THE NORMAL COMPLETION OF THE LATTICE OF CONTINUOUS FUNCTIONS

BY

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1. **Introduction.** Let S be a topological space(1) and let $C(S)$ denote the set of all real valued, bounded, continuous functions on S. It is well known that $C(S)$ is a distributive lattice under the operations sup (f, g) and inf (f, g). In general, however, $C(S)$ is not a complete lattice; that is, an arbitrary bounded set of continuous functions in $C(S)$ need not have a least upper bound in the lattice $C(S)$. Furthermore, the structure of the minimal completion of $C(S)$ by means of normal subsets has not been determined even in the simple case where S is the real interval $[0, 1]$.

The first part of the paper will be devoted to the construction of a set of functions which form a complete lattice isomorphic to the normal completion of $C(S)$. We use for this purpose a class of bounded, upper semicontinuous functions (called *normal*) which are characterized by the following property(2).

$$(f_*)^* = f.$$

It is proved that *the normal completion of $C(S)$ is isomorphic with the lattice of all normal, upper semicontinuous functions on a suitably determined completely regular space S_0. If S is completely regular, then S_0 is simply S itself.*

As an application we deduce the Stone-Nakano theorem on spaces S for which $C(S)$ is lattice complete.

In the second part of the paper it is shown that the normal completion of $C(S)$ is itself isomorphic to the lattice of all continuous functions on some compact Hausdorff space. The precise theorem is the following.

Let S be completely regular. Then the normal completion of $C(S)$ is isomorphic with the lattice of all continuous functions on the Boolean space associated with the Boolean algebra of regular open sets of S.

Birkhoff has shown that if S is a completely regular space without isolated points and satisfying the second countability axiom, then the Boolean algebra of regular open sets is isomorphic with the normal completion of the free Boolean algebra with a countably infinite set of generators. Hence specializing

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(1) The term "topological space" is used in the sense of Alexandroff and Hopf, *Topologie*, Berlin, 1936. I am indebted to Professors Bohnenblust and Karlin for their advice in connection with the topological questions arising in the work.

(2) f^* and f_* represent respectively the upper and lower limit functions of f. See formulas (3.1) and (3.2) for the precise definitions.

we get the following theorem:

If S is a completely regular space without isolated points and satisfying the second countability axiom, then the normal completion of $C(S)$ is isomorphic with the lattice of all continuous functions on the Boolean space associated with the normal completion of the free Boolean algebra with a countably infinite set of generators.

Thus the lattices of continuous functions on spaces satisfying the conditions of the theorem all have the same normal completion. In particular, this theorem gives a simple representation of the normal completion of the lattice of continuous functions on the real interval $[0, 1]$.

PART I. NORMAL UPPER SEMICONTINUOUS FUNCTIONS

2. **Preliminary reduction.** Since we shall be interested in lattice properties of $C(S)$, we may, if we wish, assume that S is completely regular(3) (Čech [2](4)). The reduction to the completely regular case can be accomplished as follows: Define

$$x \sim y \quad \text{if} \quad f(x) = f(y) \qquad \text{for all } f \in C(S).$$

The relation $x \sim y$ is clearly an equivalence relation and hence separates S into equivalence classes $X, Y, Z, \cdots$. Let S_0 denote the set of equivalence classes. To each $f \in C(S)$ there corresponds a function F on S_0 defined by $F(X) = f(x)$ where x is an element of X. If A_0 is a subset of S_0, let the closure of A_0 consist of all X such that for every F, $F(X) = 0$ whenever $F(Y) = 0$ for all Y contained in A_0. Then S_0 becomes a completely regular topological space under this definition of closure and the mapping

$$f \to F$$

is a lattice isomorphism of $C(S)$ onto $C(S_0)$.

By appealing to the Stone-Čech compactification theorem we could also assume that S is compact. However, little is gained from the additional assumption and it seems desirable that the results of part I should not depend upon transfinite methods.

We shall frequently use the fact that every completely regular space is regular; that is, if N is any open set containing x, there is an open set A containing x whose closure is contained in N.

3. **Properties of normal upper semicontinuous functions.** Let $B(S)$ denote the set of all bounded, real functions on S. If x is a point of S, let N_x denote an arbitrary open set containing x. Then the two basic unary operations on $B(S)$ which we shall use are defined as follows:

(3) A topological space S is completely regular if for each x and open set A containing x, there is a continuous function f having the value 1 at x and vanishing outside A. Replacing f by sup $(0, \inf (1, f))$ if necessary one may assume that the values of f lie between 0 and 1.

