Chapter V

ORDER STRUCTURES

The present chapter is devoted to a systematic study of order structures within the framework of topological vector spaces. No attempt has been made to give an account of the extensive literature on Banach lattices, for a survey of which we refer the reader to Day [2], nor is any special emphasis placed on ordered normed spaces. Our efforts are directed towards developing a theory that is in conformity with the modern theory of topological vector spaces, that is to say, a theory in which duality plays the central role. This approach to ordered topological vector spaces is of fairly recent origin, and thus cannot be presented in a form as definite as a mature theory; it is nonetheless hoped that the reader who has encountered parts of it in the literature (e.g., Gordon [1], [2], Kist [1], Namioka [1], Schaefer [1]–[5]) will obtain a certain survey of the methods available and of the results to which they lead. The fact that ordered topological vector spaces abound in analysis is perhaps motivation enough for a systematic study; beyond this, the present chapter is followed by an appendix intended to illustrate some applications to spectral theory. As in the preceding chapters, further information can be found in the exercises.

Section 1 is concerned with algebraic aspects only and supplies, in particular, the basic tools needed in working with vector lattices. For simplicity of exposition we restrict attention to ordered vector spaces over $\boldsymbol{R}$; Section 2 discusses briefly how these concepts can be applied to vector spaces over $\boldsymbol{C}$, which is often called for by applications (particularly to measure theory and spectral theory). Section 3 gives the basic results on the duality of convex cones. The concept of normal cone, probably the most important concept of the theory, is introduced and a number of immediate consequences are established. The discussion proceeds covering the real and complex cases simultaneously; the reader who finds this too involved may well assume first that all occurring vector spaces are defined over $\boldsymbol{R}$. Section 4 introduces ordered topological vector spaces and establishes two more properties of normal cones, among them (Theorem (4.3)) the abstract version of a classical theorem of Dini on monotone convergence. The duality of ordered vector spaces is not discussed there, since such a discussion would have amounted to a direct application of the results of Section 3, which can be left to the reader.

Section 5 is concerned with the induced order structure on spaces of linear mappings; the principal results are Theorem (5.4) on the extension of continuous positive linear forms, and Theorem (5.5) establishing the continuity of a large class of positive linear forms and mappings. The order topology, a locally convex topology accompanying every ordered vector space over $\boldsymbol{R}$, is studied in some detail in Section 6. The importance of this topology stems in part from the fact that it is the topology of many ordered t.v.s. occurring in analysis. Section 7 treats topological (in particular, locally convex) vector lattices. We obtain results especially on the strong dual of a locally convex vector lattice, and characterizations of vector lattices of minimal type in terms of order convergence and in terms of the evaluation map. (For the continuity of the lattice operations see Exercise 20.) The section concludes with a discussion of weak order units.

Section 8 is concerned with the vector lattice of all continuous real valued functions on a compact space, and with abstract Lebesgue spaces. The Stone-Weierstrass theorem is presented in both its order theoretic and its algebraic form. Further, the dual character of (AM)-spaces with unit and (AL)-spaces is studied as an illuminating example of the duality of topological vector lattices treated in Section 7. (AL)-spaces are represented as bands of Radon measures, characterized by a convergence property, on extremally disconnected compact spaces. The classical representation theorem of Kakutani for (AM)-spaces with unit is established, and an application is made to the representation of a much more general class of locally convex vector lattices.

1. ORDERED VECTOR SPACES OVER THE REAL FIELD

Throughout this section, we consider only vector spaces over the real field $\boldsymbol{R}$.

Let L be a vector space over $\boldsymbol{R}$ which is endowed with an order structure R defined by a reflexive, transitive, and anti-symmetric binary relation "$\leqq$"; L is called an **ordered vector space** over $\boldsymbol{R}$ if the following axioms are satisfied:

$(LO)_1$ *$x \leqq y$ implies $x + z \leqq y + z$ for all $x, y, z \in L$*
$(LO)_2$ *$x \leqq y$ implies $\lambda x \leqq \lambda y$ for all $x, y \in L$ and $\lambda > 0$.*

$(LO)_1$ expresses that the order of L is translation-invariant, $(LO)_2$ expresses the invariance of the order under homothetic maps $x \to \lambda x$ with ratio $\lambda > 0$. Examples of ordered vector spaces abound; for example, every vector space of real-valued functions f on a set T is naturally ordered by the relation "$f \leqq g$ if $f(t) \leqq g(t)$ for all $t \in T$"; in this fashion, one obtains a large number of ordered vector spaces from the examples given in Chapter II, Section 2, and Chapter III, Section 8, by considering real-valued functions only and taking $K = \boldsymbol{R}$.

It is immediate from the axioms above that in an ordered vector space L, the subset $C = \{x: x \geqq 0\}$ is a convex cone of vertex 0 satisfying $C \cap -C = \{0\}$; a cone in L with these properties is called a **proper cone** in L. The elements $x \in C$ are called **positive**, and C is called the **positive cone** of the ordered vector space L.

Two ordered vector spaces L_1, L_2 are **isomorphic** if there exists a linear biunivocal map u of L_1 onto L_2 such that $x \leqq y$ if and only if $u(x) \leqq u(y)$ (equivalently, such that u maps the positive cone of L_1 onto the positive cone of L_2).

If L is any vector space over $\boldsymbol{R}$, a proper cone $H \subset L$ is characterized by the properties

(i) $H + H \subset H$,
(ii) $\lambda H \subset H$ *for all* $\lambda > 0$,
(iii) $H \cap -H = \{0\}$.

It is verified without difficulty that each proper cone $H \subset L$ defines, by virtue of "$x \leqq y$ if $y - x \in H$", an order of L under which L is an ordered vector space with positive cone H. Hence for any vector space L, there is a biunivocal correspondence between the family of all proper cones in L and the family of all orderings satisfying $(LO)_1$ and $(LO)_2$. If R_1 and R_2 are two such orderings of L with respective positive cones C_1 and C_2, then the relation "R_1 is finer than R_2" is equivalent with $C_1 \subset C_2$; in particular, if $\{R_\alpha: \alpha \in A\}$ is a family of such orderings of L with respective positive cones C_α, the coarsest ordering R which is finer than all $R_\alpha (\alpha \in A)$ is determined by the proper cone $C = \bigcap_\alpha C_\alpha$. (Cf. Exercise 2.) A cone $H \subset L$ satisfying (i) and (ii) is said to be **generating** if $L = H - H$.

Let L be an ordered vector space. The order of L is called **Archimedean** (or L **Archimedean ordered**) if $x \leqq 0$ whenever there exists $y \in L$ such that $nx \leqq y$ for all $n \in \boldsymbol{N}$ (in other words, if $x \leqq 0$ whenever $\{nx: n \in \boldsymbol{N}\}$ is majorized). For example, if L is a t.v.s. and an ordered vector space whose positive cone is closed, L is Archimedean ordered; on the other hand, $\boldsymbol{R}_0^n$ is not Archimedean ordered for $n \geqq 2$ under its lexicographic ordering (see below). An **order interval** in L is a subset of the form $\{z \in L: x \leqq z \leqq y\}$, where x, y are given; it is convenient to denote this set by $[x, y]$. (There is little danger of confusing this with the inner product notation in pre-Hilbert spaces (Chapter III, Section 2, Example 5) if we avoid using the symbol in different meanings in the same context.) A subset A of L is **order bounded** if A is contained in some order interval. Every order interval is convex, and every order interval of the form $[-x, x]$ is circled. An element $e \in L$ such that $[-e, e]$ is radial is called an **order unit** of L. The set L^b of all linear forms on L that are bounded on each order interval is a subspace of L^*, called the **order bound dual** of L.

Let L be an ordered vector space over $\boldsymbol{R}$ and let M be a subspace of L.

If C is the positive cone of L, then the induced ordering on M is determined by the proper cone $C \cap M$; an ordering of L/M is determined by the canonical image $\hat{C}$ of C in L/M, provided that $\hat{C}$ is a proper cone. (Simple examples, with $L = \mathbf{R}_0^2$, show that this is not necessarily the case.) If $\{L_\alpha: \alpha \in A\}$ is a family of ordered vector spaces with respective positive cones C_α, then $C = \prod_\alpha C_\alpha$ is a proper cone in $L = \prod_\alpha L_\alpha$ which determines an ordering of L. The orderings so defined are called the **canonical orderings** of M, L/M (provided $\hat{C}$ is proper), and of $\prod_\alpha L_\alpha$. In particular, the algebraic direct sum $\oplus_\alpha L_\alpha$ is canonically ordered as a subspace of $\prod_\alpha L_\alpha$, and if T is any set, then L^T is canonically ordered by the proper cone $\{f: f(t) \in C \text{ for all } t \in T\}$.

Let L be an ordered vector space which is the algebraic direct sum of the subspaces $M_i (i = 1, \ldots, n)$; L is said to be the **ordered direct sum** of the subspaces M_i if the canonical algebraic isomorphism of L onto $\prod_i M_i$ is an order isomorphism (for the canonical ordering of $\prod_i M_i$).

If L_1, L_2 are ordered vector spaces $\neq \{0\}$ with respective positive cones C_1 and C_2, then $C = \{u: u(C_1) \subset C_2\}$ is a proper cone in the space $L(L_1, L_2)$ of linear mappings of L_1 into L_2, if and only if C_1 is generating in L_1; whenever M is a subspace of $L(L_1, L_2)$ such that $C \cap M$ is a proper cone, the ordering defined by $C \cap M$ is called the **canonical ordering** of M. A special case of importance is the following: A linear form f on an ordered vector space over $\mathbf{R}$ is **positive** if $x \geqq 0$ implies $f(x) \geqq 0$; the set C^* of all positive linear forms on L is a cone which is the polar, with respect to $\langle L, L^* \rangle$, of $-C$. The subspace $L^+ = C^* - C^*$ of L^* is called the **order dual** of L; it is immediate that $L^+ \subset L^b$. However, there exist ordered vector spaces L for which $L^+ \neq L^b$ (see Namioka [1], 6.10).

In order to use the tool of duality successfully in the study of ordered vector spaces L, one needs sufficiently many positive linear forms on L to distinguish points; we shall say that L is **regularly ordered** (or that the order of L is **regular**) if L is Archimedean ordered and L^+ distinguishes points in L (cf. (4.1) below).

As above, the canonical ordering of a subspace $M \subset L^*$ is understood to be the ordering defined by $M \cap C^*$ whenever $M \cap C^*$ is a proper cone in M.

Let us note some simple consequences of $(LO)_1$, L being an ordered vector space. The equality

$$z + \sup(x, y) = \sup(z + x, z + y) \tag{1}$$

is valid for given $x, y \in L$ and all $z \in L$ whenever $\sup(z_0 + x, z_0 + y)$ exists for some $z_0 \in L$. If A, B are subsets $\neq \emptyset$ of L such that $\sup A$ and $\sup B$ exist, then $\sup(A + B)$ exists and

$$\sup(A + B) = \sup A + \sup B. \tag{1'}$$

Also from $(LO)_1$ it follows that

$$\sup(x, y) = -\inf(-x, -y) \tag{2}$$

whenever either $\sup(x, y)$ or $\inf(-x, -y)$ exists; more generally,

$$\sup A = -\inf(-A) \tag{2'}$$

whenever either $\sup A$ or $\inf(-A)$ exists.

A **vector lattice** is defined to be an ordered vector space E over $\mathbf{R}$ such that for each pair $(x, y) \in E \times E$, $\sup(x, y)$ and $\inf(x, y)$ exist. This implies, in particular, that E is directed under the order relation $\leqq$ (equivalently, that the positive cone C of E is generating). For each $x \in E$, we define the **absolute** $|x|$ by $|x| = \sup(x, -x)$; two elements x, y of a vector lattice E are **disjoint** if $\inf(|x|, |y|) = 0$; two subsets $A \subset E$ and $B \subset E$ are **lattice disjoint** (or simply **disjoint** if no confusion is likely to result) if $x \in A$, $y \in B$ implies $\inf(|x|, |y|) = 0$. The fact that x, y are disjoint is denoted by $x \perp y$, and if A is a subset of E, $A^{\perp}$ denotes the set of all $y \in E$ such that y is disjoint from each element of A. We record the following simple but important facts on vector lattices.

