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On Banach Spaces with the Gelfand-Phillips Property

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1. Introduction and Preliminaries

If X is a Banach space, then $(X)_1$ denotes its closed unit ball, and X^* is the Banach dual space to X; X^* is often considered with its weak* topology, w^*. An "operator" always means a continuous linear operator from one Banach space into another. In general, our terminology is standard, as in [2] or [11].

Throughout, E and F denote Banach spaces.

A (bounded) subset A of E is called *limited* [2] if, for every w^*-null sequence (x_n^*) in E^*, we have $x_n^*(x) \to 0$ uniformly for $x \in A$. If all limited subsets of E are relatively (norm) compact (the converse holds trivially), then E is said to have the *Gelfand-Phillips property* [3] or to be a *Gelfand-Phillips space*; we shall often write $E \in (\mathrm{GP})$ in this case.

In Sect. 2 we prove that $E \in (\mathrm{GP})$ if $(E^*)_1$ contains a subset that is norming and weak* conditionally sequentially compact; this improves on the earlier results mentioned in [2, p. 238] and [3, p. 150], and a recent result in [7]. We also show that if $F \in (\mathrm{GP})$ and T is any topological space containing a dense and conditionally sequentially compact subset, then $C(T,F) \in (\mathrm{GP})$. This result was obtained by the author in [5] with a more direct but much longer proof, and is an improvement of similar results in [2] and [7].

In Sect. 3 we show that if E and F are Gelfand-Phillips spaces, then so is their injective tensor product $E \check{\otimes} F$. The same conclusion was obtained in [7] under the additional assumption that $\mathrm{ext}(E^*)_1$ or $\mathrm{ext}(F^*)_1$ is weak* conditionally sequentially compact. In particular, if T is a compact space, then $C(T,F) \in (\mathrm{GP})$ whenever $C(T) \in (\mathrm{GP})$ and $F \in (\mathrm{GP})$.

Section 4 brings the following result: If E^* and F are in (GP), then also $K(E,F)$, the space of compact operators from E to F, is in (GP).

Finally, in Sect. 5, the main result is that $E \in (\mathrm{GP})$ if E admits a Schauder decomposition with Gelfand-Phillips summands.

In principle, Sects. 2–5 are mutually independent.

The following two results will be frequently used below; (A) can be verified directly, while (B) follows easily from a result due to Bourgain and Diestel [1]. (B) will play a significant role in many of our arguments.

(A) *A sequence* (x_n) *in* E *is limited (i.e., the set of its terms is limited) iff* $x_n^*(x_n) \to 0$ *for each* w^**-null sequence* (x_n^*) *in* E^*.

(B) $E \in (\mathrm{GP})$ *iff every limited weakly null sequence in* E *is norm null.*

Let us also note that continuous linear images of limited sets or sequences are limited, and that any Banach space isomorphic to a subspace of a Gelfand-Phillips space is Gelfand-Phillips.

Following Bourgain and Diestel [1], we say that an operator $u{:}E \to F$ is *limited* if it maps $(E)_1$ to a limited subset of F or, equivalently, if $u^*{:}(F^*,w^*) \to (E^*,\|\cdot\|)$ is sequentially continuous. One readily verifies that

(C) $E \in (\mathrm{GP})$ *iff every limited operator with range in* E *is compact.*

2. Gelfand-Phillips Spaces and Conditional Sequential Compactness

We shall say that a subset S of a topological space $T=(T,\rho)$ is $(\rho-)$ *conditionally sequentially compact* (shortly, $(\rho-)$CSC) if every sequence in S has a subsequence converging to a limit in T.

2.1 **Lemma.** *Let* $u{:}E \to F$ *be a limited operator. If* B *is a* w^*-CSC *subset of* F^* *and* $C=\overline{\mathrm{aco}}^{w^*}(B)$ *is its* w^**-closed absolutely convex hull, then* $u^*(C)$ *is a norm compact subset of* E^*.

Proof. Since B is w^*-CSC and u^* is weak*-to-norm sequentially continuous, it follows that $u^*(B)$ is relatively norm compact, hence its norm closed absolutely convex hull $D=\overline{u^*(\mathrm{aco}B)}$ is norm compact. It is easily seen that $D=u^*(C)$.