(4) Numbers in brackets refer to the references cited at the end of the paper.

$$\phi^*(x) = \inf_{N_x} \sup_{y\in N_x} \phi(y), \tag{3.1}$$

$$\phi_*(x) = \sup_{N_x} \inf_{y\in N_x} \phi(y). \tag{3.2}$$

LEMMA 3.1. *The operations ϕ^* and ϕ_* have the following properties*:

$$\phi^* \geqq \phi \geqq \phi_*, \tag{3.3}$$

$$\phi \geqq \psi \;\rightarrow\; \phi^* \geqq \psi^* \quad and \quad \phi_* \geqq \psi_*, \tag{3.4}$$

$$(\phi^*)^* = \phi^*, \qquad (\phi_*)_* = \phi_*, \tag{3.5}$$

$$(((\phi^*)_*)^*)_* = (\phi^*)_*, \qquad (((\phi_*)^*)_*)^* = (\phi_*)^*. \tag{3.6}$$

Properties (3.3), (3.4), and (3.5) follow immediately from (3.1) and (3.2). Also by (3.3), $((\phi^*)_*)^* \geqq (\phi^*)_*$ and hence $(((\phi^*)_*)^*)_* \geqq ((\phi^*)_*)_* = (\phi^*)_*$ by (3.4) and (3.5). On the other hand $(\phi^*)_* \leqq \phi^* \rightarrow ((\phi^*)_*)^* \leqq (\phi^*)^* = \phi^* \rightarrow (((\phi^*)_*)^*)_* \leqq (\phi^*)_*$ by (3.3) and (3.4). Thus the first part of (3.6) is proved and the second part follows in a similar manner.

DEFINITION 3.1. ϕ is *upper semicontinuous* on S if $\phi^* = \phi$.

Lower semicontinuous functions are defined dually. Clearly ϕ is continuous if and only if $\phi^* = \phi_*$.

The functions of $B(S)$ which will be used to characterize the normal completion of $C(S)$ are defined as follows:

DEFINITION 3.2. An upper semicontinuous function ϕ on S is *normal* if $(\phi_*)^* = \phi$. Clearly every continuous function is normal.

Normality can be characterized as follows:

THEOREM 3.1. *An upper semicontinuous function ϕ on S is normal if and only if for each $\epsilon > 0$, $x \in S$, and open set N containing x, there exists a non-empty open set $A \subseteq N$ such that $\phi(y) > \phi(x) - \epsilon$ all $y \in A$.*

For the proof let ϕ be an upper semicontinuous function on S and let us suppose first that ϕ is normal. Let $\epsilon > 0$ and let N be an open set containing x. By (3.1),

$$\sup_{z\in N} \phi_*(z) \geqq (\phi_*)^*(x) = \phi(x).$$

For some $z \in N$

$$\phi_*(z) > \phi(x) - \epsilon.$$

By (3.2) there is a neighborhood A of z contained in N such that

$$\inf_{y\in A} \phi(y) > \phi(x) - \epsilon.$$

This gives the necessity of the condition of the theorem.

Conversely, if the condition is satisfied for all $\epsilon > 0$ and N containing x, let $z \in A$. Then

$$\phi_*(z) \geqq \inf_{y \in A} \phi(y) \geqq \phi(x) - \epsilon.$$

Hence

$$\sup_{z \in N} \phi_*(z) \geqq \phi(x) - \epsilon.$$

Thus

$$(\phi_*)^*(x) = \inf_{N_x} \sup_{z \in N_x} \phi_*(z) \geqq \phi(x) - \epsilon.$$

Since ϵ is arbitrary we have

$$(\phi_*)^* \geqq \phi.$$

Since ϕ is upper semicontinuous we have by (3.3) and (3.4)

$$(\phi_*)^* \leqq \phi^* = \phi.$$

Hence $(\phi_*)^* = \phi$ and the proof is complete.

Now a lower semicontinuous function can be characterized by the condition that $\{x \mid \phi(x) > \lambda\}$ is open for each real λ. A dual result holds for upper semicontinuous functions. Normal upper semicontinuous functions can also be characterized in a similar manner.

THEOREM 3.2. *An upper semicontinuous function ϕ on S is normal if and only if for each real λ, $\{x \mid \phi(x) > \lambda\}$ is a union of closures of open sets.*

Let us suppose first that $\phi = (\phi_*)^*$ and let $A = \{x \mid \phi(x) > \lambda\}$. Let x_0 be an arbitrary element of A. Then $\phi(x_0) > \lambda$ and hence $\phi(x_0) > \lambda + \delta$ for some $\delta > 0$. Let $B = \{x \mid \phi_*(x) > \lambda + \delta\}$. Clearly B is open since ϕ_* is lower semicontinuous. If N is an arbitrary open set containing x_0, then

$$\sup_{y \in N} \phi_*(y) \geqq (\phi_*)^*(x_0) = \phi(x_0) > \lambda + \delta.$$