1.1

Let E be a vector lattice. Then

$$x + y = \sup(x, y) + \inf(x, y) \tag{3}$$

is an identity on $E \times E$. Defining x^+ and x^- by $x^+ = \sup(x, 0)$ and $x^- = \sup(-x, 0)$ for all $x \in E$, we have $x = x^+ - x^-$ and $|x| = x^+ + x^-$; $x = x^+ - x^-$ is the unique representation of x as a difference of disjoint elements $\geqq 0$. Moreover, we have

$$|\lambda x| = |\lambda|\,|x| \tag{4}$$

$$|x + y| \leqq |x| + |y| \tag{5}$$

$$|x^+ - y^+| \leqq |x - y| \tag{6}$$

for all $x, y \in E$ and $\lambda \in \mathbf{R}$. Finally, we have

$$[0, x] + [0, y] = [0, x + y] \tag{D}$$

for all $x \geqq 0$ and $y \geqq 0$.

Proof. To prove (3), consider the more general identity

$$a - \inf(x, y) + b = \sup(a - x + b, a - y + b), \tag{3'}$$

where a, b, x, y are arbitrary elements of E. By (2) we have $-\inf(x, y) = \sup(-x, -y)$, whence (3′) follows from (1); from (3′) we obtain (3) by the substitution $a = x$, $b = y$. Letting $y = 0$ in (3), we obtain $x = x^+ - x^-$, and since $\inf(x^+, x^-) = x^- + \inf(x, 0) = x^- - \sup(-x, 0) = 0$, x^+ and x^- are disjoint elements; we now obtain via (1), $x^+ + x^- = x + \sup(-2x, 0) = \sup(-x, x) = |x|$. Let $x = y - z$, where $y \geqq 0$, $z \geqq 0$ are disjoint; we show that $y = x^+$, $z = x^-$. Note first that $x = y - z$ implies $y \geqq x$ hence $y \geqq x^+$ and, therefore, $z \geqq x^-$; it follows that $(y - x^+) \perp (z - x^-)$ which, in view of $y - x^+ = z - x^-$, implies $y = x^+$, $z = x^-$, since clearly 0 is the only element of E disjoint from itself.

If $\lambda \geqq 0$, then from $(LO)_2$ we obtain $(\lambda x)^+ = \lambda x^+$ and $(\lambda x)^- = \lambda x^-$; if $\lambda < 0$, then $(\lambda x)^+ = (-\lambda(-x))^+ = |\lambda| x^-$ and $(\lambda x)^- = |\lambda| x^+$; this proves (4). For (5), note that $\pm x \leqq |x|$, $\pm y \leqq |y|$ implies $|x + y| = \sup(x + y, -x - y) \leqq |x| + |y|$. To prove (6) we conclude from $x = y + (x - y)$ that $x \leqq y^+ + |x - y|$; hence, the right-hand side being $\geqq 0$, that $x^+ \leqq y^+ + |x - y|$; therefore, $x^+ - y^+ \leqq |x - y|$ and interchanging x and y yields $y^+ - x^+ \leqq |x - y|$, hence (6).

Finally, it is clear that $[0, x] + [0, y] \subset [0, x + y]$ whenever $x \geqq 0$ and $y \geqq 0$. Let $z \in [0, x + y]$ and define u, v by $u = \inf(z, x)$ and $v = z - u$; there remains to show that $v \in [0, y]$. But $v = z - \inf(z, x) = z + \sup(-z, -x) = \sup(0, z - x) \leqq \sup(0, x + y - x) = y$ which completes the proof of (1.1).

COROLLARY 1. *In every vector lattice E, the relation* $x \leqq y$ *is equivalent with* "$x^+ \leqq y^+$ and $y^- \leqq x^-$", *and the relation* $x \perp y$ *is equivalent with* $\sup(|x|, |y|) = |x| + |y|$. *Moreover, if* $x \perp y$, *then* $(x + y)^+ = x^+ + y^+$ *and* $|x + y| = |x| + |y|$.

Proof. In fact, if $x^+ \leqq y^+$ and $y^- \leqq x^-$, then $x = x^+ - x^- \leqq y^+ - y^- = y$. Conversely, $x \leqq y$ implies $x^+ \leqq y^+$ and $\inf(x, 0) \leqq \inf(y, 0)$; hence $-x^- \leqq -y^-$ or, equivalently, $y^- \leqq x^-$. The second assertion is immediate from (3) replacing x, y by $|x|, |y|$ respectively. Finally, $x + y = (x^+ + y^+) - (x^- + y^-)$, and $\inf(|x|, |y|) = 0$ expresses that the summands on the right are disjoint; hence $(x + y)^+ = x^+ + y^+$ by the unicity of the representation of $x + y$ as a difference of disjoint elements $\geqq 0$. The last assertion is now immediate.

COROLLARY 2. *Let E be a vector lattice and let* $A \subset E$ *be a subset for which* $\sup A = x_0$ *exists. If* $B \subset E$ *is a subset lattice disjoint from A, then B is lattice disjoint from* $\{x_0\}$.

Proof. We have to show that $z \in B$ implies $z \perp x_0$. Now $x_0^- \leqq x^- \leqq |x|$ for all $x \in A$; hence $z \perp x_0^-$ if $z \in B$. It suffices hence to show that $z \perp x_0^+$. In view of Corollary 1, we have $\sup(|z|, x^+) = |z| + x^+$ for all $x \in A$ by hypothesis, and $x_0^+ = \sup\{x^+ : x \in A\}$; (1′) implies that $\sup\{|z| + x^+ : x \in A\} = |z| + x_0^+$. Thus we obtain

$$\sup(|z|, x_0^+) = \sup_{x \in A} \sup(|z|, x^+) = \sup_{x \in A}(|z| + x^+) = |z| + x_0^+,$$

which shows that $|z| \perp x_0^+$ (Corollary 1).

The following observation sometimes simplifies the proof that a given ordered vector space is a vector lattice.

1.2

Let E be an ordered vector space over $\mathbf{R}$ whose positive cone C is generating; if for each pair $(x, y) \in C \times C$ either $\sup(x, y)$ *or* $\inf(x, y)$ *exists, then E is a vector lattice.*

The detailed verification is left to the reader; one shows that if $\sup(x, y)$ exists $(x, y \in C)$, then $z = x + y - \sup(x, y)$ proves to be $\inf(x, y)$, and conversely. If x, y are any elements of E, there exists $z \in C$ such that $x + z \in C$ and $y + z \in C$, and the existence of $\sup(x, y)$ and $\inf(x, y)$ is shown via (1).

If $\{E_\alpha: \alpha \in \mathrm{A}\}$ is a family of vector lattices, it is quickly verified that $\prod_\alpha E_\alpha$ and $\bigoplus_\alpha E_\alpha$ are vector lattices under their canonical orderings. A **vector sublattice** M of a vector lattice E is a vector subspace of E such that $x \in M$, $y \in M$ implies that $\sup(x, y) \in M$ where the supremum is formed in E; it follows that M is a vector lattice under its canonical ordering. However, it can happen that a subspace M of E is a vector lattice under its canonical order but not a sublattice of E (Exercise 14).

A subset A of a vector lattice E is called **solid** if $x \in A$ and $|y| \leqq |x|$, $y \in E$, imply that $y \in A$. It is easy to see that a solid subspace of E is necessarily a sublattice of E; for example, the algebraic direct sum $\bigoplus_\alpha E_\alpha$ of a family $\{E_\alpha: \alpha \in \mathrm{A}\}$ of vector lattices is a solid subspace of $\prod_\alpha E_\alpha$ (for the canonical ordering of the product). Also it is easy to see that if M is a solid subspace of E, then E/M is a vector lattice under its canonical order (cf. the examples below).

A subset A of a vector lattice E is called **order complete** if for each nonempty subset $B \subset A$ such that B is order bounded in A, $\sup B$ and $\inf B$ exist and are elements of A; E is **order complete** if it is order complete as a subset of itself. If E is an order complete vector lattice, a subspace M of E which is solid and such that $A \subset M$, $\sup A = x \in E$ implies $x \in M$, is called a **band** in E. E itself is a band, and clearly the intersection of an arbitrary family of bands in E is a band; hence every subset A of E is contained in a smallest band B_A, called the **band generated by** A (in E).

Examples

1. Let T be any set and consider the vector space $\mathbf{R}_0^T$ of all real-valued functions on T under its canonical order, where $\mathbf{R}_0$ is ordered as usual. Obviously $\mathbf{R}_0^T$ is an order complete vector lattice. If A is any subset of $\mathbf{R}_0^T$, denote by T_A the subset $\{t$: *there exists* $f \in A$ *such that* $f(t) \neq 0\}$ of T. Then the band generated by A is the subspace $B_A = \{f: f(t) = 0$ whenever $t \notin T_A\}$; the quotient $\mathbf{R}_0^T/B_A$, under its canonical order, is a vector lattice which is isomorphic with $\mathbf{R}_0^{T \sim T_A}$. The canonical ordering of $\mathbf{R}_0^T$ is regular (in particular, Archimedean); in fact, the order dual and the order bound dual coincide with the (ordered) direct sum of card T copies of $\mathbf{R}_0$ (Chapter IV, Section 1, Example 4).

2. Let β be any ordinal number > 0 and let $\mathbf{R}_0^\beta$ denote the vector space of all real valued functions defined on the set of all ordinals

$\alpha < \beta$, and consider the subset H of $\boldsymbol{R}_0^\beta$ defined by the property "*if there exists a smallest ordinal* $\alpha < \beta$ *such that* $f(\alpha) \neq 0$ *then* $f(\alpha) > 0$". We verify without difficulty that H is a proper cone in $\boldsymbol{R}_0^\beta$; the order determined by H is called the **lexicographical order** of $\boldsymbol{R}_0^\beta$. The lexicographical order of $\boldsymbol{R}_0^\beta$ is not Archimedean (hence not regular) if $\beta > 1$; in fact, the set of all functions f such that $f(0) = 0$ is majorized by each function f for which $f(0) > 0$. It is worth noting that the lexicographical order of $\boldsymbol{R}_0^\beta$ is a total ordering, since $\boldsymbol{R}_0^\beta = H \cup -H$; thus $\boldsymbol{R}_0^\beta$ is a vector lattice under this order which is, however, not order complete if $\beta > 1$. Moreover, (up to a positive scalar factor) $f \to f(0)$ is the only non-trivial positive linear form, hence the order dual and the order bound dual (cf. (1.4) below) are of dimension 1.

3. Let (X, Σ, μ) be a measure space (Chapter II, Section 2, Example 2). Under the ordering induced by the canonical ordering of $\boldsymbol{R}_0^X$ (Example 1 above), the spaces $\mathscr{L}^p(\mu)$ $(1 \leqq p \leqq +\infty)$ are vector lattices (take the scalar field $K = \boldsymbol{R}$) which are countably order complete (each majorized countable family has a supremum) but, in general, not order complete (Exercise 13). The subspace $\mathscr{N}_\mu$ of μ-null functions is a solid subspace but, in general, not a band in $\mathscr{L}^p(\mu)$; the quotient spaces $L^p(\mu) = \mathscr{L}^p(\mu)/\mathscr{N}_\mu$ are order complete vector lattices under their respective canonical orderings $(1 \leqq p < +\infty)$.

If E is any order complete vector lattice and A a subset of E, the set $A^\perp$ is a band in E; this is clear in view of Corollary 2 of (1.1). Concerning the bands B_A and $A^\perp$, we have the following important theorem (F. Riesz [1]).

1.3

Theorem. *Let E be an order complete vector lattice. For any subset $A \subset E$, E is the ordered direct sum of the band B_A generated by A and of the band $A^\perp$ of all elements disjoint from A.*

Proof. Since $A^{\perp\perp}$ is a band containing A, it follows that $B_A \subset A^{\perp\perp}$ and hence that $B_A \cap A^\perp = \{0\}$. Let $x \in E$, $x \geqq 0$, be given; we show that $x = x_1 + x_2$, where $x_1 \in B_A$, $x_2 \in A^\perp$, and $x_1 \geqq 0$, $x_2 \geqq 0$. Define x_1 by $x_1 = \sup\, [0, x] \cap B_A$ and x_2 by $x_2 = x - x_1$; it is clear that x_1, x_2 are positive and that $x_1 \in B_A$, since B_A is a band in E. Let us show that $x_2 \in B_A^\perp$. For any $y \in B_A$ let $z = \inf(x_2, |y|)$; then $0 \leqq z \in B_A$, since B_A is solid and $z + x_1 \leqq x_2 + x_1 = x$. This implies, by the definition of x_1 and by virtue of $z + x_1 \in B_A$, that $z + x_1 \leqq x_1$ and hence that $z = 0$. Thus $x_2 \in B_A^\perp$ and a fortiori $x_2 \in A^\perp$. Since the positive cone of E is generating, it follows that $E = B_A + A^\perp$ is the ordered direct sum of the subspaces B_A and $A^\perp$. For, the relations $x \geqq 0$ and $x = x_1 + x_2$, $x_1 \in B_A$, $x_2 \in A^\perp$ imply $x_1 \geqq 0$, $x_2 \geqq 0$.