A bounded subset B of F^* is called *norming* (for F) if $y \mapsto \sup\{|y^*(y)|{:}y^* \in B\}$ is an equivalent norm on F. It is well known that B has this property iff $\overline{\mathrm{aco}}^{w^*}(B)$ contains a ball in F^* centered at 0.

Our principal result in this section is the following.

2.2. **Theorem.** *If* F^* *has a norming* w^*-CSC *subset, then* $F \in (\mathrm{GP})$.

Proof. As noted above, if B is such a subset of F^*, then $C=\overline{\mathrm{aco}}^{w^*}(B)$ contains a ball centered at 0. On the other hand, if $u{:}E \to F$ is a limited operator then, by Lemma 2.1, $u^*(C)$ is norm compact in E^*. Thus u^* is a compact operator, and so is u, by Schauder's theorem. It follows that $F \in (\mathrm{GP})$, by (C).

2.3 **Corollary.** *If* $(F^*)_1$ *is* w^**-sequentially compact or, more generally, if it contains a* w^**-dense* w^*-CSC *subset, then* $F \in (\mathrm{GP})$.

This corollary, combined with Rosenthal's ℓ_1-theorem and Goldstine's theorem, gives the following known fact ([2, p. 150]; cf. also [6]): If F does not contain any isomorphic copy of ℓ_1 then $F^* \in (\mathrm{GP})$.

We shall say that a topological space T satisfies *condition* (DCSC) if it has a dense CSC subset S. It is easily verified that the class of (DCSC)-spaces is closed with respect to continuous images and arbitrary products. [In fact, let T be the product of a family (T_i) of (DCSC)-spaces. For each i let S_i be a dense CSC subset of T_i, and let S be any Σ-product of the family (S_i) (see [8]), i.e., fix any point $a=(a_i)$ in $\prod S_i$ and set $S=\{(s_i) \in \prod S_i{:}\ \mathrm{card}\,\{i{:}s_i \neq a_i\} \leq \aleph_0\}$. Then S is a dense CSC subset of T.]

If $T \in$(DCSC), F is a Banach space and $f: T \to F$ is a continuous function, then $f(T)$ is easily seen to be a compact subset of F; hence f is bounded. We denote by $C(T,F)$ the Banach space of all continuous functions from T to F, with the sup norm. As usual, if F is the space of scalars, we simply write $C(T)$.

If $T \in$(DCSC) with a dense CSC subset S and if $\delta: T \to C(T)^*$ is the canonical map, then $\delta(S)$ is easily seen to be a norming w^*-CSC set in $C(T)^*$. Hence $C(T) \in$(GP), by Theorem 2.2. A more general result is valid:

2.4. **Theorem.** *If T is a topological space satisfying* (DCSC) *and* $F \in$(GP), *then also* $C(T,F) \in$(GP).

Proof. In view of (*B*) it suffices to show that if (f_n) is a limited weakly null sequence in $C(T,F)$, then $\|f_n\| \to 0$. We first observe that for each t in T the evaluation operator $f \mapsto f(t)$ from $C(T,F)$ to F maps the sequence (f_n) to the sequence $(f_n(t))$, so the latter is also limited and weakly null - in F. But $F \in$(GP), so $\|f_n(t)\| \to 0$ by (B).

Now, suppose $\|f_n\| \not\to 0$. Then we may assume that for some sequence (s_n) in S (a dense CSC set in T) and some $r > 0$ we have $\|f_n(s_n)\| > 2r$ for all n. Next, applying the fact that S is CSC and passing to a subsequence if necessary, we may assume that (s_n) converges to some $t \in T$. Since $\|f_n(t)\| \to 0$, we may finally assume that $r_n = \|f_n(s_n) - f_n(t)\| > r$ for all n. Now choose for each n a norm one functional y_n^* in F^* so that $y_n^*(f_n(s_n) - f_n(t)) = r_n$, and define $\eta_n \in C(T,F)^*$ by $\eta_n(f) = y_n^*(f(s_n) - f(t))$. Since $|\eta_n(f)| \leqq \|f(s_n) - f(t)\| \to 0$ because $s_n \to t$, we see that (η_n) is a w^*-null sequence in $C(T,F)^*$. But $\eta_n(f_n) = r_n > r > 0$ for all n, contradicting the assumption that (f_n) is limited.