Hence $\phi_*(y) > \lambda + \delta$ for some $y \in N$. Thus $B \cap N \neq 0$ for all N and hence $x_0 \in \overline{B}$. Moreover, if $y_0 \in \overline{B}$, then $B \cap N \neq 0$ for every open set N containing y_0 and thus

$$\sup_{y \in N} \phi_*(y) > \lambda + \delta \quad \text{all } N \text{ containing} \quad y_0.$$

Hence $\phi(y_0) = (\phi_*)^*(y_0) \geqq \lambda + \delta > \lambda$ and thus $y_0 \in A$. But then $x_0 \in \overline{B} \subseteq A$ and it follows that A is a union of closures of open sets.

On the other hand, suppose that ϕ is upper semicontinuous and that

$\{x|\phi(x)>\lambda\}$ is a union of closures of open sets for each real λ. Let $\epsilon>0$, $x_0\in S$, and N be an arbitrary open set containing x_0. Then $\{x|\phi(x)>\phi(x_0)-\epsilon\}$ is a union of closures of open sets and hence there exists an open set $A_1\subseteq\{x|\phi(x)>\phi(x_0)-\epsilon\}$ such that $x_0\in\overline{A}_1$. But then $A=A\cap N$ is a nonempty open set contained in N such that $\phi(y)>\phi(x_0)-\epsilon$ all $y\in A$. Thus ϕ is normal by Theorem 3.1. This completes the proof of the theorem.

COROLLARY. *Every normal upper semicontinuous function on S is continuous if and only if the closure of every open subset of S is open.*

For by Theorem 3.2 the characteristic function of the closure of an open set is upper semicontinuous and normal. Hence if every normal upper semicontinuous function is continuous, the closure of every open set is open. Conversely, if the closure of every open set is open and ϕ is any normal upper semicontinuous function on S, then by Theorem 3.2, ϕ is lower semicontinuous and hence continuous.

4. **Normal subsets of** $C(S)$. Before applying these results to the completion problem we shall recall some relevant facts from the theory of partially ordered sets(5). A subset S of a partially ordered set P is normal if S contains all a for which $a\geqq x$ for every x such that $y\geqq x$ for all $y\in S$. If X is an arbitrary subset of S, the set of all x containing all elements of X is normal. In particular, for each a the set of all $x\geqq a$ is a normal subset called the *principal* normal subset generated by a. The collection of normal subsets of P form a complete lattice containing P as the partially ordered set of principal normal subsets and preserving sup and inf whenever they exist in P. This normal completion is minimal in the sense that if P is imbedded in any other complete lattice L, the lattice of normal subsets is isomorphic with a lattice within L.

In the present case P is the lattice $C(S)$ of continuous functions on S. If $\phi\in B(S)$, let L_ϕ denote the set of all $f\in C(S)$ such that $f\geqq\phi$.

LEMMA 4.1. *If $\phi\in B(S)$, then* $\inf(L_\phi)=\phi^*$.

Since $\phi^*(x)=\inf_{N_x}\sup_{y\in N_x}\phi(y)$, for $\epsilon>0$ there exists an open set N containing x such that $\phi^*(x)>\sup_{y\in N}\phi(y)-\epsilon$. By complete regularity, $g\in C(S)$ exists such that $g(x)=1$, $g(y)=0$ all $y\in{}'N$ and $g\leqq 1$. Let $m=\sup_{y\in S}\phi(y)$ and let

$$f = m - (m - \sup_{y\in N}\phi(y))g.$$

Clearly $f\in C(S)$. If $y\in N$, then $f(y)\geqq m-(m-\sup_{y\in N}\phi(y))\geqq\phi(y)$. If $y\in{}'N$, then $f(y)=m\geqq\phi(y)$. Hence $f\geqq\phi$ and thus $f\in L_\phi$. We have then

$$\phi^*(x) > \sup_{y\in N}\phi(y) - \epsilon = f(x) - \epsilon \geqq \psi(x) - \epsilon$$

(5) The reader is referred to Birkhoff [1] for an account of this theory.

where $\psi=\inf (L_\phi)$. Since ϵ is arbitrary, $\phi^*(x)\geqq\psi(x)$ for all x. On the other hand, $f\geqq\phi$ implies $f=f^*\geqq\phi^*$, which implies $\psi\geqq\phi^*$. Hence $\phi^*=\psi=\inf (L_\phi)$.