COROLLARY 1. *If A is any subset of E, the band B_A generated by A is the band $A^{\perp\perp}$.*

Proof. Applying (1.3) to the subset $A^\perp$ of E, we obtain the direct sum $E = A^{\perp\perp} + A^\perp$; since $E = B_A + A^\perp$ and $B_A \subset A^{\perp\perp}$, it follows that $B_A = A^{\perp\perp}$.

COROLLARY 2. *If x, y are disjoint elements of E and B_x, B_y are the bands generated by $\{x\}$, $\{y\}$ respectively, then B_x is disjoint from B_y.*

In fact, we have $y \in \{x\}^\perp$ and $x \in \{y\}^\perp$.

A general example of an order complete vector lattice is furnished by the order dual E^+ of any vector lattice E; however, E^+ can be finite dimensional (Example 2 above) or reduced to $\{0\}$ (Exercise 14), even if E is of infinite dimension. We prove the result in the following more general form which shows it to depend essentially on property (D) of (1.1). (Cf. Exercise 16.)

1.4

Let E be an ordered vector space over $\mathbf{R}$ whose positive cone C is generating and has property (D) *of* (1.1). *Then the order bound dual E^b of E is an order complete vector lattice under its canonical ordering; in particular, $E^b = E^+$.*

Proof. We show first that for each $f \in E^b$, $\sup(f, 0)$ exists; it follows then from (1) that $\sup(f, g) = g + \sup(f - g, 0)$ exists for any pair $(f, g) \in E^b \times E^b$; hence E^b is a vector lattice by (1.2). This implies clearly that $E^b = E^+$.

Let $f \in E^b$ be given; we define a mapping r of C into the real numbers $\geqq 0$ by

$$r(x) = \sup\{f(y) \colon y \in [0, x]\} \qquad (x \in C).$$

Since $f(0) = 0$ it follows that $r(x) \geqq 0$, and clearly $r(\lambda x) = \lambda r(x)$ for all $\lambda \geqq 0$. Also, by virtue of (1′) and (D),

$$r(x + y) = \sup\{f(z) \colon z \in [0, x] + [0, y]\} = r(x) + r(y).$$

Hence r is positive homogeneous and additive on C. By hypothesis, each $z \in E$ is of the form $z = x - y$ for suitable elements $x, y \in C$, and it is readily seen that the number $r(x) - r(y)$ is independent of the particular decomposition $z = x - y$ of z. A short computation now shows that $z \to w(z) = r(x) - r(y)$ is a linear form w on E, evidently contained in E^b. (We have, in fact, $w(x) = r(x)$ for $x \in C$.) We show that $w = \sup(f, 0)$; indeed, $w(x) \geqq \sup(f(x), 0)$ for all $x \in C$, and if $h \geqq 0$ is a linear form on E such that $x \in C$ implies $h(x) \geqq f(x)$, then $h(x) \geqq h(y) \geqq f(y)$ for all $y \in [0, x]$, which shows that $h(x) \geqq r(x) = w(x)$ whenever $x \in C$.

It remains to prove that $E^b = E^+$ is order complete; for this it suffices to show that each non-empty, majorized set A of positive linear forms on E has a supremum. Without restriction of generality, we can assume that A is directed under "$\leqq$". (This can be arranged, if necessary, by considering the set of suprema of arbitrary, non-empty finite subsets of A.) We define a mapping s of C into the real numbers by

$$s(x) = \sup\{f(x) \colon f \in A\} \qquad (x \in C).$$

The supremum is finite for all $x \in C$, since A is majorized. It is clear that $s(\lambda x) = \lambda s(x)$ for all $\lambda \geqq 0$ and, since A is directed, that $s(x + y) = s(x) + s(y)$. Hence, as before, s defines a linear form f_0 on E by means of $f_0(z) = s(x)$

$- s(y)$, where $z = x - y$ and $x, y \in C$. It is evident that $f_0 \in E^b$ (since $f_0 \geqq 0$) and that $f_0 = \sup A$.

COROLLARY. *The order dual of every vector lattice is an order complete vector lattice under its canonical ordering.*

From the construction of $f^+ = \sup(f, 0)$ in the proof of (1.4), we obtain the following useful relations; the proof of these is purely computational and will be omitted.

1.5

Let E be a vector lattice and let f, g be order bounded linear forms on E. For each $x \in E$, we have

$$\begin{aligned} \sup(f, g)(|x|) &= \sup\{f(y) + g(z): y \geqq 0, z \geqq 0, y + z = |x|\} \\ \inf(f, g)(|x|) &= \inf\{f(y) + g(z): y \geqq 0, z \geqq 0, y + z = |x|\} \end{aligned} \tag{7}$$

$$\begin{aligned} |f|(|x|) &= \sup\{f(y - z): y \geqq 0, z \geqq 0, y + z = |x|\} \\ |f(x)| &\leqq |f|(|x|). \end{aligned} \tag{8}$$

In particular, two linear forms $f \geqq 0$, $g \geqq 0$ are disjoint if and only if for each $x \geqq 0$ and each real number $\varepsilon > 0$, there exists a decomposition $x = x_1 + x_2$ with $x_1 \geqq 0$, $x_2 \geqq 0$, and such that $f(x_1) + g(x_2) \leqq \varepsilon$.

COROLLARY. *Let E be a vector lattice, and let $\langle E, G\rangle$ be a duality such that G is a sublattice of E^+. Then the polar $A^\circ \subset G$ of each solid subset $A \subset E$ is solid.*

Proof. In fact, if $x \in A$, $y \geqq 0, z \geqq 0$, and $y + z = |x|$, then $y - z \in A$, since $-|x| \leqq y - z \leqq |x|$; hence, if $f \in A^\circ$ and $|g| \leqq |f|$, then from (8) it follows that

$$|g(x)| \leqq |g|(|x|) \leqq |f|(|x|) \leqq 1,$$

which shows that $g \in A^\circ$.

If E is an ordered vector space over $\boldsymbol{R}$ such that the order dual E^+ is an ordered vector space (equivalently, if C^* is a proper cone in E^* where C is the positive cone of E), then the space $(E^+)^+$ is called the **order bidual** of E and denoted by E^{++}. Under the assumptions of (1.4) (in particular, if E is a vector lattice), E^{++} is a vector lattice, and the evaluation (or canonical) map of E into E^{++}, defined by $x \to \tilde{x}$ where $\tilde{x}(f) = f(x)$ ($f \in E^+$), is clearly order preserving. Assuming that E is a vector lattice, let us show that $x \to \tilde{x}$ is an isomorphism onto a sublattice of E^{++} if E is regularly ordered (equivalently, if $x \to \tilde{x}$ is one–to–one). For later use, we prove this result in a somewhat more general form.

1.6

Let E be a vector lattice and let G be a solid subspace of E^+ that separates points in E; the evaluation map $x \to \tilde{x}$, defined by $\tilde{x}(f) = f(x)$ ($f \in G$), is an isomorphism of E onto a sublattice of G^+.

Proof. We must show that for each $x \in E$, the element $\tilde{x}^+$ $(=\sup(0, \tilde{x})$ taken in $G^+)$ is the canonical image of $x^+ \in E$. Denote by P the subset $\bigcup\{\rho[0, x^+]: \rho \geqq 0\}$ of E and define, for each $f \geqq 0$ in G, a mapping t_f of the positive cone C of E into $\boldsymbol{R}$ by

$$t_f(y) = \sup\{f(z): z \in [0, y] \cap P\} \qquad (y \in C).$$

As in the proof of (1.4) it follows that t_f is additive and positive homogeneous, and hence defines a unique linear form $g_f \in C^*$; it is clear that $g_f \leqq f$, hence $g_f \in G$, since G is solid, and that $g_f(x^-) = 0$ because of $[0, x^-] \cap P = \{0\}$. Hence $g_f(x) = g_f(x^+)$, and we obtain $\tilde{x}^+(f) = \sup\{g(x): 0 \leqq g \leqq f\} \geqq g_f(x) = g_f(x^+) = f(x^+)$ for all $f \in C^* \cap G$. This implies $\tilde{x}^+ \geqq (x^+)^\sim$; since it is clear that $(x^+)^\sim \geqq \tilde{x}^+$ in G^+, the assertion follows.

We point out that the canonical image of E in G^+ is, in general, not an order complete sublattice of G^+ even if E is order complete (see the example following (7.4)). In particular (taking $G = E^+$), a regularly ordered, order complete vector lattice E need not be mapped onto a band in E^{++} under evaluation. If E is an order complete, regularly ordered vector lattice whose canonical image in E^{++} is order complete, E will be called **minimal** (or of **minimal type**).

If E, F are vector lattices, a linear map u of E onto F is called a **lattice homomorphism** provided that u preserves the lattice operations; in view of the linearity of u, the translation-invariance of the order and the identity (3), this condition on u is equivalent to each of the following: (i) $u(\sup(x, y)) = \sup(u(x), u(y))$ $(x, y \in E)$. (ii) $u(\inf(x, y)) = \inf(u(x), u(y))$ $(x, y \in E)$. (iii) $u(|x|) = \sup(u(x^+), u(x^-))$ $(x \in E)$. (iv) $\inf(u(x^+), u(x^-)) = 0$ $(x \in E)$. If, in addition, u is biunivocal, then u is called a **lattice isomorphism** of E onto F. It is not difficult to show that a linear map u of E onto F is a lattice homomorphism if and only if $u^{-1}(0)$ is a solid sublattice of E and $u(C_1) = C_2$, where C_1, C_2 denote the respective positive cones of E, F. In particular, if N is a solid vector sublattice of E, then E/N is a vector lattice under its canonical order and the canonical map ϕ is a lattice homomorphism of E onto E/N (Exercise 12).

The linear forms on a vector lattice E that are lattice homomorphisms onto $\boldsymbol{R}$ have an interesting geometric characterization; let us recall (Chapter II, Exercise 30) that $\{\lambda x: \lambda \geqq 0\}$, $0 \neq x \in C$ is called an extreme ray of the cone C if $x - y \in C$, $y \in C$ imply $y = \rho x$ for some ρ, $0 \leqq \rho \leqq 1$.

1.7

Let E be a vector lattice, $f \neq 0$ a linear form on E. The following assertions are equivalent:

(a) *f is a lattice homomorphism of E onto $\boldsymbol{R}$.*
(b) $\inf(f(x^+), f(x^-)) = 0$ *for all $x \in E$.*
(c) *f generates an extreme ray of the cone C^* in E^*.*
(d) $f \geqq 0$ *and $f^{-1}(0)$ is a solid hyperplane in E.*

Proof. (a)$\Leftrightarrow$(b) is clear from the preceding remarks. (b)$\Rightarrow$(d): Since $\inf(f(x^+), f(x^-)) = 0$ for each $x \in E$, it follows that $f \geqq 0$, and $f(x) = 0$ implies $f(|x|) = 0$; hence $|y| \leqq |x|$ and $f(x) = 0$ imply $|f(y)| \leqq f(|y|) \leqq f(|x|) = 0$. (d)$\Rightarrow$(c): Suppose $g \in C^*$ is such that $f - g \in C^*$ or, equivalently, that $0 \leqq g \leqq f$. Then since $f^{-1}(0)$ is solid, $f(x) = 0$ implies $|g(x)| \leqq g(|x|) \leqq f(|x|) = 0$ and hence $f^{-1}(0) \subset g^{-1}(0)$. Thus (since $f^{-1}(0)$ is a hyperplane) either $g = 0$ or $f^{-1}(0) = g^{-1}(0)$; in any case, $g = \rho f$ for some ρ, $0 \leqq \rho \leqq 1$. (c)$\Rightarrow$(b): Let f generate an extreme ray of C^*, let $x \in E$ be given, and suppose that $f(x^+) > 0$. Let $P = \bigcup\{\rho[0, x^+]: \rho \geqq 0\}$, and define $h \in E^*$ by putting, for $y \geqq 0$, $h(y) = \sup\{f(z): z \in [0, y] \cap P\}$ (see proof of (1.6)). It follows that $0 \leqq h \leqq f$ and hence $h = \rho f$ by the assumption made on f, and since $h(x^+) = f(x^+) > 0$ we must have $\rho = 1$. Thus $h = f$, and since clearly $h(x^-) = 0$, it follows that $f(x^-) = 0$, which completes the proof.