2.5. *Remarks.* 1) As observed in [5], if T satisfies (DCSC), then $C(T,F)$ is isometrically isomorphic to a space $C(K,F)$, where K is a compact Hausdorff space satisfying (DCSC).

2) Let I be a set of cardinality $2^{\aleph_0}$. Then $K = [-1,1]^I$ is a compact space satisfying (DCSC) so that $C(K) \in$(GP); however, K is not sequentially compact. Moreover, $K = (\ell_\infty(I))_1 = (\ell_1(I)^*)_1$ has a norming w^*-CSC subset (e.g., formed by the functions with a countable support) so that $\ell_1(I) \in$(GP); however, ext K is not w^*-conditionally sequentially compact. Thus the results proved above are indeed more general than those contained in [7].

3. Injective Tensor Products of Gelfand-Phillips Spaces

We refer to [3, Ch.VIII] for the definition and basic properties of injective tensor products of Banach spaces. Our main result here is the following

3.1. **Theorem.** *If both E and F are Gelfand-Phillips spaces, then so is their injective tensor product $E \check{\otimes} F$.*

Proof. According to (B), we have to prove that if (z_n) is a limited weakly null sequence in $E \check{\otimes} F$, then $\|z_n\| \to 0$.

Let $i: E \to E$ and $j: F \to F$ be the identity operators.

For every x^* in E^* consider the operator $v_{x^*} = x^* \otimes j: E \check{\otimes} \mathrm{F} \to \mathrm{F}$; thus $v_{x^*}(x \otimes y) = x^*(x) y$ for all $x \in E$, $y \in F$. Then $(v_{x^*}(z_n))$ is a limited and weakly null sequence in F; hence, by (B),

$$\|v_{x^*}(z_n)\| \to 0.$$

Since $\|z_n\| = \sup\{|(x^* \otimes y^*)(z_n)| : \|x^*\| \leqq 1, \ \|y^*\| \leqq 1\}$, we can find sequences (x_n^*) in E^* and (y_n^*) in F^* of functionals of norm $\leqq 1$ such that, for each n,

$$|(x_n^* \otimes y_n^*)(z_n)| \geqq \tfrac{1}{2} \|z_n\|.$$

For any y^* in F^* consider the operator $w_{y^*} = i \otimes y^* : E \check{\otimes} F \to E$; thus $w_{y^*}(x \otimes y) = y^*(y)x$ for all $x \in E$, $y \in F$. Let, for each n,

$$x_n = w_{y_n^*}(z_n).$$

Since $x^*(w_{y^*}(z)) = (x^* \otimes y^*)(z)$, we have

$$(*) \qquad \|x_n\| \geqq |x_n^*(x_n)| = |(x_n^* \otimes y_n^*)(z_n)| \geqq \tfrac{1}{2} \|z_n\|.$$

(x_n) is weakly null in E: Indeed, if $x^* \in E^*$, then

$$|x^*(x_n)| = |(x^* \otimes y_n^*)(z_n)| = |y_n^*(v_{x^*}(z_n))| \leqq \|v_{x^*}(z_n)\| \to 0.$$

(x_n) is limited in E: Let (u_n^*) be a w^*-null sequence in E^*. Then $(u_n^* \otimes y_n^*)$ is a w^*-null sequence in $(E \check{\otimes} F)^*$. In fact, it is bounded and for any $x \in E$ and $y \in F$ we have $((u_n^* \otimes y_n^*)(x \otimes y) = u_n^*(x) y_n^*(y) \to 0$. Since $E \otimes F$ is dense in $E \check{\otimes} F$, we must have $(u_n^* \otimes y_n^*)(z) \to 0$ for every z in $E \check{\otimes} F$. Now,

$$u_n^*(x_n) = u_n^*(w_{y_n^*}(z_n)) = (u_n^* \otimes y_n^*)(z_n) \to 0$$

because (z_n) is limited in $E \check{\otimes} F$ and $(u_n^* \otimes y_n^*)$ is w^*-null in $(E \check{\otimes} F)^*$. It follows that (x_n) is limited in E, as claimed.

Thus the sequence (x_n) is limited and weakly null in E. Since $E \in$ (GP), $\|x_n\| \to 0$, by (B); now, using $(*)$ we get $\|z_n\| \to 0$ which concludes the proof.