LEMMA 4.2. *Let ϕ be a normal, upper semicontinuous function on S. Then L_ϕ is a normal subset of $C(S)$.*

For let $f\geqq g$ for all g contained in the functions of L_ϕ. We must show that $f\in L_\phi$. Let $x\in S$ and let $\epsilon>0$. Since f is continuous, there exists an open set N containing x such that $f(y)-f(x)<\epsilon/2$ all $y\in N$. Since ϕ is normal there exists a non-empty open set $A\subseteq N$ such that $\phi(y)>\phi(x)-\epsilon/2$ all $y\in A$. Let y_0 be a point of A. By complete regularity, there exists a continuous function $h(y)$ such that $h\leqq 1$, $h(y_0)=1$, and $h(y)=0$ all $y\in 'A$. Let $m_\epsilon=\inf_{y\in S}\phi(y)-\epsilon/2$ and set $g=m_\epsilon+(\phi(x)-\epsilon/2-m_\epsilon)h$. Now if $y\in A$, then

$$g(y) \leqq m_\epsilon + (\phi(x) - \epsilon/2 - m_\epsilon) = \phi(x) - \epsilon/2 < \phi(y).$$

But if $y\in 'A$, then $g(y)=m_\epsilon<\phi(y)$. Hence $g\leqq\phi$ and thus g is a continuous function contained in all of the functions of L_ϕ. It follows that $f\geqq g$. But then

$$f(y_0) \geqq g(y_0) = m_\epsilon + (\phi(x) - \epsilon/2 - m_\epsilon) = \phi(x) - \epsilon/2.$$

Since $y_0\in A\subseteq N$ we have

$$f(x) = f(y_0) + (f(x) - f(y_0)) > \phi(x) - \epsilon.$$

Since ϵ is arbitrary, $f(x)\geqq\phi(x)$ for all x and hence $f\in L_\phi$. This completes the proof of the lemma.

We need also a converse result.

LEMMA 4.3. *Let $\mathfrak{A}$ be a normal subset of $C(S)$. Then* $\inf (\mathfrak{A})$ *is a normal, upper semicontinuous function on S.*

For let $\phi=\inf (\mathfrak{A})$ and let $\phi_*\leqq f$ where $f\in C(S)$. Then if g is contained in all of the functions of $\mathfrak{A}$, we have $g\leqq\phi$ and hence $g\leqq\phi_*\leqq f$. Hence $f\in\mathfrak{A}$ since $\mathfrak{A}$ is normal. But then by Lemma 4.1

$$(\phi_*)^* = \inf (L_{\phi_*}) = \inf (\mathfrak{A}) = \phi.$$

Thus ϕ is a normal, upper semicontinuous function and the lemma follows.

With these lemmas we are ready to prove the fundamental isomorphism theorem.

THEOREM 4.1. *Let S be a completely regular topological space. Then the completion of $C(S)$ by normal subsets is isomorphic with the lattice of all normal, upper semicontinuous real functions on S.*

For the proof let us recall that $B(S)$ is a complete lattice containing $C(S)$ as a sublattice and hence it follows from the general theory of the normal completion of a partially ordered set that if $\mathfrak{A}$ is a normal subset $C(S)$ the mapping $\mathfrak{A}\rightarrow\inf (\mathfrak{A})$ is an isomorphism. By Lemma 4.3, $\mathfrak{A}$ is mapped into the set of normal, upper semicontinuous real functions on S. But by Lemmas 4.1 and 4.2 every normal upper semicontinuous function is an image of a normal subset of $C(S)$. The proof is thus complete.

If $\psi \in B(S)$ and $\phi = (\psi_*)^*$, then $(\phi_*)^* = \phi$ by (3.6). Conversely, if $(\phi_*)^* = \phi$, then ϕ trivially has the form $(\psi_*)^*$ with $\psi \in B(S)$. Hence Theorem 4.1 can also be stated in the following way.

COROLLARY. *If S is completely regular, then the normal completion of $C(S)$ is isomorphic to the lattice of all functions of the form $(\psi_*)^*$ where ψ is a bounded real function on S.*

Now it is clear from Theorem 3.2 that sup (ϕ_1, ϕ_2) where ϕ_1 and ϕ_2 are normal upper semicontinuous functions is also upper semicontinuous and normal. Hence sup (ϕ_1, ϕ_2) is the lattice union of ϕ_1 and ϕ_2. However, if $\mathfrak{A}$ is a bounded class of normal upper semicontinuous functions, sup $(\mathfrak{A})$ need not be normal. For example, let ϕ be defined over the real interval $[0, 1]$ by $\phi(x) = 1$ when $x \neq 1/2$ and $\phi(1/2) = 0$. Let $\mathfrak{A}$ be the set of all continuous functions f such that $f \leqq \phi$. Then sup $(\mathfrak{A}) = \phi$ and ϕ is not normal. Also it should be noted that inf (ϕ_1, ϕ_2) need not be normal if ϕ_1 and ϕ_2 are normal. For example, let ϕ_1, ϕ_2 be the characteristic functions of the closed intervals $[0, 1/2]$ and $[1/2, 1]$ respectively. Then $\{x \mid \inf (\phi_1, \phi_2) > 0\}$ consists of the single point $x = 1/2$ and hence is not a union of closures of open sets.