2. ORDERED VECTOR SPACES OVER THE COMPLEX FIELD

It is often useful to have the concept of an ordered vector space over the complex field $\boldsymbol{C}$. Such is the case, for instance, in spectral theory and in measure theory. It is the purpose of this section to agree on a definite terminology. We define a vector space L over $\boldsymbol{C}$ to be **ordered** if its underlying real space L_0 (Chapter I, Section 7) is an ordered vector space over $\boldsymbol{R}$; thus by definition, order properties of L are order properties of L_0. The usefulness of this (otherwise trivial) definition lies in the fact that the transition to L_0 does not have to be mentioned continually.

The canonical orderings of products, subspaces, direct sums, quotients, function spaces, and spaces of linear maps are then defined with reference to the respective underlying real spaces; only the term "positive linear form" on L has to be additionally specified when L is an ordered vector space over $\boldsymbol{C}$. We define $f \in L^*$ to be **positive** if $\operatorname{Re} f(x) \geqq 0$ whenever $x \geqq 0$ in L; this definition guarantees that whenever the canonical ordering of $(L_0)^*$ is defined, then L^* is ordered, and the canonical isomorphism of (I, 7.2) is an order isomorphism (a corresponding statement holding for subspaces of L^*). The **order bound dual** L^b of an ordered vector space L over $\boldsymbol{C}$ is then defined as the subspace of L^* containing exactly the linear forms bounded on each order interval in L; the **order dual** L^+ is the (complex) subspace of L^* which is the linear hull of the cone C^* of positive linear forms. In accordance with the definition given above, the order of L is called regular if L_0 is regularly ordered; we point out that this is not implied by the fact that C^* separates points in L, and that in general $(L^+)_0$ cannot be identified with $(L_0)^+$ by virtue of (I, 7.2) (Exercise 4).

The term *vector lattice* will not be extended to complex spaces; we shall, however, say that an ordered vector space L over $\boldsymbol{C}$ with positive cone C is **lattice ordered** if the real subspace $C - C$ of L is a vector lattice. For example, the complexification (Chapter I, Section 7) of a vector lattice L is a lattice ordered vector space L_1 over $\boldsymbol{C}$.

3. DUALITY OF CONVEX CONES

Let L be a vector space (over $\boldsymbol{R}$ or $\boldsymbol{C}$); by a **cone** in L we shall henceforth understand a convex cone C of vertex 0 and such that $0 \in C$. Let C be a fixed cone in L; for any pair $(x, y) \in L \times L$, we shall write $[x, y] = (x + C) \cap (y - C)$. This notation is consistent with the notation introduced for order intervals in Section 1; if C is the positive cone of an ordering of L, then $(x + C) \cap (y - C)$ is the order interval $\{z: x \leqq z \leqq y\}$. For any subset $A \subset L$, define

$$[A] = (A + C) \cap (A - C) = \bigcup\{[x, y]: x \in A, y \in A\}.$$

A subset $B \subset L$ is called **C-saturated** if $B = [B]$; it is immediate that for any $A \subset L$, $[A]$ is the intersection of all C-saturated subsets containing A, and hence called the **C-saturated hull** of A. It is also quickly verified that $A \to [A]$ is monotone: $A \subset B$ implies $[A] \subset [B]$, that $[A]$ is convex if A is convex, and that $[A]$ is circled with respect to $\boldsymbol{R}$ if A is circled with respect to $\boldsymbol{R}$. Finally we note that if $\mathfrak{F}$ is a filter (more generally, a filter base) in L, then the family $\{[F]: F \in \mathfrak{F}\}$ is a filter base in L; the corresponding filter will be denoted by $[\mathfrak{F}]$.

Assume now that L is a t.v.s. A cone C in L is said to be **normal** if $\mathfrak{U} = [\mathfrak{U}]$ where $\mathfrak{U}$ is the neighborhood filter of 0. Hence C is a normal cone in the t.v.s. L if and only if there exists a base of C-saturated neighborhoods of 0 (equivalently, if and only if the family of all C-saturated 0-neighborhoods is a base at 0). It will be useful to have a number of alternative characterizations of normal cones.

3.1

Let L be a t.v.s. over K and let C be a cone in L. The following propositions are equivalent:

(a) *C is a normal cone.*
(b) *For every filter $\mathfrak{F}$ in L, $\lim \mathfrak{F} = 0$ implies $\lim[\mathfrak{F}] = 0$.*
(c) *There exists a 0-neighborhood base $\mathfrak{B}$ in L such that $V \in \mathfrak{B}$ implies $[V \cap C] \subset V$.*

If $K = \boldsymbol{R}$ and the topology of L is locally convex, then (a) *is equivalent to each of the following:*

(d) *There exists a 0-neighborhood base consisting of convex, circled, and C-saturated sets.*
(e) *There exists a generating family $\mathscr{P}$ of semi-norms on L such that $p(x) \leqq p(x + y)$ whenever $x \in C$, $y \in C$ and $p \in \mathscr{P}$.*

Proof. Denote by $\mathfrak{U}$ the neighborhood filter of 0 in L. (a) $\Rightarrow$ (b): If $\mathfrak{F}$ is a filter on L which is finer than $\mathfrak{U}$, then $[\mathfrak{F}]$ is finer than $[\mathfrak{U}]$; hence the assertion follows from $\mathfrak{U} = [\mathfrak{U}]$. (b) $\Rightarrow$ (c): (b) implies that $[\mathfrak{U}]$ is the neighborhood filter of 0 in L; hence $\mathfrak{B} = \{[U]: U \in \mathfrak{U}\}$ is a neighborhood base of 0 such that $V \in \mathfrak{B}$ implies $[V \cap C] \subset [V] = V$. (c) $\Rightarrow$ (a): Given $U \in \mathfrak{U}$, it suffices to show there exists $W \in \mathfrak{U}$ such that $[W] \subset U$. Let $\mathfrak{B}$ be a 0-neighborhood base as described in (c); select $V \in \mathfrak{B}$ such that $V + V \subset U$ and a circled $W \in \mathfrak{U}$ such that $W + W \subset V$. We obtain

$$[W] = \bigcup_{x, y \in W} [x, y] = \bigcup_{x, y \in W} (x + [0, y - x]) \subset W + [(W + W) \cap C] \subset V + [V \cap C] \subset V + V \subset U,$$

which proves the implication (c) $\Rightarrow$ (a).

Assume now that $K = \boldsymbol{R}$ and the topology of L is locally convex. (a) $\Rightarrow$ (d): If $\mathfrak{U}_1$ is the family of all convex, circled 0-neighborhoods in L, then $\mathfrak{W} = \{[U]: U \in \mathfrak{U}_1\}$ is a base at 0 consisting of convex, circled, and C-saturated sets. (d) $\Rightarrow$ (e): If $\mathfrak{W}$ is a 0-neighborhood base as in (d) and p_W is the gauge function of $W \in \mathfrak{W}$, the family $\{p_W: W \in \mathfrak{W}\}$ is of the desired type. (e) $\Rightarrow$ (c): If $\mathscr{P}$ is as in (e) then the family of all finite intersections of the sets $V_{p,\varepsilon} = \{x \in L: p(x) \leqq \varepsilon\}$ $(p \in \mathscr{P}, \varepsilon > 0)$ is a neighborhood base $\mathfrak{B}$ of 0 having the property stated in (c). This completes the proof.

COROLLARY 1. *If L is a Hausdorff t.v.s., every normal cone C in L is a proper cone.*

Proof. In fact, if $x \in C \cap -C$, then $x \in [\{0\}] \subset [U]$ for each 0-neighborhood U, and it follows that $x = 0$.

COROLLARY 2. *If C is a normal cone in L and $B \subset L$ is bounded, then $[B]$ is bounded; in particular, each set $[x, y]$ is bounded.*

Proof. If B is bounded and U is a 0-neighborhood in L, there exists $\lambda > 0$ such that $B \subset \lambda U$; it follows that $[B] \subset [\lambda U] = \lambda[U]$.

COROLLARY 3. *If the topology of L is locally convex, the closure $\bar{C}$ of a normal cone is a normal cone.*

Proof. It is immediate that $\bar{C}$ is a cone in L, and $\bar{C}$ is also the closure of C in the real space L_0; the assertion follows now from proposition (e) of (3.1).

It will become evident from the results in this chapter and the Appendix that the concept of a normal cone is an important (and perhaps the most important) notion in the theory of ordered topological vector spaces; for cones in normed spaces over $\boldsymbol{R}$ it goes back to M.G. Krein [2]. The original definition of Krein postulates the existence of a constant $\gamma (\geqq 1)$ such that $\|x\| \leqq \gamma \|x + y\|$ for all $x, y \in C$; it follows at once that this definition is equivalent, for normed spaces $(L, \| \; \|)$ over $\boldsymbol{R}$, with the one given above, and (3.1) (e) implies that there exists an equivalent norm on L for which one can suppose $\gamma = 1$.

If M is a subspace of the t.v.s. L and C is a normal cone in L, it is clear that $M \cap C$ is a normal cone in M; it is also easy to verify that if $\{L_\alpha: \alpha \in \mathrm{A}\}$ is a family of t.v.s., C_α a cone in L_α, and $L = \prod_\alpha L_\alpha$, then $C = \prod_\alpha C_\alpha$ is a normal cone in L if and only if C_α is normal in $L_\alpha (\alpha \in \mathrm{A})$. Let us record the following result on locally convex direct sums.

3.2

If $\{L_\alpha: \alpha \in \mathrm{A}\}$ is a family of l.c.s., C_α a cone in $L_\alpha(\alpha \in \mathrm{A})$, and $L = \bigoplus_\alpha L_\alpha$ the locally convex direct sum of this family, then $C = \bigoplus_\alpha C_\alpha$ is a normal cone in L if and only if C_α is normal in L_α $(\alpha \in \mathrm{A})$.

Proof. The necessity of the condition is immediate, since each L_α can be identified with a subspace of L such that C_α is identified with $L_\alpha \cap C$ $(\alpha \in \mathrm{A})$. To prove that the condition is sufficient assume that $K = \boldsymbol{R}$ (which can be arranged, if necessary, by transition to the underlying real space L_0 of L). Let $\mathfrak{B}_\alpha$ be a neighborhood base of 0 in L_α $(\alpha \in \mathrm{A})$ satisfying (3.1) (d); the family of all sets $V = \Gamma_\alpha V_\alpha$ $(V_\alpha \in \mathfrak{B}_\alpha, \alpha \in \mathrm{A})$ is a neighborhood base of 0 in L (Chapter II, Section 6). Now it is clear that $[V \cap C]$ is the convex hull of $\bigcup_\alpha [V_\alpha \cap C_\alpha]$; since $[V_\alpha \cap C_\alpha] \subset V_\alpha$ for all $V_\alpha \in \mathfrak{B}_\alpha$ $(\alpha \in \mathrm{A})$, it follows that $[V \cap C] \subset V$, which proves the assertion in view of (3.1) (c).

It can be shown in a similar fashion that a corresponding result holds for the direct sum topology introduced in Exercise 1, Chapter I (in this case, the spaces L_α need not be supposed to be locally convex). On the other hand, if C is a normal cone in L and M is a subspace of L, then the canonical image $\hat{C}$ of C in L/M is, in general, not a proper cone, let alone normal. (For a condition under which $\hat{C}$ is normal, see Exercise 3.)

Intuitively speaking, normality of a cone C in a t.v.s. L restricts the "width" of C and hence, in a certain sense, is a gauge of the pointedness of C. For example, a normal cone in a Hausdorff space cannot contain a straight line ((3.1), Corollary 1); a cone C in a finite-dimensional Hausdorff space L is normal if its closure $\bar{C}$ is proper (cf. (4.1) below). In dealing with dual pairs of cones, one also needs a tool working in the opposite direction and gauging, in an analogous sense, the bluntness of C. The requirement that $L = C - C$ goes in this direction; in fact, it indicates that every finite subset S of L can be recovered from C in the sense that $S \subset S_0 - S_0$ for a suitable finite subset $S_0 \subset C$. The precise definition of the property we have in mind is as follows.

