18] Zippin, M. (1977). The separable extension problem. *Israel J. Math.*, **26**(3-4) 372–387.

Bibliography

Alon, N. and Naor, A. (2004). Approximating the cut-norm via Grothendieck's inequality. In *Proceedings of the 36th Annual ACM Symposium on Theory of Computing*, pages 72–80 (electronic). ACM, New York.

Amemiya, I. and Shiga, K. (1957). On tensor products of Banach spaces. *Kōdai Math. Sem. Rep.*, **9** 161–178.

Amir, D. (1962). Continuous functions' spaces with the bounded extension property. *Bull. Res. Council Israel Sect. F*, **10F** 133–138 (1962).

Argyros, S. A. (1983). On the space of bounded measurable functions. *Quart. J. Math. Oxford Ser. (2)*, **34**(134) 129–132.

Argyros, S. A., Castillo, J. F., Granero, A. S., Jiménez, M., and Moreno, J. P. (2002). Complementation and embeddings of $c_0(I)$ in Banach spaces. *Proc. London Math. Soc. (3)*, **85**(3) 742–768.

Arregui, J. L. and Blasco, O. (2002). (p,q)-summing sequences. *J. Math. Anal. Appl.*, **274**(2) 812–827.

Bade, W. G. (1971). *The Banach space $C(S)$*. Lecture Notes Series, No. 26. Matematisk Institut, Aarhus Universitet, Aarhus.

Bishop, E. and Phelps, R. R. (1963). The support functionals of a convex set. In *Proc. Sympos. Pure Math., Vol. VII*, pages 27–35. Amer. Math. Soc., Providence, R.I.

Blecher, D. P. (1992). Tensor products of operator spaces. II. *Canad. J. Math.*, **44**(1) 75–90.

Blecher, D. P. and Paulsen, V. I. (1991). Tensor products of operator spaces. *J. Funct. Anal.*, **99**(2) 262–292.

Blecher, D. P. and Smith, R. R. (1992). The dual of the Haagerup tensor product. *J. London Math. Soc. (2)*, **45**(1) 126–144.

Bochnak, J. (1970). Analytic functions in Banach spaces. *Studia Math.*, **35** 273–292.

Bombal, F., Pérez-García, D., and Villanueva, I. (2004). Multilinear extensions of Grothendieck's theorem. *Q. J. Math.*, **55**(4) 441–450.

Bourgain, J. (1984). New Banach space properties of the disc algebra and H^∞. *Acta Math.*, **152**(1-2) 1–48.

Bourgain, J. (1986). Real isomorphic complex Banach spaces need not be complex isomorphic. *Proc. Amer. Math. Soc.*, **96**(2) 221–226.

Bourgain, J. and Davis, W. J. (1986). Martingale transforms and complex uniform convexity. *Trans. Amer. Math. Soc.*, **294**(2) 501–515.

Bu, Q. (2003). On Banach spaces verifying Grothendieck's theorem. *Bull. London Math. Soc.*, **35**(6) 738–748.

Bu, Q. and Diestel, J. (2001). Observations about the projective tensor product of Banach spaces. I. $l^p\widehat{\otimes}X,\ 1 < p < \infty$. *Quaest. Math.*, **24**(4) 519–533.

Buhvalov, A. V., Veksler, A. I., and Lozanovskiĭ, G. J. (1979). Banach lattices — some Banach aspects of the theory. *Russian Math. Surveys*, **34** 159–212.

Carne, T. K. (1978). Tensor products and Banach algebras. *J. London Math. Soc. (2)*, **17**(3) 480–488.

Carne, T. K. (1980). Banach lattices and extensions of Grothendieck's inequality. *J. London Math. Soc. (2)*, **21**(3) 496–516.

Casazza, P. G. (2001). Approximation properties. In *Handbook of the geometry of Banach spaces, Vol. I*, pages 271–316. North-Holland, Amsterdam.

Chevet, S. (1969). Sur certains produits tensoriels topologiques d'espaces de Banach. *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete*, **11** 120–138.

Cohen, H. B. (1964). Injective envelopes of Banach spaces. *Bull. Amer. Math. Soc.*, **70** 723–726.

Cohen, J. S. (1973). Absolutely p-summing, p-nuclear operators and their conjugates. *Math. Ann.*, **201** 177–200.

