

EXTENSIONS OF MEASURES AND THE VON NEUMANN SELECTION THEOREM¹

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ABSTRACT. Let (X, B_X) be a Blackwell space, where B_X is the σ -algebra of Borel sets. Then if σ is a finite measure defined on a countably generated sub- σ -algebra $B \subset B_X$, σ can be extended to a Borel measure τ . Equivalently, if X and Y are Blackwell and $f: X \rightarrow Y$ is Borel, and μ is a Borel measure carried on $f(X) \subset Y$, then there exists a Borel measure τ on X with $\tau' = \sigma$, where $\tau'(E) = \tau(f^{-1}(E))$. We characterize $\{\tau | \tau' = \sigma\}$ if f is semischlicht.

Let B_X denote the Borel sets of a topological space X . We consider the following measure extension (or equivalently restriction) problem: given a measure (we will always mean finite measure) σ defined on a σ -algebra $B \subset B_X$, can σ be extended to all of B_X , i.e., does there exist a Borel measure τ such that $\tau(E) = \sigma(E)$ for all $E \in B$? It is well known (see [1, p. 71], for details) that if B_1 and B_2 are σ -algebras, and B_2 is generated by B_1 and finitely many additional sets, then any measure on B_1 can be extended to B_2 . The result is not known for countably generated extensions. We show below (Theorem 5) that if X is a Blackwell space and B is a countably generated sub- σ -algebra of B_X , then any measure on B extends to B_X .

A Blackwell space is a measure space (X, B_X) , where X is an analytic subset of a complete separable metric space (c.s.m.). A subset A of a c.s.m. is analytic iff A is the continuous image of a c.s.m. We note that the analytic sets form a proper subset of U_X , the set of absolutely measurable subsets of X , where $E \in U_X$ iff E is $\bar{\mu}$ -measurable for all finite Borel measures μ , where " $\bar{\mu}$ " denotes the completion of μ , i.e., given μ , there exist $E_1, E_2 \in B_X$ such that $E_1 \subset E \subset E_2$ and $\mu(E_2 - E_1) = 0$. A function g is said to be absolutely measurable if $g^{-1}(V) \in U_X$ for all open V . Details may be found in [3], [4], or [5]. We note that if $X \subset S$, X analytic, S a c.s.m., then $B_X = \{E \cap X | E \in B_S\}$, so elements of B_X are topologically analytic, and not necessarily Borel in S .

We begin by considering a special class of sub- σ -algebras of B_X . Let $f: X \rightarrow Y$ be Borel measurable, and let $B_f = \{f^{-1}(E) | E \in B_Y\}$. Given a Borel

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measure τ on X , let τ^f be the Borel measure in Y defined by $\tau^f(E) = \tau(f^{-1}(E))$. It follows from a well known isomorphism theorem (see [7, p. 270]) that if τ is a positive Borel measure on a c.s.m. X , and μ is a positive Borel measure on a c.s.m. Y with $\mu(Y) = \tau(X)$ and $\tau(\{x\}) = 0$ for all $x \in X$, then there exists a Borel function $f: X \rightarrow Y$ with $\tau^f = \mu$. The following shows that a dual of this is equivalent to our extension problem.

PROPOSITION 1. *Let $f: X \rightarrow Y$ be Borel measurable. Then every measure on B_f is the restriction of a Borel measure iff for every μ defined on B_Y with $\bar{\mu}(f(X)^c) = 0$ (i.e., μ is carried on $f(X)$), there exists τ on B_X with $\tau^f = \mu$.*

PROOF. Assume that all Borel μ carried on $f(X) \subset Y$ are of the form $\mu = \tau^f$, and let σ be a measure on $B_f \subset B_X$. Define μ on B_Y by $\mu(E) = \sigma(f^{-1}(E))$, $E \subset Y$. By assumption, there exists a Borel measure τ with $\tau^f = \mu$. Thus, $\tau(f^{-1}(E)) = \mu(E) = \sigma(f^{-1}(E))$, so τ extends σ .

Conversely, suppose the extension property holds. Then given a Borel measure μ carried on $f(X) \subset Y$, define σ on B_f by $\sigma(f^{-1}(E)) = \mu(E)$. (Note σ is well-defined since μ is carried on $f(X)$.) By assumption, σ is the restriction of some τ defined on B_X , and clearly $\tau^f = \mu$.

The proof of our main result relies on the following “selection” theorem of von Neumann.