Let L be a t.v.s., let C be a cone in L, and let $\mathfrak{S}$ be a family of bounded subsets of L (Chapter III, Section 3); for each $S \in \mathfrak{S}$, define S_C to be the subset $S \cap C - S \cap C$ of L. We say that C is an **$\mathfrak{S}$-cone** if the family $\{\bar{S}_C: S \in \mathfrak{S}\}$ is a fundamental subfamily of $\mathfrak{S}$; C is called a **strict $\mathfrak{S}$-cone** if $\{S_C: S \in \mathfrak{S}\}$ is fundamental for $\mathfrak{S}$. If L is a l.c.s. over $\boldsymbol{R}$ and $\mathfrak{S}$ is a saturated family, in place of S_C we can use the convex, circled hull of $S \cap C$ in the preceding definitions. A case of particular importance is the case where $\mathfrak{S} = \mathfrak{B}$ is the family of all bounded subsets of L: C is a $\mathfrak{B}$-cone in L. The notion of a $\mathfrak{B}$-cone in a normed space $(L, \| \ \|)$ appears to have been first used by Bonsall

[2]; Bonsall defines L to have the decomposition property if each z, $\|z\| \leqq 1$, can be approximated with given accuracy by differences $x - y$, where $x \in C$, $y \in C$ and $\|x\| \leqq k$, $\|y\| \leqq k$ for a fixed constant $k > 0$.

The property of being an $\mathfrak{S}$-cone satisfies certain relations of permanence (Exercise 5); since these are consequences of (3.3), below, the permanence properties of normal cones, and the duality theorems (IV, 4.1) and (IV, 4.3), they will be omitted here. Let us point out that, as the concept of a normal cone, the concept of an $\mathfrak{S}$-cone is independent of the scalar field ($\boldsymbol{R}$ or $\boldsymbol{C}$) over which L is defined.

Examples

1. The set of real-valued, non-negative functions determines a normal cone in each of the Banach spaces enumerated in Chapter II, Examples 1–3. If E is any one of these spaces and C the corresponding cone, then $E = C - C$ if $K = \boldsymbol{R}$; this implies that C is a strict $\mathfrak{B}$-cone in E (see (3.5) below). If the functions (or classes of functions) that constitute E are complex valued, then C and $C + iC$ are normal cones and $C + iC$ is a strict $\mathfrak{B}$-cone.

2. Let C denote the set of all non-negative functions in the space $\mathscr{D}$ of L. Schwartz (Chapter II, Section 6, Example 2). C is not a normal cone in $\mathscr{D}$, but $C + iC$ is a strict $\mathfrak{B}$-cone. The cone C_1 of all distributions T such that $(Tf) \geqq 0$ for $f \in C$ (which can be identified with the set of all positive Radon measures on $\boldsymbol{R}^n$ (cf. L. Schwartz [1])) is a normal cone in $\mathscr{D}'$, but $C_1 + iC_1$ is not a $\mathfrak{B}$-cone (Exercise 6).

3. Let E be the space of complex-valued, continuous functions with compact support on a locally compact space X with its usual topology (Chapter II, Section 6, Example 3), and let C be the cone of non-negative functions in E. C is a normal cone in E, $C + iC$ is normal and a strict $\mathfrak{B}$-cone. If C_1 denotes the set of all positive Radon measures on X, $C_1 + iC_1$ is normal and a strict $\mathfrak{B}$-cone in the strong dual E'.

The proofs for these assertions will become clear from the following results and are therefore omitted.

If C is a cone in the t.v.s. E, the **dual cone** C' of C is defined to be the set $\{f \in E' \colon \operatorname{Re} f(x) \geqq 0 \text{ if } x \in C\}$; hence C' is the polar of $-C$ with respect to $\langle E, E' \rangle$. In the following proofs it will often be assumed that the scalar field K of E is $\boldsymbol{R}$; whenever this is done, implicit reference is made to (I, 7.2) (cf. also Section 2). Before proving the principal result of this section, we establish this lemma which is due to M. G. Krein [2].

Lemma 1. *If C is a normal cone in the normed space E, then $E' = C' - C'$.*

Proof. We can assume that $K = \boldsymbol{R}$. Let $f \in E'$ and define the real function $p \geqq 0$ on C by $p(x) = \sup\{f(z) \colon z \in [0, x]\}$. Then it is clear that $p(\lambda x) = \lambda p(x$n if $\lambda \geqq 0$ and that $p(x + y) \geqq p(x) + p(y)$, since $[0, x] + [0, y] \subset [0, x + y]$ for) all $x, y \in C$. It follows that the set

$$V = \{(t, x) \colon 0 \leqq t \leqq p(x)\}$$

is a cone in the product space $\boldsymbol{R}_0 \times E$. Let $\{x_n: n \in N\}$ be a null sequence in E and suppose that $\{t_n: n \in N\}$ is a sequence of real numbers such that $(t_n, x_n) \in V$ $(n \in N)$. Since C is a normal cone and f is continuous, it follows that $p(x_n) \to 0$ and hence that $t_n \to 0$; this implies that $(1, 0)$ is not in the closure $\bar{V}$ of V in the normable space $\boldsymbol{R}_0 \times E$. By (II, 9.2) there exists a closed hyperplane H strictly separating $\{(1, 0)\}$ and V; it can be arranged that $H = \{(t, x): h(t, x) = -1\}$, where $h(1, 0) = -1$ and h is $\geqq 0$ on V. By (IV, 4.3) h is of the form $(t, x) \to -t + g(x)$; since $g \in E'$ and $(0, x) \in V$ for each $x \in C$, it follows that $g \in C'$. Now $(p(x), x) \in V$ for all $x \in C$; hence we have $-p(x) + g(x) \geqq 0$ if $x \in C$. Since $f(x) \leqq p(x) \leqq g(x)$ for $x \in C$, we obtain $f = g - (g - f)$, where $g \in C'$, $g - f \in C'$, and the lemma is proved.

3.3

Theorem. *Let E be a l.c.s., let C be a cone in E with dual cone $C' \subset E'$, and let $\mathfrak{S}$ be a saturated family of weakly bounded subsets of E'. If C' is an $\mathfrak{S}$-cone, then C is normal for the $\mathfrak{S}$-topology on E; conversely, if C is normal for an $\mathfrak{S}$-topology consistent with $\langle E, E' \rangle$, then C' is a strict $\mathfrak{S}$-cone in E'.*

Proof. We can assume that $K = \boldsymbol{R}$. If C' is an $\mathfrak{S}$-cone in E', then the saturated hull of the family $\{\Gamma(S \cap C'): S \in \mathfrak{S}\}$ equals $\mathfrak{S}$; hence the $\mathfrak{S}$-topology is generated by the semi-norms

$$x \to p_S(x) = \sup\{|\langle x, x' \rangle| : x' \in S \cap C'\} \qquad (S \in \mathfrak{S})$$

which are readily seen to satisfy proposition (e) of (3.1).

Suppose now that $\mathfrak{T}$ is an $\mathfrak{S}$-topology on E consistent with $\langle E, E' \rangle$ and that C is normal with respect to $\mathfrak{T}$. By (3.1) (d) there exists a 0-neighborhood base $\mathfrak{U}$ in $(E, \mathfrak{T})$ consisting of convex, circled, and C-saturated sets. Since $\{U^\circ: U \in \mathfrak{U}\}$ is a fundamental subfamily of $\mathfrak{S}$, it suffices to show that there exists, for each $U \in \mathfrak{U}$, an integer n_0 such that $U^\circ \subset n_0(U^\circ \cap C' - U^\circ \cap C')$. Let $U \in \mathfrak{U}$ be fixed.

Now the dual of the normed space E_U (for notation, see Chapter III, Section 7) can be identified with E'_{U° and the cone $C' \cap E'_{U^\circ}$ can be identified with the dual cone of $C_U = \phi_U(C)$ where, as usual, ϕ_U is the canonical map $E \to \tilde{E}_U$. Using the fact that U is C-saturated, it is readily seen that C_U is a normal cone in E_U; hence if we define the set $M \subset E'$ by $M = U^\circ \cap C' - U^\circ \cap C'$, Lemma 1 implies that $E'_{U^\circ} = \bigcup_{n \in N} nM$. Now M is $\sigma(E', E)$-compact by (I, 1.1) (iv), hence $\sigma(E', E)$-closed and a fortiori closed in the Banach space E'_{U°; since the latter is a Baire space, it follows that M has an interior point and hence (being convex and circled) is a neighborhood of 0 in E'_{U°; it follows that $U^\circ \subset n_0 M$ for a suitable $n_0 \in N$ and the proof is complete.

COROLLARY 1. *Let C be a cone in the l.c.s. E. The following assertions are equivalent:*

(a) *C is a normal cone in E.*

(b) *For any equicontinuous set $A \subset E'$, there exists an equicontinuous set $B \subset C'$ such that $A \subset B - B$.*
(c) *The topology of E is the topology of uniform convergence on the equicontinuous subsets of C'.*

COROLLARY 2. *If $\mathfrak{S}$ is a saturated family, covering E', of $\sigma(E', E)$-relatively compact sets and if H is an $\mathfrak{S}$-cone in E', then the $\sigma(E', E)$-closure $\bar{H}$ of H is a strict $\mathfrak{S}$-cone.*

Proof. In fact, the cone $C = -H^0$ is normal for the $\mathfrak{S}$-topology which is consistent with $\langle E, E' \rangle$, (IV, 3.2), and $C' = \bar{H}$ by (IV, 1.5).

COROLLARY 3. *If C is a cone in the l.c.s. E, then $E' = C' - C'$ if and only if C is weakly normal; in particular, every normal cone in E is weakly normal.*

We obtain this corollary by taking $\mathfrak{S}$ to be the saturated hull of the family of all finite subsets of E'. Let us point out that if C is a cone in a l.c.s. E over $\mathbf{C}$, it is sometimes of interest to consider the cone $H \subset E'$ of linear forms whose real *and* imaginary parts are $\geqq 0$ on C; we have $H = C' \cap (-iC)'$, and it follows from Corollary 3 above and (IV, 1.5), Corollary 2, that $E' = H - H$ if and only if $C + iC$ (equivalently, $C - iC$) is weakly normal in E (for, $H = (C - iC)'$).

REMARK. In normed spaces, weak normality and normality of cones are equivalent (see (3.5) below).

The following is an application of (3.3) to the case where $\mathfrak{S}$ is the family of all strongly bounded subsets of the dual of an infrabarreled space E; recall that this class comprises all barreled and all bornological (hence all metrizable l.c.) spaces.

3.4

Let E be an infrabarreled l.c.s., C a cone in E, $\mathfrak{B}$ the family of all strongly bounded subsets of E'. The following assertions are equivalent:

(a) *C is a normal cone in E.*
(b) *The topology of E is the topology of uniform convergence on strongly bounded subsets of C'.*
(c) *C' is a $\mathfrak{B}$-cone in E'.*
(d) *C' is a strict $\mathfrak{B}$-cone in E'.*

The proof is clear from the preceding in view of the fact that $\mathfrak{B}$ is the family of all equicontinuous subsets of E' (Chapter IV, Section 5).

COROLLARY. *If E is a reflexive space, normal cones and $\mathfrak{B}$-cones correspond dually to each other (with respect to $\langle E, E' \rangle$).*

It is an interesting fact that the complete symmetry between normal and $\mathfrak{B}$-cones under the duality $\langle E, E' \rangle$ remains in force, without reflexivity assumptions, when E is a Banach space (cf. Ando [2]). From the proof of this result we isolate the following lemma, which will be needed later, and is of some interest in itself.

LEMMA 2. *Let $(E, \mathfrak{T})$ be a metrizable t.v.s. over $\mathbf{R}$, let C be a cone in E which is complete, and let $\{U_n: n \in N\}$ be a neighborhood base of 0 consisting of closed, circled sets such that $U_{n+1} + U_{n+1} \subset U_n$ $(n \in N)$. Then the sets*

$$V_n = U_n \cap C - U_n \cap C \qquad (n \in N)$$

form a 0-neighborhood base for a topology $\mathfrak{T}_1$ on $E_1 = C - C$ such that $(E_1, \mathfrak{T}_1)$ is a complete (metrizable) t.v.s. over $\mathbf{R}$.