The general determination of the lattice operations in the set of normal upper semicontinuous functions is contained in the following theorem.

THEOREM 4.2. *Let S be an arbitrary topological space and let $\mathfrak{A}$ be a bounded collection of normal upper semicontinuous functions on S. Then the unique minimal normal upper semicontinuous function containing the functions of $\mathfrak{A}$ is* (sup $\mathfrak{A})^*$, *while the unique maximal normal upper semicontinuous function contained in the functions of $\mathfrak{A}$ is* $((\inf \mathfrak{A})_*)^*$.

For by (3.3), (3.4), and (3.5) we have $(((\sup \mathfrak{A})^*)_*)^* \leqq (\sup \mathfrak{A})^*$. On the other hand, since sup $\mathfrak{A} \geqq \phi$ all $\phi \in \mathfrak{A}$, we have $(((\sup \mathfrak{A})^*)_*)^* \geqq ((\phi^*)_*)^* = (\phi_*)^* = \phi$ for all $\phi \in \mathfrak{A}$. Hence $(((\sup \mathfrak{A})^*)_*)^* \geqq \sup \mathfrak{A}$ and thus $(((\sup \mathfrak{A})^*)_*)^* \geqq (\sup \mathfrak{A})^*$. We conclude that $(\sup \mathfrak{A})^*$ is normal. If ψ is a normal upper semicontinuous function such that $\psi \geqq \phi$ all $\phi \in \mathfrak{A}$, then $\psi \geqq \sup \mathfrak{A}$ and hence $\psi = \psi^* \geqq (\sup \mathfrak{A})^*$. Thus the first conclusion of the theorem holds. Now if $\psi \leqq \phi$ all $\phi \in \mathfrak{A}$, then $\psi \leqq \inf \mathfrak{A}$ and hence $\psi = (\psi_*)^* \leqq ((\inf \mathfrak{A})_*)^*$ and $((\inf \mathfrak{A})_*)^*$ is a normal, upper semicontinuous function by (3.6). The proof is thus complete.

5. **An application.** We show now that the results of §§3 and 4 contain as a special case the theorem of Stone [5, 6] and Nakano [3] on complete lattices of continuous functions.

THEOREM 5.1 (Stone-Nakano). *If S is a topological space in which the* *closure of every open set is open, then $C(S)$ is complete. Conversely, if $C(S)$ is complete and S is completely regular, then the closure of every open set is open.*

For if the closure of every open set is open, by the corollary to Theorem 3.2, every normal upper semicontinuous function is continuous and by Theorem 4.2, $C(S)$ is complete. Conversely, if $C(S)$ is complete and S is completely regular, by Theorem 4.1 every normal upper semicontinuous function is continuous and hence by the corollary to theorem 3.2, the closure of every open set is open.

PART II. THE BOOLEAN SPACE ASSOCIATED WITH THE NORMAL COMPLETION

6. The second representation theorem. In this section it will be shown that the normal completion of the lattice of continuous functions on a topological space is isomorphic to the lattice of all continuous functions on another suitably determined topological space. Now it is well known (Birkhoff [1]) that the regular open sets(6) of a topological space form a complete Boolean algebra under set inclusion. Furthermore, with any Boolean algebra there is associated the Boolean space of minimal dual ideals. The precise theorem to be proved is the following:

THEOREM 6.1. *Let S be completely regular. Then the normal completion of $C(S)$ is isomorphic with the lattice of all continuous functions on the Boolean space(7) associated with the Boolean algebra of regular open sets of S.*

Let $\mathfrak{S}$ denote the Boolean space associated with the Boolean algebra Σ of regular open sets of S. Thus $\mathfrak{S}$ is the set of all minimal dual ideals(8) of Σ. The topology in $\mathfrak{S}$ is such that the closure of a subset $\mathfrak{A}$ of $\mathfrak{S}$ consists of all minimal dual ideals $\mathfrak{p}$ of $\mathfrak{S}$ for which $\bigcup\mathfrak{A}\supseteq\mathfrak{p}$ in the lattice of dual ideals.