Davie, A. M. (1975). The Banach approximation problem. *J. Approximation Theory*, **13** 392–394. Collection of articles dedicated to G. G. Lorentz on the occasion of his sixty-fifth birthday, IV.

Davis, W. J., Figiel, T., Johnson, W. B., and Pełczyński, A. (1974). Factoring weakly compact operators. *J. Functional Analysis*, **17** 311–327.

Davis, W. J. and Johnson, W. B. (1974). Compact, nonnuclear operators. *Studia Math.*, **51** 81–85.

Defant, A. and Floret, K. (1993). *Tensor norms and operator ideals*. North-Holland Publishing Co., Amsterdam.

Diestel, J., Fourie, J., and Swart, J. (2003). The projective tensor product. I. In *Trends in Banach spaces and operator theory (Memphis, TN, 2001)*, volume 321 of *Contemp. Math.*, pages 37–65. Amer. Math. Soc., Providence, RI.

Diestel, J., Jarchow, H., and Tonge, A. (1995). *Absolutely summing operators*, volume 43 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge.

Diestel, J. and Uhl, Jr., J. J. (1977). *Vector measures*. American Mathematical Society, Providence, R.I. With a foreword by B. J. Pettis, Mathematical Surveys, No. 15.

Dixmier, J. (1950). Les fonctionnelles linéaires sur l'ensemble des opérateurs bornés d'un espace de Hilbert. *Ann. of Math. (2)*, **51** 387–408.

Dubinsky, E., Pełczyński, A., and Rosenthal, H. P. (1972). On Banach spaces X for which $\Pi_2(\mathcal{L}_\infty, X) = B(\mathcal{L}_\infty, X)$. *Studia Math.*, **44** 617–648. Collection of articles honoring the completion by Antoni Zygmund of 50 years of scientific activity, VI.

Dunford, N. and Pettis, B. J. (1940). Linear operations on summable functions. *Trans. Amer. Math. Soc.*, **47** 323–392.

Effros, E. and Ruan, Z.-J. (1994a). The Grothendieck-Pietsch and Dvoretzky-Rogers theorems for operator spaces. *J. Funct. Anal.*, **122**(2) 428–450.

Effros, E. G., Junge, M., and Ruan, Z.-J. (2000). Integral mappings and the principle of local reflexivity for noncommutative L^1-spaces. *Ann. of Math. (2)*, **151**(1) 59–92.

Effros, E. G., Ozawa, N., and Ruan, Z.-J. (2001). On injectivity and nuclearity for operator spaces. *Duke Math. J.*, **110**(3) 489–521.

Effros, E. G. and Ruan, Z.-J. (1990). On approximation properties for operator spaces. *Internat. J. Math.*, **1**(2) 163–187.

Effros, E. G. and Ruan, Z.-J. (1991a). A new approach to operator spaces. *Canad. Math. Bull.*, **34**(3) 329–337.

Effros, E. G. and Ruan, Z.-J. (1991b). Self-duality for the Haagerup tensor product and Hilbert space factorizations. *J. Funct. Anal.*, **100**(2) 257–284.

Effros, E. G. and Ruan, Z.-J. (1994b). Mapping spaces and liftings for operator spaces. *Proc. London Math. Soc. (3)*, **69**(1) 171–197.

Effros, E. G. and Ruan, Z.-J. (1997). On the analogues of integral mappings and local reflexivity for operator spaces. *Indiana Univ. Math. J.*, **46**(4) 1289–1310.

Effros, E. G. and Ruan, Z.-J. (2000). *Operator spaces*, volume 23 of *London Mathematical Society Monographs. New Series*. The Clarendon Press Oxford University Press, New York.

Enflo, P. (1973a). A Banach space with basis constant > 1. *Ark. Mat.*, **11** 103–107.

Enflo, P. (1973b). A counterexample to the approximation problem in Banach spaces. *Acta Math.*, **130** 309–317.

Figiel, T. (1973). Factorization of compact operators and applications to the approximation problem. *Studia Math.*, **45** 191–210. (errata insert).

Figiel, T. and Johnson, W. B. (1973). The approximation property does not imply the bounded approximation property. *Proc. Amer. Math. Soc.*, **41** 197–200.

Fishburn, P. C. and Reeds, J. A. (1994). Bell inequalities, Grothendieck's constant, and root two. *SIAM J. Discrete Math.*, **7**(1) 48–56.