Proof. It is clear that each set V_n is radial and circled in E_1, and obviously $V_{n+1} + V_{n+1} \subset V_n$ for all $n \in N$. It follows from (I, 1.2) that $\{V_n: n \in N\}$ is a 0-neighborhood base for a (unique translation invariant) topology $\mathfrak{T}_1$ on E_1 under which E_1 is a t.v.s. Of course $(E_1, \mathfrak{T}_1)$ is metrizable, and there remains to prove that $(E_1, \mathfrak{T}_1)$ is complete. In fact, given a Cauchy sequence in $(E_1, \mathfrak{T}_1)$, there exists a subsequence $\{z_n\}$ such that $z_{n+1} - z_n \in V_n (n \in N)$; we have, consequently, $z_{n+1} - z_n = x_n - y_n$, where x_n and y_n are elements of $U_n \cap C$, and it is evidently sufficient to show that the series $\sum_{n=1}^{\infty} x_n$ and $\sum_{n=1}^{\infty} y_n$ converge in $(E_1, \mathfrak{T}_1)$. Let us show this for $\sum_{n=1}^{\infty} x_n$. Letting $u_n = \sum_{\nu=1}^{n} x_\nu$ $(n \in N)$, we obtain

$$u_{n+p} - u_n \in (U_{n+1} + \cdots + U_{n+p}) \cap C \subset (U_n \cap C) \subset V_n$$

for all $p \in N$ and $n \in N$. Since C is complete in $(E, \mathfrak{T})$, $\{u_n\}$ converges for $\mathfrak{T}$ to some $u \in C$ and we have $u - u_{n+1} \in U_n \cap C \subset V_n$, since $U_n \cap C$ is closed in $(E, \mathfrak{T})$. Now the last relation shows that $u_n \to u$ in $(E_1, \mathfrak{T}_1)$, and the proof is complete.

3.5

Theorem. *Let E be a Banach space and let C be a closed cone in E. Then C is normal (respectively, a strict $\mathfrak{B}$-cone) if and only if C' is a strict $\mathfrak{B}$-cone (respectively, normal) in E'_β.*

Proof. The assertion concerning normal cones $C \subset E$ is a special case of (3.4), and if C is a $\mathfrak{B}$-cone, then C' is normal in E'_β by (3.3). Hence suppose that C' is normal in E'_β and denote by U the unit ball of E. The bipolar of $U \cap C$ (with respect to $\langle E', E''\rangle$) is $U^{\circ\circ} \cap C''$ and by (3.3) C'' is a strict $\mathfrak{B}$-cone in the strong bidual E''; hence $U^{\circ\circ} \cap C'' - U^{\circ\circ} \cap C''$ is a 0-neighborhood in E''. It follows that $V = U \cap C - U \cap C$ is dense in a 0-neighborhood V_1 in E; if $E_1 = C - C$ and $\mathfrak{T}_1$ is the topology on E_1 defined in Lemma 2, this means precisely that the imbedding ψ of $(E_1, \mathfrak{T}_1)$ into E is nearly open and continuous, with dense range. Consequently, Banach's homomorphism theorem (III, 2.1) implies that ψ is a topological isomorphism of $(E_1, \mathfrak{T}_1)$ onto E, and hence C is a strict $\mathfrak{B}$-cone in E.

COROLLARY. *Let E be a Banach space and let C be a cone in E with closure $\bar{C}$. The following assertions are equivalent:*

(a) *C is a $\mathfrak{B}$-cone in E.*
(b) $E = \bar{C} - \bar{C}$.
(c) *$\bar{C}$ is a strict $\mathfrak{B}$-cone in E.*

Proof. (a) $\Rightarrow$ (c) is clear from the preceding since C' is normal in E' whenever C is a $\mathfrak{B}$-cone, by (3.3). (c) $\Rightarrow$ (b) is trivial. (b) $\Rightarrow$ (a): Let M denote the closure of $U \cap C - U \cap C$, where U is the unit ball of E. Then M is convex, circled (over $\boldsymbol{R}$), and such that $E = \bigcup_1^\infty nM$; since E is a Baire space, it follows that M is a 0-neighborhood in E and hence C is a $\mathfrak{B}$-cone.

4. ORDERED TOPOLOGICAL VECTOR SPACES

Let L be a t.v.s. (over $\boldsymbol{R}$ or $\boldsymbol{C}$) and an ordered vector space; we say that L is an **ordered topological vector space** if the following axiom is satisfied:

(LTO) *The positive cone $C = \{x\colon x \geqq 0\}$ is closed in L.*

Recall that an Archimedean ordered vector space is called regularly ordered if the real bilinear form $(x, x^*) \to \operatorname{Re}\langle x, x^*\rangle$ places L_0 and L_0^+ in duality, where L_0 is the real underlying space of L (Chapter I, Section 7). In order to prove some alternative characterizations, we need the following lemma which is of interest in itself. (Cf. Exercise 21.)

LEMMA. *Let E be an ordered vector space of finite dimension over $\boldsymbol{R}$. The order of E is Archimedean if and only if the positive cone C is closed for the unique topology under which E is a Hausdorff t.v.s.*

Proof. If C is closed, then clearly the order of E is Archimedean. Conversely, suppose that E is Archimedean ordered; without restriction of generality we can assume that $E = C - C$. If the dimension of E is n ($\geqq 1$), then C contains n linearly independent elements $x_1, \ldots, x_n$ and hence the n-dimensional simplex with vertices $0, x_1, \ldots, x_n$; since the latter has non-empty interior, so does C. Now let $x \in \bar{C}$ and let y be interior to C; by (II, 1.1) $n^{-1}y + x$ is interior to C ($n \in \boldsymbol{N}$), and hence we have $-x \leqq n^{-1}y$ for all n. This implies $-x \leqq 0$ or, equivalently, $x \in C$.

4.1

If L is an ordered vector space over $\boldsymbol{R}$ with positive cone C, the following propositions are equivalent:

(a) *The order of L is regular.*
(b) *C is sequentially closed for some Hausdorff l.c. topology on L, and L^+ distinguishes points in L.*

(c) *The order of L is Archimedean, and C is normal for some Hausdorff l.c. topology on L.*

Proof. (a) $\Rightarrow$ (b): It suffices to show that the intersection of C with every finite dimensional subspace M is closed (Chapter II, Exercise 7), and this is immediate from the preceding lemma, since the order of L is Archimedean, and hence the canonical order of each subspace $M \subset L$ is Archimedean. (b) $\Rightarrow$ (c): If $\mathfrak{T}$ is a Hausdorff l.c. topology under which C is sequentially closed, then, clearly, L is Archimedean ordered. Moreover, since L^+ separates points in L, the canonical bilinear form on $L \times L^*$ places L and $L^+ = C^* - C^*$ in duality, and by (3.3) C is normal for the Hausdorff l.c. topology $\sigma(L, L^+)$. (c) $\Rightarrow$ (a): It suffices to show that L^+ separates points in L. If $\mathfrak{T}$ is a Hausdorff l.c. topology for which C is normal, then $(L, \mathfrak{T})' = C' - C'$ by (3.3); hence $C' - C'$, and a fortiori $L^+ = C^* - C^*$ separates points in L. This completes the proof.

COROLLARY 1. *The canonical orderings of subspaces, products, and direct sums of regularly ordered vector spaces are regular.*

COROLLARY 2. *Every ordered locally convex space is regularly ordered.*

Proof. If $(E, \mathfrak{T})$ is an ordered l.c.s., then C is closed by definition, and the bipolar theorem (IV, 1.5) shows $-C$ to be the polar of C' with respect to $\langle E, E' \rangle$. Since $C \cap -C = \{0\}$, it follows that $C' - C'$ is weakly dense in $(E, \mathfrak{T})'$, hence $L^+ = C^* - C^*$ separates points in L.

If A is an ordered set and $S \subset A$ is a subset ($\neq \emptyset$) directed for $\leqq$, recall that the section filter $\mathfrak{F}(S)$ is the filter on A determined by the base $\{S_x \colon x \in S\}$, where $S_x = \{y \in S \colon y \geqq x\}$, and S_x is called a section of S. In particular, if S is a monotone sequence in A, then $\mathfrak{F}(S)$ is the filter usually associated with S.

4.2

Let L be an ordered t.v.s. and let S be a subset of L directed for $\leqq$. If the section filter $\mathfrak{F}(S)$ converges to $x_0 \in L$, then $x_0 = \sup S$.

Proof. Let $x \in S$ and let z be any element of L majorizing S; we have $x \leqq y \leqq z$ for all $y \in S_x$, and from $x_0 \in \bar{S}_x$ it follows that $x \leqq x_0 \leqq z$, since the positive cone is closed in L. This proves that $x_0 = \sup S$.

A deeper result is the following monotone convergence theorem, which can be viewed as an abstract version of a classical theorem of Dini. Although it can be derived from Dini's theorem, using (4.4) below (Exercise 9), we give a direct proof based on the Hahn-Banach theorem.

4.3

Theorem. *Let E be an ordered l.c.s. whose positive cone C is normal, and suppose that S is a subset of L directed for $\leqq$. If the section filter $\mathfrak{F}(S)$ converges for $\sigma(E, E')$, then it converges in E.*

Proof. Without loss in generality we can suppose that S is directed for $\geqq$, and that $\lim \mathfrak{F}(S) = 0$ for $\sigma(E, E')$; it follows now from (4.2) that $S \subset C$. Assume that the assertion is false; then there exists a 0-neighborhood U in E that contains no section of S, and since C is normal we can suppose that U is convex and C-saturated. Since $x \in S \cap U$ implies $S_x \subset U$, it follows that $S \cap U = \varnothing$; moreover, $(S + C) \cap U = \varnothing$, since U is C-saturated, and $S + C$ is convex, since it is the union of the family $\{x + C: x \in S\}$ of convex sets which is directed under inclusion. Hence by (II, 9.2) U and S can be separated by a closed real hyperplane in E, and this contradicts the weak convergence of $\mathfrak{F}(S)$ to 0.

COROLLARY 1. *Let S be a directed ($\leqq$) subset of E such that $x_0 = \sup S$, where E is an ordered l.c.s. with normal positive cone. If for every real linear form f which is positive and continuous on E one has $f(x_0) = \sup\{f(x): x \in S\}$, then $\lim_{x \in S} g(x) = g(x_0)$ ($g \in E'$) uniformly on each equicontinuous subset of E'.*

Proof. In fact, in view of (3.3), Corollary 3, the weak convergence of $\mathfrak{F}(S)$ to x_0 is equivalent to the relation $f(x_0) = \sup\{f(x): x \in S\}$ for every real linear form f on E which is positive and continuous (cf. (I, 7.2)).

The reader will note that the preceding corollary is equivalent with (4.3). The following result can be viewed as a partial converse of (4.2).

COROLLARY 2. *Let E be a semi-reflexive, ordered l.c.s. whose positive cone is normal. If S is a directed ($\leqq$) subset of E which is majorized or (topologically) bounded, then $x_0 = \sup S$ exists and $\mathfrak{F}(S)$ converges to x_0.*

Proof. Let S_x be any fixed section of S; it suffices to show that $\sup S_x$ exists in E. If S is majorized by some $z \in E$, then $S_x \subset [x, z]$ and hence S_x is bounded in E by (3.1), Corollary 2; hence assume that S_x is bounded in E. The weak normality of C implies that $E' = C' - C'$, and hence that the section filter $\mathfrak{F}(S_x)$ is a weak Cauchy filter in E which is bounded. It follows from (IV, 5.5) that $\mathfrak{F}(S_x)$ converges to some $x_0 \in E$, and (4.2) implies that $x_0 = \sup S_x$, since C is closed and hence (being convex) weakly closed in E.

The following result is an imbedding (or representation) theorem for ordered l.c.s. over $\mathbf{R}$; let us denote by X a (separated) locally compact space, and by $R(X)$ the space of all real-valued continuous functions on X under the topology of compact convergence and endowed with its canonical order (Section 1).

4.4

Let E be an ordered l.c.s. over $\mathbf{R}$. If (and only if) the positive cone C of E is normal, there exists a locally compact space X such that E is isomorphic (as an ordered t.v.s.) with a subspace of $R(X)$.

Proof. The condition is clearly necessary, for the positive cone of $R(X)$ (and hence of every subspace of $R(X)$) is normal. To show that the condition is sufficient, we note first from (3.3), Corollary 1, that the topology of E is the topology of uniform convergence on the equicontinuous subsets of the dual cone $C' \subset E'$. Let $\{B_\alpha : \alpha \in \mathrm{A}\}$ be a fundamental family of $\sigma(E', E)$-closed equicontinuous subsets of C'; under the topology induced by $\sigma(E', E)$, each B_α is a compact space. We define X as follows: Endow A with the discrete topology, C' with the topology induced by $\sigma(E', E)$, and let X_α be the subspace $\{\alpha\} \times B_\alpha$ of the topological product $\mathrm{A} \times C'$; then X is defined to be the subspace $\bigcup_\alpha X_\alpha$ of $\mathrm{A} \times C'$. The space X is the topological sum of the family $\{B_\alpha : \alpha \in \mathrm{A}\}$; clearly, X is a locally compact space in which every X_α is open and compact, and hence every compact subset of X is contained in the union of finitely many sets X_α. For each $x \in E$, we define an element $f_x \in R(X)$ by putting $f_x(t) = \langle x, x' \rangle$ for every $t = (\alpha, x') \in X$; it is clear that $x \to f_x$ is an algebraic and order isomorphism of E into $R(X)$. Finally, since a closed subset of X is compact if and only if it is contained in a finite union $\bigcup_{\alpha \in \mathrm{H}} X_\alpha$, it is also evident that $x \to f_x$ is a homeomorphism.