We next define a pair of correspondences, σ and τ, one of which maps $B(S)$ into $B(\mathfrak{S})$ while the other maps $B(\mathfrak{S})$ into $B(S)$. The mapping σ is defined by

$$\sigma f(\mathfrak{p}) = \inf_{P\in\mathfrak{p}} \sup_{y\in P} f(y). \tag{6.1}$$

Thus for each regular open set $P\in\mathfrak{p}$, the upper bound of f on P is calculated and the lower bound of these values for all $P\in\mathfrak{p}$ is $\sigma f(\mathfrak{p})$. The mapping τ is defined by

$$\tau F(x) = \inf_{x\in A} \sup_{A\in\mathfrak{q}} F(\mathfrak{q}). \tag{6.2}$$

Thus for each regular open set A containing x, the upper bound of $F(\mathfrak{q})$ for all

(6) See Birkhoff [1, p. 177].

(7) See Stone [4].

(8) The minimal dual ideals of the lattice Σ are in one-to-one correspondence with the maximal ring ideals of Σ as a Boolean ring.

$\mathfrak{q}$ containing A is calculated and the lower bound of these values for all A containing x is $\tau F(x)$.

The proof of Theorem 6.1 will rest on a series of lemmas concerning the mappings σ, τ.

LEMMA 6.1. *If $f^* \geqq g$, then $\sigma f \geqq \sigma g$. Dually, if $F^* \geqq G$, then $\tau F \geqq \tau G$.*

For if N is any open set, we have

$$\sup_{x \in N} f(x) = \sup_{x \in N} f^*(x).$$

Hence

$$\sigma f(\mathfrak{p}) = \inf_{P \in \mathfrak{p}} \sup_{x \in P} f(x) = \inf_{P \in \mathfrak{p}} \sup_{x \in P} f^*(x) \geqq \inf_{P \in \mathfrak{p}} \sup_{x \in P} g(x) = \sigma g(\mathfrak{p}).$$

If A is any regular open set of S, then the set of all $\mathfrak{q}$ containing A is both open and closed and hence

$$\sup_{A \in \mathfrak{q}} F(\mathfrak{q}) = \sup_{A \in \mathfrak{q}} F^*(\mathfrak{q}).$$

Thus if $F^* \geqq G$, we have

$$\tau F(x) = \inf_{x \in A} \sup_{A \in \mathfrak{q}} F(\mathfrak{q}) = \inf_{x \in A} \sup_{A \in \mathfrak{q}} F^*(\mathfrak{q}) \geqq \inf_{x \in A} \sup_{A \in \mathfrak{q}} G(\mathfrak{q}) = \tau G(x).$$

LEMMA 6.2. *σf and τF are upper semicontinuous for each $f \in B(S)$ and $F \in B(\mathfrak{S})$.*

For let $\sigma f(\mathfrak{p}) < \lambda$. Then $P \in \mathfrak{p}$ exists such that $\sup_{y \in P} f(y) < \lambda$. If $P \in \mathfrak{q}$, then $\sigma f(\mathfrak{q}) \leqq \sup_{y \in P} f(y) < \lambda$. Since the set of all $\mathfrak{q}$ containing P is both open and closed, it follows that $\{\mathfrak{p} | \sigma f(\mathfrak{p}) < \lambda\}$ is open and hence σf is upper semicontinuous.

Similarly if $\tau F(x) < \lambda$, then there exists a regular open set A containing x such that $\sup_{A \in \mathfrak{p}} F(\mathfrak{p}) < \lambda$. Hence if $y \in A$, we have $\tau F(y) \leqq \sup_{A \in \mathfrak{p}} F(\mathfrak{p}) < \lambda$. Since A is open $\{x / \tau F(x) < \lambda\}$ is open and τF is thus upper semicontinuous.

LEMMA 6.3. *If f is a normal, upper semicontinuous function on S, then σf is a continuous function on $\mathfrak{S}$.*

For let $\sigma f(\mathfrak{p}) > \lambda$ and suppose that for each $P \in \mathfrak{p}$ there exists a $\mathfrak{q}$ containing P such that $\sigma f(\mathfrak{q}) \leqq \lambda$. Let $\sigma f(\mathfrak{p}) > \lambda_1 > \lambda$. Then $\sigma f(\mathfrak{q}) < \lambda_1$, and hence there exists $Q \in \mathfrak{q}$ such that $f(y) < \lambda_1$ all $y \in Q$. Now $P \cap Q$ belongs to $\mathfrak{q}$ and hence is non-empty. Let

$$W = \{y | f(y) < \lambda_1\}.$$

Then $P \cap Q \subseteq W \subseteq \overline{W}$ and if B denotes the interior of $\overline{W}$ we have $P \cap Q \subseteq B$. Hence $B \cap P \neq 0$ for every $P \in \mathfrak{p}$ and thus $(B) \cap \mathfrak{p} \neq 0$. But then $B \in \mathfrak{p}$. On the other hand, since $B \subseteq \overline{W}$, $f(y) < \lambda_1$, on a set dense in B. Hence $f_*(y) \leqq \lambda_1$