Fourie, J. H. and Röntgen, I. M. (2003). Banach space sequences and projective tensor products. *J. Math. Anal. Appl.*, **277**(2) 629–644.

Fourie, J. H. and Swart, J. (1981). Tensor products and Banach ideals of p-compact operators. *Manuscripta Math.*, **35**(3) 343–351.

Freudenthal, H. (1936). Teilweise geordnete Moduln. *Proc. Acad Sci. Amsterdam*, **39** 641–651.

Gamelin, T. W. and Kislyakov, S. V. (2001). Uniform algebras as Banach spaces. In *Handbook of the geometry of Banach spaces, Vol. I*, pages 671–706. North-Holland, Amsterdam.

Garling, D. J. H. and Gordon, Y. (1971). Relations between some constants associated with finite dimensional Banach spaces. *Israel J. Math.*, **9** 346–361.

Gilbert, J. E. and Leih, T. J. (1980). Factorization, tensor products, and bilinear forms in Banach space theory. In *Notes in Banach spaces*, pages 182–305. Univ. Texas Press, Austin, Tex.

Gleason, A. M. (1958). Projective topological spaces. *Illinois J. Math.*, **2** 482–489.

Godefroy, G., Kalton, N. J., and Saphar, P. D. (1993). Unconditional ideals in Banach spaces. *Studia Math.*, **104**(1) 13–59.

Godefroy, G. and Saphar, P. D. (1989). Three-space problems for the approximation properties. *Proc. Amer. Math. Soc.*, **105**(1) 70–75.

Goodner, D. B. (1950). Projections in normed linear spaces. *Trans. Amer. Math. Soc.*, **69** 89–108.

Gordon, Y. and Lewis, D. R. (1974). Absolutely summing operators and local unconditional structures. *Acta Math.*, **133** 27–48.

Gordon, Y., Lewis, D. R., and Retherford, J. R. (1973). Banach ideals of operators with applications. *J. Functional Analysis*, **14** 85–129.

Grothendieck, A. (1953). Sur les applications linéaires faiblement compactes d'espaces du type $C(K)$. *Canadian J. Math.*, **5** 129–173.

Grothendieck, A. (1953/1956a). Résumé de la théorie métrique des produits tensoriels topologiques. *Bol. Soc. Mat. São Paulo*, **8** 1–79.

Grothendieck, A. (1953/1956b). Sur certaines classes de suites dans les espaces de Banach et le théorème de Dvoretzky-Rogers. *Bol. Soc. Mat. São Paulo*, **8** 81–110 (1956).

Grothendieck, A. (1955a). Produits tensoriels topologiques et espaces nucléaires. *Mem. Amer. Math. Soc.*, **1955**(16) 140.

Grothendieck, A. (1955b). Une caractérisation vectorielle-métrique des espaces L^1. *Canad. J. Math.*, **7** 552–561.

Grothendieck, A. (1955c). Une caractérisation vectorielle-métrique des espaces L^1. *Canad. J. Math.*, **7** 552–561.

Gurariĭ, V. I. (1965). The index of sequences in $\tilde{C}$ and the existence of infinite-dimensional separable Banach spaces having no orthogonal basis. *Rev. Roumaine Math. Pures Appl.*, **10** 967–971.

Haagerup, U. (1985). The Grothendieck inequality for bilinear forms on C^*-algebras. *Adv. in Math.*, **56**(2) 93–116.

Haagerup, U. (1987). A new upper bound for the complex Grothendieck constant. *Israel J. Math.*, **60**(2) 199–224.

Haagerup, U. and Musat, M. (2007). The Effros-Ruan conjecture for bilinear forms on C*-algebras. Preprint.

Hasumi, M. (1958). The extension property of complex Banach spaces. *Tôhoku Math. J. (2)*, **10** 135–142.

Havin, V. P. (1973). Weak completeness of the space L^1/H_0^1. *Vestnik Leningrad. Univ.*, **13**(Mat. Meh. Astronom. Vyp. 3) 77–81, 172.

Haydon, R. (1978). On dual L^1-spaces and injective bidual Banach spaces. *Israel J. Math.*, **31**(2) 142–152.

Isbell, J. R. and Semadeni, Z. (1963). Projection constants and spaces of continuous functions. *Trans. Amer. Math. Soc.*, **107** 38–48.

Jarchow, H. and John, K. (1994). Bilinear forms and nuclearity. *Czechoslovak Math. J.*, **44(119)**(2) 367–373.