THEOREM 2 [6]. *Let A be an analytic subset of a c.s.m. S , and let F be a continuous real-valued function on A . Then there exists an absolutely measurable $G: F(A) \rightarrow A$ such that $F \circ G$ is the identity on $F(A)$.*

COROLLARY 3. *If X and Y are Blackwell and $F: X \rightarrow Y$ is Borel, there exists an absolutely measurable $G: Y \rightarrow X$ such that $F \circ G$ is the identity on $F(X)$.*

PROOF. Since Y is Blackwell, we may assume $Y \subseteq \mathbb{R}$, the real line. Let $(F \times I): (X \times Y) \rightarrow (Y \times X)$ be defined by $(F \times I)((x, y)) = (F(x), y)$, and let P_j be projection on the j th coordinate, $j = 1, 2$. Then

$$A = \{(x, F(x)) \mid x \in X\} = (F \times I)^{-1}(\{(y, y)\})$$

is Borel in $X \times Y$, and is thus an analytic subset of a c.s.m. Since $P_2: A \rightarrow Y$ is continuous, there exists an absolutely measurable $g: P_2(A) \rightarrow A$ with $P_2 \circ g$ the identity on $P_2(A) = F(X)$. Then $G = P_1 \circ g$ satisfies the theorem.

REMARK 4. (i) If $f: X \rightarrow Y$ is Borel and B_Y is countably generated, then B_f is countably generated.

(ii) If X is Blackwell and $B \subset B_X$ is countably generated, then there exists a Borel $f: X \rightarrow \mathbb{R}$ with $B = B_f$. (This was pointed out to me by the referee.)

PROOF. (i) follows immediately.

Suppose $\{E_n\}$ generates the σ -algebra B . Consider $f(x) = \sum_n 3^{-n} \chi_{E_n(x)}$. Then since B and B_f clearly have the same atoms and X is Blackwell, it follows that $B = B_f$ (see [5, p. 38]).

THEOREM 5. *Let X be Blackwell and let σ be a finite measure on a countably generated sub- σ -algebra B . Then σ has an extension τ to the full σ -algebra B_X .*

We first prove the following

LEMMA. *If $f: X \rightarrow Y$ is absolutely measurable and $S \subset Y$ is absolutely measurable, then $f^{-1}(S) \subset X$ is absolutely measurable. In particular, a composition of absolutely measurable functions is absolutely measurable.*

PROOF. Given any μ on B_X , consider $\nu = (\bar{\mu})^f$ defined on B_Y . Since S is absolutely measurable, there exist $S_1, S_2 \in B_Y$ with $S_1 \subset S \subset S_2$, $\nu(S_2 - S_1) = 0$. Let $T_i = f^{-1}(S_i)$. Then $T_1 \subset f^{-1}(S) \subset T_2$, T_1, T_2 are absolutely measurable, and $\bar{\mu}(T_2 - T_1) = \bar{\mu}(f^{-1}(S_2 - S_1)) = \nu(S_2 - S_1) = 0$. Thus, $f^{-1}(S)$ is $\bar{\mu}$ -measurable.

PROOF OF THEOREM 5. Given σ on B , we have $B = B_f$ where $f: X \rightarrow \mathbf{R}$ is Borel. Then there exists an absolutely measurable $g: f(X) = Y \rightarrow X$ with $f \circ g = i_Y$. Define τ on B_X by $\tau(S) = \bar{\sigma}((g \circ f)^{-1}(S))$. (This makes sense, since $g \circ f$ is absolutely measurable.) Clearly, if $S \in B_f$, there exists $T \subset Y$ such that $S = f^{-1}(T)$ and $T \in B_Y$.

Then

$$\begin{aligned} \tau(S) &= \bar{\sigma}((g \circ f)^{-1}(S)) = \bar{\sigma}(f^{-1}(g^{-1}(S))) \\ &= \bar{\sigma}(f^{-1}((f \circ g)^{-1}(T))) = \bar{\sigma}(f^{-1}(T)) \\ &= \bar{\sigma}(S) = \sigma(S). \end{aligned}$$

Thus, τ extends σ to B_X .

COROLLARY 6. *Let $f: X \rightarrow Y$ be Borel, where X and Y are Blackwell, and let μ be a Borel measure carried on $f(X) \subset Y$. Then there exists τ on B_X with $\tau^f = \mu$.*

The proof of the corollary is explicit in the proof of the theorem. We note that the τ obtained is of the form $\tau(E) = \mu(g^{-1}(E)) = \mu(f(E \cap S))$, where $S = g(Y)$. S thus consists of one point "selected" from each pre-image set $f^{-1}(\{y\})$. Hence, τ is carried on a 1-1 set, i.e., a section, of f .

If f is 1-1, then τ is clearly unique. Suppose that f is semischlicht, i.e., that $f^{-1}(\{y\})$ is a countable set for all $y \in Y$. Then there is a countable collection of disjoint Borel sets $\{S_n\}$ such that $\bigcup S_n = X$, $f_n = f|_{S_n}$ is 1-1, and $f(S_n) = f(X - \bigcup_{j < n} S_j)$. (See [2, p. 335].) Given a Borel measure σ on Y , let τ_n be defined on B_X by $\tau_n(E) = \sigma(f(E \cap S_n))$. Then $\tau_n^f = \sigma|_{f(S_n)}$.