for all $y \in B$. But then $(f_*)^*(y) \leqq \lambda_1$ all $y \in B$. From the normality of f we get

$$f(y) = (f_*)^*(y) \leqq \lambda_1 \qquad \text{all} \quad y \in B.$$

Hence $\sigma f(\mathfrak{p}) \leqq \sup_{y \in B} f(y) \leqq \lambda_1 < \sigma f(\mathfrak{p})$ which is impossible. Thus for some $P \in \mathfrak{p}$, $\sigma f(\mathfrak{q}) > \lambda$ all A containing P. It follows that $\{\mathfrak{p} | \sigma f(\mathfrak{p}) > \lambda\}$ is open for each λ and hence σf is lower semicontinuous. But then Lemma 6.2 implies that σf is continuous.

LEMMA 6.4. *If S is regular and F is a lower semicontinuous function on $\mathfrak{S}$, then τF is a normal, upper semicontinuous function on S.*

Now τF is upper semicontinuous by Lemma 6.2. Hence if τF is not normal, by Theorem 3.1 there exists $\epsilon > 0$, $x \in S$ and open set N containing x such that $U = \{y | \tau F(y) \leqq \tau F(x) - \epsilon\}$ is dense in N. By regularity there exists a regular open set A such that $x \in A \subseteq N$. It follows that $A \cap U$ is dense in A. Let $\tau F(x) > \lambda > \tau F(x) - \epsilon$. If $y \in A \cap U$ then $\tau F(y) < \lambda$, and hence a regular open set A_y containing y exists such that $F(\mathfrak{p}) < \lambda$ all $\mathfrak{p}$ containing A_y. Let $\mathfrak{A}$ be the collection of all $\mathfrak{p}$ for which $A_y \in \mathfrak{p}$ for some $y \in A \cap U$. Let $B \in \cup \mathfrak{A}$. Then $B \supseteq A_y$ all $y \in A \cap U$ and hence $B \supseteq A \cap U$. Since $A \cap U$ is dense in A we have $\overline{B} \supseteq A$. But B is a regular open set and hence $B \supseteq A$. Thus $\cup \mathfrak{A} \supseteq \mathfrak{p}$ all $\mathfrak{p}$ containing A. But $F(\mathfrak{q}) < \lambda$ all $\mathfrak{q} \in \mathfrak{A}$ and hence by the lower semicontinuity of F, $F(\mathfrak{p}) \leqq \lambda$ all $\mathfrak{p}$ containing A. But then

$$\tau F(x) \leqq \sup_{A \in \mathfrak{p}} F(\mathfrak{p}) \leqq \lambda < \tau F(x),$$

which is impossible. It follows that τF is a normal, upper semicontinuous function on S.

LEMMA 6.5. *If S is regular and $f \in B(S)$, then $\tau\sigma f \leqq f^*$.*

For all regular open sets A containing x we have

$$\sup_{A \in \mathfrak{p}} \sigma f(\mathfrak{p}) \geqq \tau\sigma f(x).$$

Thus if $\epsilon > 0$, for each A containing x, there exists a $\mathfrak{p}$ containing A such that $\sigma f(\mathfrak{p}) > \tau\sigma f(x) - \epsilon$. But then

$$\sup_{y \in A} f(y) \geqq \sigma f(\mathfrak{p}) > \tau\sigma f(x) - \epsilon.$$

Thus $f(y) > \tau\sigma f(x) - \epsilon$ for some y in each A containing x. Hence if S is regular, $f^*(x) \geqq \tau\sigma f(x) - \epsilon$. Since ϵ is arbitrary we have $f^* \geqq \tau\sigma f$.

LEMMA 6.6. *If f is a normal, upper semicontinuous function on S, then $\tau\sigma f \geqq f$.*

For if $\epsilon > 0$ and x is any element of S, there exists a regular open set A such that $\sigma f(\mathfrak{p}) < \tau\sigma f(x) + \epsilon$ for all $\mathfrak{p}$ containing A. But then for some $P \in \mathfrak{p}$

we have $f(y) < \tau\sigma f(x) + \epsilon$ all $y \in P$. Let

$$W = \{y \mid f(y) < \tau\sigma f(x) + \epsilon\}.$$

Then for each $\mathfrak{p}$ containing A, there is a $P \in \mathfrak{p}$ such that $W \supseteq P$. Thus if B denotes the interior of $\overline{W}$, $B \supseteq P$ and hence $B \in \mathfrak{p}$ all $\mathfrak{p}$ containing A. But then $B \supseteq A$ and hence $x \in B$. Now $f(y) < \tau\sigma f(x) + \epsilon$ on a set dense in $\overline{W}$ and hence dense in B. Thus $f_*(y) \leqq \tau\sigma f(x) + \epsilon$ for all $y \in B$. By the normality of f we have

$$f(y) = (f_*)^*(y) \leqq \tau\sigma f(x) + \epsilon \qquad \text{all} \quad y \in B.$$

In particular, $f(x) \leqq \tau\sigma f(x) + \epsilon$ for each $\epsilon > 0$. Thus $f \leqq \tau\sigma f$.