John, F. (1948). Extremum problems with inequalities as subsidiary conditions. In *Studies and Essays Presented to R. Courant on his 60th Birthday, January 8, 1948*, pages 187–204. Interscience Publishers, Inc., New York, N. Y.

John, K. (1983). Counterexample to a conjecture of Grothendieck. *Math. Ann.*, **265**(2) 169–179.

John, K. (1984). Tensor product of several spaces and nuclearity. *Math. Ann.*, **269**(3) 333–356.

John, K. (1986). Tensor powers of operators and nuclearity. *Math. Nachr.*, **129** 115–121.

John, K. (1990). On the compact nonnuclear operator problem. *Math. Ann.*, **287**(3) 509–514.

Johnson, J. (1979). Remarks on Banach spaces of compact operators. *J. Funct. Anal.*, **32**(3) 304–311.

Johnson, W. B. (1971). Factoring compact operators. *Israel J. Math.*, **9** 337–345.

Johnson, W. B. (1974). On finite dimensional subspaces of Banach spaces with local unconditional structure. *Studia Math.*, **51** 225–240.

Johnson, W. B., Rosenthal, H. P., and Zippin, M. (1971). On bases, finite dimensional decompositions and weaker structures in Banach spaces. *Israel J. Math.*, **9** 488–506.

Jones, P. W. (1985). BMO and the Banach space approximation problem. *Amer. J. Math.*, **107**(4) 853–893.

Kadets, M. Ĭ. and Snobar, M. G. (1971). Certain functionals on the Minkowski compactum. *Mat. Zametki*, **10** 453–457.

Kahane, J.-P. (1968). *Some random series of functions*. D. C. Heath and Co. Raytheon Education Co., Lexington, Mass.

Kaijser, S. (1983). A simple-minded proof of the Pisier-Grothendieck inequality. In *Banach spaces, harmonic analysis, and probability theory (Storrs, Conn., 1980/1981)*, volume 995 of *Lecture Notes in Math.*, pages 33–44. Springer, Berlin.

Kaijser, S. and Sinclair, A. M. (1984). Projective tensor products of C^*-algebras. *Math. Scand.*, **55**(2) 161–187.

Kaiser, R. J. and Retherford, J. R. (1983). Eigenvalue distribution of nuclear operators: a survey. In *Proceedings of the conferences on vector measures and integral representations of operators, and on functional analysis/Banach space geometry (Essen, 1982)*, volume 10 of *Vorlesungen Fachbereich Math. Univ. Essen*, pages 245–287. Univ. Essen, Essen.

Kaiser, R. J. and Retherford, J. R. (1984). Eigenvalue distribution of nuclear operators: a survey. In *Proceedings of the second international conference on operator algebras, ideals, and their applications in theoretical physics (Leipzig, 1983)*, volume 67 of *Teubner-Texte Math.*, pages 173–177. Teubner, Leipzig.

Kakutani, S. (1941a). Concrete representation of abstract (L)-spaces and the mean ergodic theorem. *Ann. of Math. (2)*, **42** 523–537.

Kakutani, S. (1941b). Concrete representation of abstract (M)-spaces. (A characterization of the space of continuous functions.). *Ann. of Math. (2)*, **42** 994–1024.

Kalton, N. J. (1974). Spaces of compact operators. *Math. Ann.*, **208** 267–278.

Kalton, N. J. (1995). An elementary example of a Banach space not isomorphic to its complex conjugate. *Canad. Math. Bull.*, **38**(2) 218–222.

Kelley, J. L. (1952). Banach spaces with the extension property. *Trans. Amer. Math. Soc.*, **72** 323–326.

Kislyakov, S. V. (1975). Sobolev imbedding operators, and the nonisomorphism of certain Banach spaces. *Funkcional. Anal. i Priložen.*, **9**(4) 22–27.

Kislyakov, S. V. (1976). On spaces with "small" annihilators. *Funkcional. Anal. i Priložen.*, **65** 192–195.

Kislyakov, S. V. (1989). Proper uniform algebras are uncomplemented. *Dokl. Akad. Nauk SSSR*, **309**(4) 795–798.

Kislyakov, S. V. (1991). Absolutely summing operators on the disc algebra. *Algebra i Analiz*, **3**(4) 1–77.