Choose $a_n \in L^1(\tau_n)$ such that $\sum_n a_n(f_n^{-1}(y))=1$ a.e. $[\sigma]$, where

$$a_n(f_n^{-1}(y)) = 0 \quad \text{if } y \notin f_n(S_n),$$

and the series is absolutely convergent. Let τ be defined on B_X by $\tau(E)=\sum_n \int a_n(x) d\tau_n(x)$, i.e., $d\tau(x)=\sum_n a_n(x) d\tau_n(x)$ so τ is a ‘‘convex combination’’ of the measures τ_n . Then it is easy to see that $\tau^f=\sigma$, and we see below that this characterizes $\{\tau|\tau^f=\sigma\}$.

PROPOSITION 7. *Let X and Y be Blackwell, and let $f: X \rightarrow Y$ be a semischlicht Borel map. Let σ be a measure defined on B_Y , and suppose that $\{S_n\}$, $\{f_n\}$ and $\{\tau_n\}$ are as above. Then there exists $\{a_n(x)\}$ such that $a_n \in L^1(\tau_n)$, $\sum_n a_n(f_n^{-1}(y))=1$ a.e. $[\sigma]$, and $d\tau(x)=\sum_n a_n(x) d\tau_n(x)$.*

PROOF. Using the Hahn decomposition, we may assume that σ and τ are positive. Then for $E \subset S_n \subset X$,

$$\tau(E) = \tau(f_n^{-1}(f_n(E))) \leq \tau(f^{-1}(f_n(E))) = \sigma(f_n(E)) = \tau_n(E),$$

so by the Radon-Nikodym theorem, there exists $a_n \in L^1(\tau_n)$ with

$$\tau(E) = \int_E a_n(x) d\tau_n(x) = \int_{f_n(E)} a_n(f_n^{-1}(y)) d\sigma(y) \quad \text{if } E \subset S_n.$$

Thus, for $E \subset X$,

$$\tau(E) = \sum_n \tau(E \cap S_n) = \int_E \sum_n a_n(x) d\tau_n(x),$$

so

$$d\tau(x) = \sum_n a_n(x) d\tau_n(x).$$

For $E \subset Y$,

$$\begin{aligned} \sigma(E) &= \tau(f^{-1}(E)) = \sum_n \int_{f_n(f^{-1}(E) \cap S_n)} a_n(f_n^{-1}(y)) d\sigma(y) \\ &= \int_E \left(\sum_n a_n(f_n^{-1}(y)) \right) d\sigma(y). \end{aligned}$$

Hence,

$$\sum_n a_n(f_n^{-1}(y)) = 1 \text{ a.e. } [\sigma].$$

We point out that if, in fact, $\tau(E)=\sigma(f(E \cap S))$ for $E \subset X$, where $f|_S$ is 1-1, then for each n , there exists $D_n \subset S_n$ such that $a_n = \chi_{D_n}$ and $S = \bigcup_n D_n$. Further, the following example shows that the assumption f semischlicht cannot be dropped.

For $a \in [0, 1]$ an irrational number, consider the unique decimal representation $a=0.a_1a_2 \dots$, and let $f: [0, 1] \rightarrow [0, 1]$ be defined by $f(0.a_1a_2a_3 \dots) = 0.a_1a_3a_5 \dots$. Then, after defining f in an appropriate manner on the rational points, which we can ignore since we will be using

nonatomic measures, f maps $[0, 1]$ onto $[0, 1]$, and

$$f^{-1}(0.a_1a_2 \cdots) = \{0.a_1x_1a_2x_2 \cdots \mid x_i \text{ arbitrary}\}.$$

Let $\Phi: [0, 1] \rightarrow ([0, 1] \times [0, 1])$ be defined by

$$\Phi(0.a_1a_2a_3 \cdots) = (0.a_1a_3a_5 \cdots, 0.a_2a_4a_6 \cdots).$$

Let m be Lebesgue measure in $[0, 1]$. Let σ be a singular nonatomic measure on $[0, 1]$. We now define τ on $[0, 1]$ by $\tau(E) = (\sigma \times m)(\Phi(E))$. We note that $\Phi(E)$ is measurable since Φ restricted to the irrationals is a homeomorphism.

Suppose $E = f^{-1}(B)$ for some Borel set $B \subset [0, 1]$. Then

$$E = \{0.a_1x_1a_2x_2 \cdots \mid 0.a_1a_2 \cdots \in B\} \quad \text{and} \quad \Phi(E) = B \times [0, 1].$$

Thus,

$$\begin{aligned} \tau(E) &= \tau(f^{-1}(B)) = (\sigma \times m)(\Phi(E)) \\ &= (\sigma \times m)(B \times [0, 1]) = \sigma(B). \end{aligned}$$

Hence, $\tau^f = \sigma$, and it is easy to see that if $f|_S$ is 1-1, then $\tau(S) = 0$, so τ cannot be written as a "combination" of measures supported on 1-1 sets of f .

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