LEMMA 6.7. *If $F \in B(\mathfrak{S})$, then $F_* \leqq \sigma\tau F \leqq F^*$.*

For let $\epsilon > 0$ and let $\mathfrak{p}$ be an arbitrary element of $\mathfrak{S}$. Then $P \in \mathfrak{p}$ exists such that $\tau F(y) < \sigma\tau F(\mathfrak{p}) + \epsilon$ all $y \in P$. But for each $y \in P$ there exists a regular open set A_y containing y, such that $F(\mathfrak{q}) < \sigma\tau F(\mathfrak{p}) + \epsilon$ all $\mathfrak{q} \in A_y$. Let $\mathfrak{A}_1$ be the set of all $\mathfrak{q}$ containing A_y for some y. If $B \in \bigcup\mathfrak{A}_1$, then $y \in A_y \subseteq B$ for all y and hence $P \subseteq B$. But then $B \in \mathfrak{p}$ and hence $\bigcup\mathfrak{A}_1 \supseteq \mathfrak{p}$. Since $\mathfrak{p} \in \overline{\mathfrak{A}}_1$, we have

$$F_*(\mathfrak{p}) \leqq \sigma\tau F(\mathfrak{p}) + \epsilon.$$

But ϵ is arbitrary, and hence $F_* \leqq \sigma\tau F$.

On the other hand, for every $P \in \mathfrak{p}$ we have

$$\sup_{y \in P} \tau F(y) \geqq \sigma\tau F(\mathfrak{p}).$$

Hence if $\epsilon > 0$, there is a $y \in P$ such that $\tau F(y) > \sigma\tau F(\mathfrak{p}) - \epsilon$ and thus $\sup_{P \in \mathfrak{q}} F(\mathfrak{q}) > \sigma\tau F(\mathfrak{p}) - \epsilon$. Let $\mathfrak{A}_2 = \{\mathfrak{q} \mid F(\mathfrak{q}) > \sigma\tau F(\mathfrak{p}) - \epsilon\}$. Then $\bigcup\mathfrak{A}_2 \cap (P) \neq 0$ for every $P \in \mathfrak{p}$. Thus $\bigcup\mathfrak{A}_2 \cap \mathfrak{p} \neq 0$ and hence $\bigcup\mathfrak{A}_2 \supseteq \mathfrak{p}$. Since $\mathfrak{p}$ is a limit point of $\mathfrak{A}_2$, we have

$$F^*(\mathfrak{p}) \geqq \sigma\tau F(p) - \epsilon.$$

But ϵ is arbitrary, and hence $F^* \geqq \sigma\tau F$.

Proof of Theorem 6.1. By Lemma 6.3, σ maps normal, upper semicontinuous functions on S into continuous functions on $\mathfrak{S}$. By Lemmas 6.5 and 6.6, distinct normal semi-continuous functions on S map into different continuous functions. By Lemmas 6.4 and 6.7, every continuous function on $\mathfrak{S}$ is an image of a normal, upper semicontinuous function on S. Finally, Lemma 6.1 shows that the mapping is an isomorphism. Hence the theorem follows from Theorem 4.1 of Part I.

It should be noted that if $C(S)$ is lattice complete, then the regular open sets are simply the open and closed set of S and Boolean space of Theorem 6.1 is the Stone-Čech compactification of S.

7. **Special cases.** Birkhoff [1, p. 177] has shown that if S is a completely regular space without isolated points and satisfying the second

countability axiom, then the Boolean algebra of regular open sets is isomorphic with the normal completion of the free Boolean algebra with a countably infinite set of generators. Applying Theorem 6.1 to this case we obtain the following theorem.

THEOREM 7.1. *Let S be a completely regular space without isolated points and satisfying the second countability axiom. Then the normal completion of $C(S)$ is isomorphic with the lattice of all continuous functions on the Boolean space associated with the normal completion of the free Boolean algebra with a countably infinite set of generators.*

As an immediate consequence we have the following corollary.

COROLLARY. *All completely regular spaces without isolated points and satisfying the second countability axiom have the same normal completion for their lattices of continuous functions.*

In particular, Theorem 7.1 gives a simple representation of the normal completion of the lattice of continuous functions on the interval $[0, 1]$. According to the corollary, the Cantor set and the real line also have lattices of continuous functions with this same normal completion.

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