Kislyakov, S. V. (1995). Some more spaces for which an analogue of the Grothendieck theorem holds. *Algebra i Analiz*, **7**(1) 62–91.

Kislyakov, S. V. (1998). Bourgain's analytic projection revisited. *Proc. Amer. Math. Soc.*, **126**(11) 3307–3314.

Kislyakov, S. V. and Sidorenko, N. G. (1988). Absence of local unconditional structure in anisotropic spaces of smooth functions. *Sibirsk. Mat. Zh.*, **29**(3) 64–77, 220.

König, H. (1986). *Eigenvalue distribution of compact operators*. Birkhäuser Verlag, Basel.

Krivine, J. L. (1974). Théorèmes de factorisation dans les espaces réticulés. In *Séminaire Maurey-Schwartz 1973–1974: Espaces L^p, applications radonifiantes* *et géométrie des espaces de Banach, Exp. Nos. 22 et 23*, page 22. Centre de Math., École Polytech., Paris.

Krivine, J.-L. (1978). Constantes de Grothendieck et fonctions de type positif sur les sphères. In *Séminaire sur la Géométrie des Espaces de Banach (1977–1978)*, pages Exp. No. 1–2, 17. École Polytech., Palaiseau.

Krivine, J.-L. (1979). Constantes de Grothendieck et fonctions de type positif sur les sphères. *Adv. in Math.*, **31**(1) 16–30.

Kwapień, S. and Pełczyński, A. (1980). Absolutely summing operators and translation-invariant spaces of functions on compact abelian groups. *Math. Nachr.*, **94** 303–340.

Lapresté, J.-T. (1976). Opérateurs sommants et factorisations. À travers les espaces L^p. *Studia Math.*, **57**(1) 47–83.

Lewis, D. R. (1973). Conditional weak compactness in certain inductive tensor products. *Math. Ann.*, **201** 201–209.

Lima, Å., Nygaard, O., and Oja, E. (2000). Isometric factorization of weakly compact operators and the approximation property. *Israel J. Math.*, **119** 325–348.

Lindenstrauss, J. (1964). Extension of compact operators. *Mem. Amer. Math. Soc. No.*, **48** 112.

Lindenstrauss, J. (1971). On James's paper "Separable conjugate spaces". *Israel J. Math.*, **9** 279–284.

Lindenstrauss, J. and Pełczyński, A. (1968). Absolutely summing operators in $\mathcal{L}_p$-spaces and their applications. *Studia Math.*, **29** 275–326.

Lindenstrauss, J. and Rosenthal, H. P. (1969). The $\mathcal{L}_p$ spaces. *Israel J. Math.*, **7** 325–349.

Lindenstrauss, J. and Tzafriri, L. (1977). *Classical Banach spaces. I*. Springer-Verlag, Berlin. Sequence spaces, Ergebnisse der Mathematik und ihrer Grenzgebiete, Vol. 92.

Lindenstrauss, J. and Tzafriri, L. (1979). *Classical Banach spaces. II*, volume 97 of *Ergebnisse der Mathematik und ihrer Grenzgebiete [Results in Mathematics and Related Areas]*. Springer-Verlag, Berlin. Function spaces.

Lomonosov, V. (2000a). A counterexample to the Bishop-Phelps theorem in complex spaces. *Israel J. Math.*, **115** 25–28.

Lomonosov, V. (2000b). On the Bishop-Phelps theorem in complex spaces. *Quaest. Math.*, **23**(2) 187–191.

Lomonosov, V. (2001). The Bishop-Phelps theorem fails for uniform non-selfadjoint dual operator algebras. *J. Funct. Anal.*, **185**(1) 214–219.

Luxemburg, W. A. J. and Zaanen, A. C. (1971). *Riesz spaces. Vol. I*. North-Holland Publishing Co., Amsterdam. North-Holland Mathematical Library.

Mathes, B. (1994). Characterizations of row and column Hilbert space. *J. London Math. Soc. (2)*, **50**(1) 199–208.

Maurey, B. (1974). Type et cotype dans les espaces munis de structures locales inconditionnelles. In *Séminaire Maurey-Schwartz 1973–1974: Espaces L^p, applications radonifiantes et géométrie des espaces de Banach, Exp. Nos. 24 et 25*, page 25. Centre de Math., École Polytech., Paris.