3.2 **Corollary.** *Let T be a compact space and F a Banach space. If both $C(T)$ and F are Gelfand-Phillips spaces, then so is $C(T, F) = C(T) \check{\otimes} F$.*

3.3. *Remarks.* 1) Of course, Theorem 3.1 is best possible: If $\{0\} \neq E \check{\otimes} F \in$ (GP), then $E \check{\otimes} F$ contains isometric copies of both E and F and so $E, F \in$ (GP).

2) A direct proof of 3.2 (which was discovered first) is a bit simpler: In this case, starting with a limited weakly null sequence (f_n) in $C(T, F)$. we can find a sequence (y_n^*) of norm one functionals in F^* so that $\|f_n\| = \|y_n^* f_n\|$. Therefore, proceeding as in the proof of 3.1, we do not need the sequence (x_n^*). [The relevant operators used in this particular situation are: $v_\mu(f) = \int_T f \, d\mu$ for $\mu \in C(T)^*$, the space of Radon measures on T; $w_{y^*}(f) = y^* f$; $(\mu \otimes y^*)(f) = \int_T y^* f \, d\mu$.]

3) The present author does not know whether $C(T) \in$ (GP) may occur for a compact space T *not* satisfying (DCSC). Consequently, it is not clear at the moment if Corollary 3.2 is a genuine improvement of Theorem 2.4.

4. The Gelfand-Phillips Property in Spaces of Compact Operators

Let $K(E,F)$ denote the Banach space of all compact operators from E to F. We start with an application of Theorem 2.4 although, in a moment, a more general (and, in fact, best possible) result will be proved, using arguments similar to those applied in Sect. 3.

4.1. **Proposition.** *Let E and F be Banach spaces such that*

(i) *E does not have subspaces isomorphic to ℓ_1 and $F\in$(GP), or*

(ii) *$E^*\in$(GP) and $(F^*)_1$ is w^*-sequentially compact.*

Then $K(E,F)$ is a Gelfand-Phillips space.

Proof. (i): $T=(E^{**})_1\in$(DCSC) (by Rosenthal's ℓ_1-theorem and Goldstine's theorem), hence $C(T,F)\in$(GP) by 2.4. Now it is enough to observe that the map $u\mapsto u^{**}|_T$ is a linear isometry from $K(E,F)$ into $C(T,F)$.

(ii): In this case $u\mapsto u^*|_T$, where $T=(F^*)_1$, is a linear isometric embedding of $K(E,F)$ into $C(T,E^*)$, and the latter space is Gelfand-Phillips, by 2.4 again.

[Of course, we have considered the unit balls above with their respective weak* topologies.]

4.2. **Theorem.** *If the Banach spaces E and F are such that both E^* and F are Gelfand-Phillips spaces, then also $K(E,F)$ is a Gelfand-Phillips space.*

Proof. Let (u_n) be a limited weakly null sequence in $K=K(E,F)$. We have to show that $\|u_n\|\to 0$ (cf. (B)).

Choose a sequence (x_n) in E so that $\|x_n\|=1$ and $\|u_n(x_n)\|\geq\frac{1}{2}\|u_n\|$ for all n. We claim that $(y_n)=(u_n(x_n))$ is a weakly null limited sequence in F.

For every y^* in F^*, applying the operator $u\mapsto y^*u:K\to E^*$, we see that (y^*u_n) is a limited weakly null sequence in E^*. Since $E^*\in$(GP), $\|y^*u_n\|\to 0$ obtains by (B); hence $y^*(y_n)=(y^*u_n)(x_n)\to 0$. Thus (y_n) is weakly null.

Now let (y_n^*) be w^*-null in F^*, and define a sequence (η_n) in K^* by $\eta_n(u)=y_n^*(u(x_n))$. If $u\in K$, then $u[(E)_1]$ is relatively compact in F; therefore, $y_n^*(u(x))\to 0$ uniformly for x in $(E)_1$, i.e., $\|y_n^*u\|\to 0$. It follows that $y_n^*(u(x_n))\to 0$ and so (η_n) is w^*-null in K^*. Since (u_n) is limited, we have $y_n^*(y_n)=\eta_n(u_n)\to 0$. Thus (y_n) is limited.