Remarks. It is easy to see that $R(X)$ is complete; hence the image of E under $x \to f_x$ is closed in $R(X)$ if and only if E is complete. Moreover, if E is metrizable, then the family $\{B_\alpha : \alpha \in \mathrm{A}\}$ can be assumed to be countable, and hence X countable at infinity; if E is normable, one can take $X = U^\circ \cap C'$ (under $\sigma(E', E)$), where U is any bounded neighborhood of 0 in E (in particular, the unit ball if E is normed). If E is a separable normed space, $U^\circ \cap C'$ is a compact metrizable space for $\sigma(E', E)$ by (IV, 1.7) and hence a continuous image of the Cantor set (middle third set) in $[0, 1] \subset \boldsymbol{R}$; in this case X can be taken to be the Cantor set itself, or $[0, 1]$ (for details, see Banach [1], chap. XI, § 8, theor. 9).

Finally, proposition (4.4) can be specialized to the case $C = \{0\}$; we obtain thus a representation of an arbitrary l.c.s. E over $\boldsymbol{R}$ as a subspace of a suitable space $R(X)$; it is immediate that in this particular case, the restriction to the scalar field $\boldsymbol{R}$ can be dropped.

5. POSITIVE LINEAR FORMS AND MAPPINGS

The present section is concerned with special properties of linear maps $u \in L(E, F)$ which map the positive cone C of E into the positive cone D of F, where E, F are ordered vector spaces (respectively, ordered t.v.s.); these mappings are called **positive**. It is clear that the set H of all positive maps is a cone in $L(E, F)$; whenever M is a subspace of $L(E, F)$ such that $H \cap M$ is a proper cone, $H \cap M$ defines the canonical ordering of M (Section 1). Recall also (Section 2) that a linear form f on an ordered vector space E is called positive if $\mathrm{Re} f(x) \geqq 0$ for each x in the positive cone C of E.

We begin our investigation with some simple but useful observations concerning the properties of the cone $\mathscr{H} \subset \mathscr{L}(E, F)$ of continuous positive maps, where E, F are supposed to be ordered t.v.s. over K. We point out that in view of the agreements made in Section 2, it suffices in general to consider the case $K = \mathbf{R}$.

5.1

Let E, F be ordered t.v.s. and let $\mathfrak{S}$ be a family of bounded subsets of E that covers E. Then the positive cone $\mathscr{H} \subset \mathscr{L}(E, F)$ is closed for the $\mathfrak{S}$-topology. For $\mathscr{H}$ to be a proper cone, it is sufficient (and, if E is a l.c.s. and $F \neq \{0\}$, necessary) that the positive cone C of E be total in E.

Proof. In fact, by definition of the $\mathfrak{S}$-topology (Chapter III, Section 3) the bilinear map $(u, x) \to u(x)$ is separately continuous on $\mathscr{L}_{\mathfrak{S}}(E, F) \times E$ into F; hence the partial map f_x: $u \to u(x)$ is continuous for each $x \in E$. Since $\mathscr{H} = \bigcap \{f_x^{-1}(D): x \in C\}$ and the positive cone D of F is closed, $\mathscr{H}$ is closed in $\mathscr{L}_{\mathfrak{S}}(E, F)$. Further, since D is proper, $u \in \mathscr{H} \cap -\mathscr{H}$ implies that $u(x) = 0$ for $x \in C$; hence $u = 0$ if C is total in E. Finally, if E is a l.c.s. and C is not total in E, there exists an $f \in E'$ such that $f \neq 0$ but $f(C) = \{0\}$, by virtue of the Hahn-Banach theorem; if y is any element $\neq 0$ of F, the mapping $u = f \otimes y$ (defined by $x \to f(x)y$) satisfies $u \in \mathscr{H} \cap -\mathscr{H}$.

COROLLARY. *If C is total in E and if F is a l.c.s., the (canonical) ordering of $\mathscr{L}(E, F)$ defined by $\mathscr{H}$ is regular.*

Proof. In fact, $\mathscr{H}$ is a closed proper cone for the topology of simple convergence which is a Hausdorff l.c. topology by (III, 3.1), Corollary; the assertion follows from (4.1), Corollary 2.

5.2

Let E, F be ordered l.c.s. with respective positive cones C, D and let $\mathfrak{S}$ be a family of bounded subsets of E. If C is an $\mathfrak{S}$-cone in E and D is normal in F, the positive cone $\mathscr{H} \subset \mathscr{L}(E, F)$ is normal for the $\mathfrak{S}$-topology.

Proof. Since D is normal in F, there exists, by (3.1), a family $\{q_\alpha: \alpha \in \mathrm{A}\}$ of real semi-norms on F that generate the topology of F, and which are monotone (for the order of F) on D. Since C is an $\mathfrak{S}$-cone in E, it follows that the real semi-norms

$$u \to p_{\alpha,S}(u) = \sup\{q_\alpha(ux): x \in S \cap C\} \qquad (\alpha \in \mathrm{A},\ S \in \mathfrak{S})$$

generate the $\mathfrak{S}$-topology on $\mathscr{L}(E, F)$. Now, evidently, each $p_{\alpha,S}$ is monotone on $\mathscr{H}$ (for the canonical order of $\mathscr{L}(E, F)$); hence $\mathscr{H}$ is a normal cone in $\mathscr{L}_{\mathfrak{S}}(E, F)$, as asserted.

On the other hand, there are apparently no simple conditions guaranteeing that $\mathscr{H}$ is a $\mathfrak{T}$-cone in $\mathscr{L}_{\mathfrak{S}}(E, F)$, even for the most frequent types of families $\mathfrak{T}$ of bounded subsets of $\mathscr{L}_{\mathfrak{S}}(E, F)$, except in every special cases (cf. Exercise 7). At any rate, the following result holds where E, F are ordered l.c.s. with respective positive cones C, D, and $\mathscr{L}_s(E, F)$ denotes $\mathscr{L}(E, F)$ under the topology of simple convergence.

5.3

If C is weakly normal in E and if $F = D - D$, then $\mathscr{H} - \mathscr{H}$ is dense in $\mathscr{L}_s(E, F)$.

Proof. Since the weak normality of C is equivalent with $E' = C' - C'$ by (3.3), Corollary 3, the assumptions imply that $\mathscr{H} - \mathscr{H}$ contains the subspace $E' \otimes F$ of $\mathscr{L}(E, F)$. On the other hand, the dual of $\mathscr{L}_s(E, F)$ can be identified with $E \otimes F'$ by (IV, 4.3), Corollary 4, and it is known that (under the duality between $\mathscr{L}(E, F)$ and $E \otimes F'$) $E' \otimes F$ separates points in $E \otimes F'$ (Chapter IV, Section 1, Example 3); it follows from (IV, 1.3) that $E' \otimes F$ is weakly dense in $\mathscr{L}_s(E, F)$ and hence (being convex) dense in $\mathscr{L}_s(E, F)$.

We turn to the question of extending a continuous positive linear form, defined on a subspace of an ordered t.v.s. E, to the entire space E. The following extension theorem is due to H. Bauer [1], [2] and, independently, to Namioka [1].

5.4

Theorem. *Let E be an ordered t.v.s. with positive cone C and let M be a subspace of E. For a linear form f_0 on M to have an extension f to E which is a continuous positive linear form, it is necessary and sufficient that $\mathrm{Re} f_0$ be bounded above on $M \cap (U - C)$, where U is a suitable convex 0-neighborhood in E.*

Proof. It suffices to consider the case $K = \boldsymbol{R}$. If f is a linear extension of f_0 to E which is positive and continuous and if $U = \{x: f(x)\} < 1\}$, it is clear that $f_0(x) < 1$ whenever $x \in M \cap (U - C)$; hence the condition is necessary. Conversely, suppose that U is an open convex 0-neighborhood such that $x \in M \cap (U - C)$ implies $f_0(x) < \gamma$ for some $\gamma \in \boldsymbol{R}$. Then $\gamma > 0$ and $N = \{x \in M: f_0(x) = \gamma\}$ is a linear manifold in E not intersecting the open convex set $U - C$. By the Hahn-Banach theorem (II, 3.1) there exists a closed hyperplane H containing N and not intersecting $U - C$, which, consequently, can be assumed to be of the form $H = \{x: f(x) = \gamma\}$; clearly, f is a continuous extension of f_0. Furthermore, since $0 \in U - C$ it follows that $f(x) < \gamma$ when $x \in U - C$ and hence when $x \in -C$; thus $x \in C$ implies $f(x) \geqq 0$.

COROLLARY 1. *Let f_0 be a linear form defined on the subspace M of an ordered vector space L. f_0 can be extended to a positive linear form f on L if and only if $\mathrm{Re}\, f_0$ is bounded above on $M \cap (W - C)$, where W is a suitable convex radial subset of L.*

In fact, it suffices to endow L with its finest locally convex topology for which W is a neighborhood of 0, and to apply (5.4). The same specialization can be made in the following result which is due to Krein-Rutman [1].

COROLLARY 2. *Let E be an ordered t.v.s. with positive cone C, and suppose that M is a subspace of E such that $C \cap M$ contains an interior point of C. Then every continuous, positive linear form on M can be extended to E under preservation of these properties.*

Proof. If f_0 is the linear form in question and $x_0 \in M$ is an interior point of C, choose a convex 0-neighborhood U in E such that $x_0 + U \subset 2x_0 - C$. Then $\operatorname{Re} f_0$ is bounded above on $M \cap (U - C)$, for we have $M \cap (U - C) \subset (x_0 - C) \cap M$.

REMARK. The condition of Corollary 2 can, in general, not be replaced by the assumption that $C \cap M$ possesses an interior point (Exercise 14). For another condition guaranteeing that every linear form f_0, defined and positive on a subspace M of an ordered vector space L, can be extended to a positive linear form f on L see Exercise 11.

There is a comparatively large class of ordered t.v.s. on which every positive linear form is necessarily continuous; we shall see (Section 7 below) that this class includes all topological vector lattices that are at least sequentially complete (semi-complete). It is plausible that in spaces with this property, the positive cone must be sufficiently "wide" (cf. the discussion following (3.2)). More precisely, one has the following result (condition (ii) is due to Klee [2], condition (iii) to the author [2]).

5.5

Theorem. *Let E be an ordered t.v.s. with positive cone C. Each of the following conditions is sufficient to ensure the continuity of every positive linear form on E:*

(i) *C has non-empty interior.*
(ii) *E is metrizable and complete, and $E = C - C$.*
(iii) *E is bornological, and C is a semi-complete strict $\mathfrak{B}$-cone.*

Proof. It is again sufficient to consider real linear forms on E. The sufficiency of condition (i) is nearly trivial, for if f is positive, then $f^{-1}(0)$ is a hyperplane in E lying on one side of the convex body C, and hence closed which is equivalent with the continuity of f by (I, 4.2). Concerning condition (ii), we use Lemma 2 of Section 3: The topology $\mathfrak{T}_1$ on E, determined by the neighborhood base of 0, $\{V_n : n \in N\}$, where $V_n = U_n \cap C - U_n \cap C$, is evidently finer than the given topology $\mathfrak{T}$ of E, and hence we have $\mathfrak{T} = \mathfrak{T}_1$ by Banach's theorem (III, 2.1), Corollary 2. Now if f is a positive, real linear form on E which is not continuous, then f is unbounded on each set $U_n \cap C$ hence there exists $x_n \in U_n \cap C$ such that $f(x_n) > 1$ $(n \in N)$. On the other hand, since $U_{n+1} + U_{n+1} \subset U_n$ for all n, the sequence $\{x_n\}$ is summable in E with sum $\sum_{n \in N} x_n = z \in C$ (C being closed), and from $z \geqq \sum_{n=1}^{p} x_n$ we obtain $f(z) > p$ for each $p \in N$, which is contradictory. Finally, concerning condition (iii) we observe that since E is bornological and C is a strict $\mathfrak{B}$-cone, a linear form on E which is bounded on the bounded subsets of C is necessarily continuous by (II, 8.3); now if f is a positive, real linear form on E which is not continuous, there exists a bounded sequence $\{x_n\}$ in C such that $f(x_n) > n$ $(n \in N)$. Since E is locally convex by definition, we conclude that $\{n^{-2}x_n: n \in N\}$ is a summable sequence in C with sum $z \in C$, say, and it follows that

$$f(z) \geqq \sum_{n=1}^{p} n^{-2} f(x_n) > \sum_{n=1}^{p} n^{-1} \qquad \text{for all } p,$$

which is impossible. The proof is complete.