Maurey, B. and Pisier, G. (1976). Séries de variables aléatoires vectorielles indépendantes et propriétés géométriques des espaces de Banach. *Studia Math.*, **58**(1) 45–90.

Meyer-Nieberg, P. (1991). *Banach lattices.* Universitext. Springer-Verlag, Berlin.

Michal, A. D. and Wyman, M. (1941). Characterization of complex couple spaces. *Ann. of Math. (2)*, **42** 247–250.

Milne, H. (1972). Banach space properties of uniform algebras. *Bull. London Math. Soc.*, **4** 323–326.

Moedomo, S. and Uhl, Jr., J. J. (1971). Radon-Nikodým theorems for the Bochner and Pettis integrals. *Pacific J. Math.*, **38** 531–536.

Mooney, M. C. (1972). A theorem on bounded analytic functions. *Pacific J. Math.*, **43** 457–463.

Muñoz, G. A., Sarantopoulos, Y., and Tonge, A. (1999). Complexifications of real Banach spaces, polynomials and multilinear maps. *Studia Math.*, **134**(1) 1–33.

Nachbin, L. (1950). A theorem of the Hahn-Banach type for linear transformations. *Trans. Amer. Math. Soc.*, **68** 28–46.

Nakano, H. (1941). Über das System aller stetigen Funktionen auf einem topologischen Raum. *Proc. Imp. Acad. Tokyo*, **17** 308–310.

Paulsen, V. (2002). *Completely bounded maps and operator algebras*, volume 78 of *Cambridge Studies in Advanced Mathematics.* Cambridge University Press, Cambridge.

Pełczyński, A. (1971). Any separable Banach space with the bounded approximation property is a complemented subspace of a Banach space with a basis. *Studia Math.*, **40** 239–243.

Pełczyński, A. (1974). Sur certaines propriétés isomorphiques nouvelles des espaces de Banach de fonctions holomorphes A et H^∞. *C. R. Acad. Sci. Paris Sér. A*, **279** 9–12.

Pełczyński, A. (1977). *Banach spaces of analytic functions and absolutely summing operators.* American Mathematical Society, Providence, R.I. Expository lectures from the CBMS Regional Conference held at Kent State University, Kent, Ohio, July 11–16, 1976, Conference Board of the Mathematical Sciences Regional Conference Series in Mathematics, No. 30.

Pełczyński, A. and Wojciechowski, M. (2002). Sobolev spaces in several variables in L^1-type norms are not isomorphic to Banach lattices. *Ark. Mat.*, **40**(2) 363–382.

Pełczyński, A. and Wojciechowski, M. (2003a). Sobolev spaces. In *Handbook of the geometry of Banach spaces, Vol. 2*, pages 1361–1423. North-Holland, Amsterdam.

Pełczyński, A. and Wojciechowski, M. (2003b). Spaces of functions with bounded variation and Sobolev spaces without local unconditional structure. *J. Reine Angew. Math.*, **558** 109–157.

Pérez-García, D. and Villanueva, I. (2004). There is no lattice preserving natural tensor norm. *Quaest. Math.*, **27**(3) 267–273.

Persson, A. and Pietsch, A. (1969). p-nukleare une p-integrale Abbildungen in Banachräumen. *Studia Math.*, **33** 19–62.

Phillips, R. S. (1940). On linear transformations. *Trans. Amer. Math. Soc.*, **48** 516–541.

Pietsch, A. (1966/1967). Absolut p-summierende Abbildungen in normierten Räumen. *Studia Math.*, **28** 333–353.

Pietsch, A. (1980). *Operator ideals.* North-Holland Publishing Co., Amsterdam. Translated from German by the author.

Pietsch, A. (1987). *Eigenvalues and s-numbers.* Cambridge University Press, Cambridge.

Pisier, G. (1978a). Grothendieck's theorem for noncommutative C^*-algebras, with an appendix on Grothendieck's constants. *J. Funct. Anal.*, **29**(3) 397–415.

Pisier, G. (1978b). Some results on Banach spaces without local unconditional structure. *Compositio Math.*, **37**(1) 3–19.

Pisier, G. (1978c). Une nouvelle classe d'espaces de Banach vérifiant le théorème de Grothendieck. *Ann. Inst. Fourier (Grenoble)*, **28**(1) x, 69–90.

Pisier, G. (1980). Un théorème sur les opérateurs linéaires entre espaces de Banach qui se factorisent par un espace de Hilbert. *Ann. Sci. École Norm. Sup. (4)*, **13**(1) 23–43.