Since $F\in$(GP), we appeal to (B) again to get $\|y_n\|\to 0$ which, in turn, implies $\|u_n\|\to 0$.

Remark. $K(E,F)$ is linearly isometric to the ε-product $E^*\varepsilon F$, and $E^*\varepsilon F$ contains $E^*\check{\otimes}F$ (see [9] or [10]). Hence 4.2 implies 3.1 when one of the spaces in 3.1 is a dual Banach space.

5. Schauder Decompositions with Gelfand-Phillips Summands

5.1. **Theorem.** *If a Banach space E has a Schauder decomposition* (cf. [11]) $E=\sum_{n=1}^{\infty} E_n$, *where each summand E_n is a Gelfand-Phillips space, then E itself is a Gelfand-Phillips space.*

Proof. For every $x \in E$ and $n=1,2,\ldots$, we denote by $x(n)$ the natural projection of x in E_n; thus $x=\sum_{n=1}^{\infty} x(n)$ and $x(n) \in E_n$ for all n.

Suppose $E \notin (\mathrm{GP})$; Then there exists a limited sequence (x_k) in E such that for some $r>0$ we have $\|x_k - x_j\| > 2r$ whenever $k \neq j$. Since, for each n, the projection $(x_k(n))$ is a limited sequence in E_n and $E_n \in (\mathrm{GP})$, we may assume (by passing to a subsequence if needed) that $\lim_k x_k(n)$ exists in E_n for each n. Then $(y_k)=(x_{k+1}-x_k)$ is a limited sequence in E, and

$$\|y_k\| > 2r, \quad \forall k \quad \text{and} \quad \lim_k y_k(n)=0, \quad \forall n.$$

By applying a standard sliding hump argument, we may find a subsequence (z_k) of (y_k), and a sequence $1=m_1<m_2<\ldots$ of integers such that if $n_k=m_{k+1}-1$ and

$$w_k=\sum_{n=m_k}^{n_k} z_k(n),$$

then

$$\|w_k\|>r, \quad \forall k \quad \text{and} \quad \|z_k - w_k\| \to 0.$$

The latter relation implies that (w_k) is a limited sequence in E.

Now, for each k, choose a norm 1 functional w_k^* in F_k^*, where

$$F_k=\sum_{n=m_k}^{n_k} E_n$$

(with the norm induced from E) such that $w_k^*(w_k)=\|w_k\|$. Also, let Q_k be the natural projection from E onto F_k, and set $v_k^*=w_k^* \circ Q_k$. Then, since the projections Q_k are uniformly bounded, we have for every $x \in E$

$$|v_k^*(x)| \leqq \|w_k^*\| \cdot \|Q_k\| \cdot \left\| \sum_{n=m_k}^{n_k} x(n) \right\| \to 0 \quad \text{as } k \to \infty.$$

Thus (v_k^*) is a w^*-null sequence in E^*. But $|v_k^*(w_k)|=|w_k^*(w_k)|>r$ for all k, which contradicts the limitedness of (w_k).

The above result is also valid for uncountable unconditional Schauder decompositions; in order to see this, we need the following simple observation.

5.2. **Proposition.** *If for every separable subspace L of the Banach space E there exists a complemented subspace M of E such that $L \subset M$ and $M \in (\mathrm{GP})$, then $E \in (\mathrm{GP})$.*

Proof. It suffices to show that every countable limited subset A of E is relatively compact. Given such a set A, let L be the (separable) closed linear span of A, and choose a complemented subspace $M \in (\mathrm{GP})$ containing L. Let P be a projection from E onto M. Then $A=P(A)$ is limited in M, hence relatively compact because $M \in (\mathrm{GP})$.

5.3. **Corollary.** *If a Banach space E admits an unconditional (possibly uncountable Schauder decomposition $E=\sum_{i \in I} E_i$, where $E_i \in (\mathrm{GP})$ for all $i \in I$, then E is a Gelfand-Phillips space.*

5.4. **Corollary.** *If* $(E_i)_{i\in I}$ *is a family of Gelfand-Phillips spaces, then*

$$(\sum_{i\in I} E_i)_{\ell_p} \quad \text{for } 1 \leqq p < \infty \quad \text{and} \quad (\sum_{i\in I} E_i)_{c_0}$$

are Gelfand-Phillips spaces.

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