Pisier, G. (1982a). Holomorphic semigroups and the geometry of Banach spaces. *Ann. of Math. (2)*, **115**(2) 375–392.

Pisier, G. (1982b). Holomorphic semigroups and the geometry of Banach spaces. *Ann. of Math. (2)*, **115**(2) 375–392.

Pisier, G. (1983). Counterexamples to a conjecture of Grothendieck. *Acta Math.*, **151**(3-4) 181–208.

Pisier, G. (1986). *Factorization of linear operators and geometry of Banach spaces*, volume 60 of *CBMS Regional Conference Series in Mathematics*. Published for the Conference Board of the Mathematical Sciences, Washington, DC.

Pisier, G. (1989). *The volume of convex bodies and Banach space geometry*. Cambridge University Press, Cambridge.

Pisier, G. (1992). A simple proof of a theorem of Jean Bourgain. *Michigan Math. J.*, **39**(3) 475–484.

Pisier, G. (2003). *Introduction to operator space theory*, volume 294 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge.

Pisier, G. and Shlyakhtenko, D. (2002). Grothendieck's theorem for operator spaces. *Invent. Math.*, **150**(1) 185–217.

Rieffel, M. A. (1968). The Radon-Nikodym theorem for the Bochner integral. *Trans. Amer. Math. Soc.*, **131** 466–487.

Rietz, R. E. (1974). A proof of the Grothendieck inequality. *Israel J. Math.*, **19** 271–276.

Rosenthal, H. (2000). The complete separable extension property. *J. Operator Theory*, **43**(2) 329–374.

Rosenthal, H. P. (1970). On injective Banach spaces and the spaces $L^\infty(\mu)$ for finite measure μ. *Acta Math.*, **124** 205–248.

Rosenthal, H. P. (1973). On subspaces of L^p. *Ann. of Math. (2)*, **97** 344–373.

Ruan, Z.-J. (1988). Subspaces of C^*-algebras. *J. Funct. Anal.*, **76**(1) 217–230.

Ryan, R. A. (2002). *Introduction to tensor products of Banach spaces*. Springer Monographs in Mathematics. Springer-Verlag London Ltd., London.

Saphar, P. (1970). Produits tensoriels d'espaces de Banach et classes d'applications linéaires. *Studia Math.*, **38** 71–100. (errata insert).

Schaefer, H. H. (1974). *Banach lattices and positive operators*. Springer-Verlag, New York. Die Grundlehren der mathematischen Wissenschaften, Band 215.

Schatten, R. (1960). *Norm ideals of completely continuous operators*. Ergebnisse der Mathematik und ihrer Grenzgebiete. N. F., Heft 27. Springer-Verlag, Berlin.

Schneider, R. (1993). *Convex bodies: the Brunn-Minkowski theory*, volume 44 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge.

Schütt, C. (1978). Unconditionality in tensor products. *Israel J. Math.*, **31**(3-4) 209–216.

Sobczyk, A. (1941). Projection of the space (m) on its subspace (c_0). *Bull. Amer. Math. Soc.*, **47** 938–947.

Stegall, C. (1981). The Radon-Nikodým property in conjugate Banach spaces. II. *Trans. Amer. Math. Soc.*, **264**(2) 507–519.

Stone, M. H. (1949). Boundedness properties in function-lattices. *Canadian J. Math.*, **1** 176–186.

Szankowski, A. (1976). A Banach lattice without the approximation property. *Israel J. Math.*, **24**(3-4) 329–337.

Szankowski, A. (1978). Subspaces without the approximation property. *Israel J. Math.*, **30**(1-2) 123–129.

Szankowski, A. (1981). $B(\mathcal{H})$ does not have the approximation property. *Acta Math.*, **147**(1-2) 89–108.

Szarek, S. J. (1978). On Kashin's almost Euclidean orthogonal decomposition of l_n^1. *Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys.*, **26**(8) 691–694.

Szarek, S. J. (1986). A superreflexive Banach space which does not admit complex structure. *Proc. Amer. Math. Soc.*, **97**(3) 437–444.

Szarek, S. J. (1987). A Banach space without a basis which has the bounded approximation property. *Acta Math.*, **159**(1-2) 81–98.

Tomczak-Jaegermann, N. (1974). The moduli of smoothness and convexity and the Rademacher averages of trace classes $S_p(1 \leq p < \infty)$. *Studia Math.*, **50** 163–182.

Tomczak-Jaegermann, N. (1989). *Banach-Mazur distances and finite-dimensional operator ideals*. Longman Scientific & Technical, Harlow.

Van Zyl, A. J. (2007). Complexifications of tensor products of Banach spaces and Banach ideals of operators. Preprint.

Varopoulos, N. T. (1974). On an inequality of von Neumann and an application of the metric theory of tensor products to operators theory. *J. Functional Analysis*, **16** 83–100.

Wenzel, J. (1995). Real and complex operator ideals. *Quaestiones Math.*, **18**(1-3) 271–285. First International Conference in Abstract Algebra (Kruger Park, 1993).

Wojtaszczyk, P. (1991). *Banach spaces for analysts*. Cambridge University Press, Cambridge.

Wolfe, J. (1978). Injective Banach spaces of continuous functions. *Trans. Amer. Math. Soc.*, **235** 115–139.

Zaanen, A. C. (1983). *Riesz spaces. II*, volume 30 of *North-Holland Mathematical Library*. North-Holland Publishing Co., Amsterdam.

Zippin, M. (1977). The separable extension problem. *Israel J. Math.*, **26**(3-4) 372–387.

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Index of Notation

Generalities

$\mathbb{R}$	The field of real numbers (scalars)
$\mathbb{C}$	The field of complex numbers (scalars)
$\mathbb{K}$	The generic scalar field $\mathbb{R}$ or $\mathbb{C}$
$\mathbb{R}^n$	The n-dimensional Euclidean space
$T^{\leftarrow}(E)$	The inverse image of the set E under the operator T
$\ker(T)$	The kernel of T $\big(=T^{\leftarrow}(\{0\})\big)$
$\overline{A}$	The closure of the set A
A°	The interior of the set A
$\mathrm{co}(A)$	The convex hull of the set A
$\overline{\mathrm{co}}(A)$	The closed convex hull of the set A
$T : X \twoheadrightarrow Y$	T is a surjective linear operator
$T : X \hookrightarrow Y$	T is an injective linear operator
$\mathcal{F}(X)$	The set of all finite dimensional subspaces of the Banach space X
B_X	The unit ball of a Banach space X
id_X	The identity operator on the vector space X
$\mathrm{ext}A$	The set of all extreme points of a set A in a vector space
χ_A	The indicator or characteristic function of A
$r_n(\cdot)$	The n-th Rademacher function defined on $[0,1]$: $r_n(t) = \mathrm{sign}\big(\sin 2^n \pi t\big)$

Vector spaces; Banach spaces

X'	The algebraic dual of a vector space X	
X^*	The (continuous) dual of a Banach space	
ℓ^p	The Banach space of all absolutely p-summable scalar sequences: $\big(\{(\lambda_n)_n : \sum_n \lvert\lambda_n\rvert^p < \infty\}\big)$; $\lVert(\lambda_n)_n\rVert = \big(\sum_n \lvert\lambda_n\rvert^p\big)^{\frac{1}{p}}$	
ℓ^∞	The Banach space of all bounded scalar sequences; $\lVert(\lambda_n)_n\rVert = \sup_n \lvert\lambda_n\rvert$	
c_0	The Banach space of all scalar null sequences; $\lVert(\lambda_n)_n\rVert = \sup_n \lvert\lambda_n\rvert$	
ℓ^p_X or $\ell^p(X)$	The Banach space of all absolutely p-summable sequences in a Banach space X	16
ℓ^∞_X or $\ell^\infty(X)$	The Banach space of all bounded sequences in a Banach space X	16
$c_0(X)$	The Banach space of all null sequences in a Banach space X	16

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Compact and convex sets

Banach lattices

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Grothendieck's Resumé is a landmark in functional analysis. Despite having appeared more than a half century ago, its techniques and results are still not widely known nor appreciated. This is due, no doubt, to the fact that Grothendieck included practically no proofs, and the presentation is based on the theory of the very abstract notion of tensor products. This book aims at providing the details of Grothendieck's constructions and laying bare how the important classes of operators are a consequence of the abstract operations on tensor norms. Particular attention is paid to how the classical Banach spaces ($C(K)$'s, Hilbert spaces, and the spaces of integrable functions) fit naturally within the mosaic that Grothendieck constructed.

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