COROLLARY. *Let E be an ordered l.c.s. which is the inductive limit of a family $\{E_\alpha: \alpha \in A\}$ of ordered* (F)-*spaces with respect to a family of positive linear maps, and suppose that $E_\alpha = C_\alpha - C_\alpha$ $(\alpha \in A)$. Then each positive linear form on E is continuous.*

This is immediate in view of (II, 6.1). For locally convex spaces, an important consequence of (5.5) is the automatic continuity of rather extensive classes of positive linear maps.

5.6

Let E, F be ordered l.c.s. with respective positive cones C, D. Suppose that E is a Mackey space on which every positive linear form is continuous, and assume that D is a weakly normal cone in F. Then every positive linear map of E into F is continuous.

Proof. Let u be a linear map of E into F such that $u(C) \subset D$, and consider the algebraic adjoint u^* of u (Chapter IV, Section 2). For each $y' \in D'$, $x \to \langle x, u^*y' \rangle$ is a positive linear form on E, hence continuous by assumption; since $F' = D' - D'$ by (3.3), Corollary 3, it follows that $u^*(F') \subset E'$; hence u is weakly continuous by (IV, 2.1). Thus $u \in \mathscr{L}(E, F)$ by (IV, 7.4).

We conclude this section with an application of several of the preceding results to the convergence of directed families of continuous linear maps.

5.7

Let E be an ordered barreled space such that $E = C - C$, and let F be an ordered semi-reflexive space whose positive cone D is normal. Suppose that $\mathscr{U}$ is a subset of $\mathscr{L}(E, F)$ which is directed upward for the canonical order of $\mathscr{L}(E, F)$, and either majorized or simply bounded. Then $u_0 = \sup \mathscr{U}$ exists, and the section filter $\mathfrak{F}(\mathscr{U})$ converges to u_0 uniformly on every precompact subset of E.

Proof. In fact, (5.1) and (5.2) show that $\mathscr{K}$ is a closed normal cone in $\mathscr{L}_s(E, F)$ and hence is the positive cone for the canonical order of $\mathscr{L}_s(E, F)$. For each $x \in C$, the family $\{u(x): u \in \mathscr{U}\}$ satisfies the hypotheses of (4.3), Corollary 2, and hence of (4.3), so $\mathfrak{F}(\mathscr{U})$ converges simply to a linear map $u_0 \in \mathscr{L}(E, F)$. By (III, 4.6) u_0 is continuous, and the convergence of $\mathfrak{F}(\mathscr{U})$ is uniform on every precompact subset of E. Since $\mathscr{K}$ is closed in $\mathscr{L}_s(E, F)$, (4.2) implies that $u_0 = \sup \mathscr{U}$.

6. THE ORDER TOPOLOGY

If S is an ordered set, the order of S gives rise to various topologies on S (cf. Birkhoff [1]); however, in general, the topologies so defined do not satisfy axioms $(LT)_1$ and $(LT)_2$ (Chapter I, Section 1) if S is a vector space, even if (LTO) holds (cf. Exercise 17). On the other hand, if L is an ordered vector space over $\boldsymbol{R}$, there is a natural locally convex topology which, as will be seen below, is the topology of many (if not all) ordered vector spaces occuring in analysis. The present section is devoted to a study of the principal properties of this topology. (See also Gordon [2].)

Let L be an ordered vector space over $\boldsymbol{R}$; we define the **order topology** $\mathfrak{T}_O$ of L to be the finest locally convex topology on L for which every order interval is bounded. The family of locally convex topologies on L having this property is not empty, since it contains the coarsest topology on L, and $\mathfrak{T}_O$ is the upper bound of this family (Chapter II, Section 5); a subset $W \subset L$ is a 0-neighborhood for $\mathfrak{T}_O$ if and only if W is convex and absorbs every order interval $[x, y] \subset L$. (W is necessarily radial, since $\{x\} = [x, x]$ for each $x \in L$.) Although $\mathfrak{T}_O$ is a priori defined for ordered vector spaces over $\boldsymbol{R}$ only, it can happen (cf. the corollaries of (6.2) and (6.4) below) that $(L, \mathfrak{T})$ is an ordered vector space over $\boldsymbol{C}$ such that $(L_0, \mathfrak{T}) = (L_0, \mathfrak{T}_O)$, where L_0 is the underlying real space of L. We begin with the following simple result.

6.1

The dual of $(L, \mathfrak{T}_O)$ is the order bound dual L^b of L. If L^b separates points in L (in particular, if the order of L is regular), $(L, \mathfrak{T}_O)$ is a bornological l.c.s. If L, M are ordered vector spaces, each positive linear map of L into M is continuous for the respective order topologies.

Proof. It is clear from the definition of $\mathfrak{T}_O$ that each order interval is bounded for $\mathfrak{T}_O$; hence if $f \in (L, \mathfrak{T}_O)'$ then $f \in L^b$. Conversely, if $f \in L^b$, then $f^{-1}([-1, 1])$ is convex and absorbs each order interval, and hence is a 0-neighborhood for $\mathfrak{T}_O$. $\mathfrak{T}_O$ is a Hausdorff topology if and only if L^b distinguishes points in L. Let W be a convex subset of L that absorbs each bounded subset of $(L, \mathfrak{T}_O)$; since W a fortiori absorbs all order intervals in L, W is a $\mathfrak{T}_O$-neighborhood of 0. Hence $(L, \mathfrak{T}_O)$ is bornological if (and only if) $\mathfrak{T}_O$ is a Hausdorff topology. Finally, if u is a positive linear map of L into M, then $u([x, y]) \subset [u(x), u(y)]$ for each order interval in L; hence if V is convex and absorbs order intervals in M, $u^{-1}(V)$ has the same properties in L and thus u is continuous for the order topologies. (Cf. Exercise 12.)

COROLLARY. *Let L_i $(i = 1, \ldots, n)$ be a finite family of ordered vector spaces, and endow $L = \prod_i L_i$ with its canonical order. Then the order topology of L is the product of the respective order topologies of the L_i.*

Proof. We show that the projection p_i of $(L, \mathfrak{T}_O)$ onto $(L_i, \mathfrak{T}_O)$ $(i = 1, \ldots, n)$ is a topological homomorphism. In fact, p_i is continuous by (6.1); if I_i is an order interval in L_i then $I_i \times \{0\}$ is an order interval in L, hence if W is a convex 0-neighborhood in $(L, \mathfrak{T}_O)$ then $p_i(W)$ is convex and absorbs I_i which proves the assertion.

The order topology is most easily analyzed when L is an Archimedean ordered vector space with an order unit e. For convenience of expression, let us introduce the following terminology: A sequence $\{x_n : n \in N\}$ of elements $\geqq 0$ of an ordered vector space L is **order summable** if $\sup_n u_n$ exists in L, where $u_n = \sum_{p=1}^{n} x_p$. We shall say that a positive sequence $\{x_n : n \in N\}$ is *of type* l^1 if there exists an $a \geqq 0$ in L and a sequence $(\lambda_n) \in l^1$ such that $(0 \leqq) x_n \leqq \lambda_n a$ $(n \in N)$.

6.2

Let L be an Archimedean ordered vector space over $\mathbf{R}$, possessing an order unit e. Then $(L, \mathfrak{T}_O)$ is an ordered t.v.s. which is normable, $\mathfrak{T}_O$ is the finest locally convex topology on L for which the positive cone C is normal, and the following assertions are equivalent:

(a) *$(L, \mathfrak{T}_O)$ is complete.*

(b) *Each positive sequence of type l^1 in L is order summable.*

Proof. The order interval $[-e, e]$ is convex, circled, and (by the definition of order unit) radial in L; since L is Archimedean ordered, the gauge p_e of $[-e, e]$ is a norm on L. The topology generated by p_e is finer than $\mathfrak{T}_O$ since $[-e, e]$ is $\mathfrak{T}_O$-bounded, and it is coarser than $\mathfrak{T}_O$, since it is locally convex and $[-e, e]$ absorbs order intervals; hence p_e generates $\mathfrak{T}_O$. To see that C is closed in $(L, \mathfrak{T}_O)$, note that e is an interior point of C; the fact that C is closed follows then, as in the proof of the lemma preceding (4.1), from the hypothesis that L is Archimedean ordered. Moreover, since by (3.1), Corollary 2, $\mathfrak{T}_O$ is finer than any l.c. topology on L for which C is normal, the second assertion follows from the fact that the family $\{\varepsilon[-e, e] : \varepsilon > 0\}$ is a 0-neighborhood base for $\mathfrak{T}_O$ that consists of C-saturated sets.

Further, it is clear that (a) $\Rightarrow$ (b), since every positive sequence of type l^1 in L is of type l^1 with respect to $a = e$, and hence even absolutely summable in $(L, \mathfrak{T}_O)$; the assertion follows from (4.2). (b) $\Rightarrow$ (a): We have to show that $(L, \mathfrak{T}_O)$ is complete. Given a Cauchy sequence in $(L, \mathfrak{T}_O)$, there exists a subsequence $\{x_n : n \in N\}$ such that for all n, $p_e(x_{n+1} - x_n) < \lambda_n$, where $(\lambda_n) \in l^1$; hence $x_{n+1} - x_n \in \lambda_n[-e, e]$ and we have $x_{n+1} - x_n = u_n - v_n$, where $u_n = \lambda_n e + (x_{n+1} - x_n)$ and $v_n = \lambda_n e$ $(n \in N)$. To show that $\{x_n\}$ converges, it suffices to show that $\sum_{n=1}^{\infty} u_n$ converges. Now $0 \leqq u_n \leqq 2\lambda_n e$; hence $\{u_n\}$ is of type l^1 and $\sup_n \sum_{p=1}^{n} u_p = u \in C$ exists by hypothesis. Since for all n

$$0 \leqq u - \sum_{p=1}^{n} u_p = \sup_k \sum_{p=n+1}^{n+k} u_p \leqq 2\left(\sum_{p=n+1}^{\infty} \lambda_p\right) e,$$

it follows that $\sum_{n=1}^{\infty} u_n = u$ for $\mathfrak{T}_O$ and hence $(L, \mathfrak{T}_O)$ is complete.

COROLLARY 1. *If L is Archimedean ordered and has an order unit, the order of L is regular and we have $L^b = L^+$.*

This is immediate in view of (6.1) and (3.3).

COROLLARY 2. *Let $(E, \mathfrak{T})$ be an ordered Banach space possessing an order unit. Then $\mathfrak{T} = \mathfrak{T}_O$ if and only if the positive cone C of E is normal in $(E, \mathfrak{T})$.*

Proof. In fact, the order of E is Archimedean, since C is closed in $(E, \mathfrak{T})$; if $\mathfrak{T} = \mathfrak{T}_O$, then C is normal by (6.2). Conversely, if C is normal, then $\mathfrak{T}$ is coarser than $\mathfrak{T}_O$; since $[-e, e]$ is a barrel in $(E, \mathfrak{T})$ (we can suppose that $K = \boldsymbol{R}$), it follows that $\mathfrak{T} = \mathfrak{T}_O$.

Examples to which the preceding corollary applies are furnished by the spaces $\mathscr{C}(X)$ (X compact) and $L^\infty(\mu)$ (Chapter II, Section 2, Examples 1 and 2) and, more generally, by every ordered Banach space whose positive cone is normal and has non-empty interior. It is readily verified that each interior point of the positive cone C of an ordered t.v.s. L is an order unit, and each order unit is interior to C for $\mathfrak{T}_O$.

However, most of the ordered vector spaces occurring in analysis do not have order units, so that the description of $\mathfrak{T}_O$ given in (6.2) does not apply. Let L be an Archimedean ordered vector space over $\boldsymbol{R}$ and denote, for each $a \geqq 0$, by L_a the ordered subspace $L_a = \bigcup_{n=1}^{\infty} n[-a, a]$ endowed with its order topology; L_a is a normable space. The family $\{L_a \colon a \geqq 0\}$ is evidently directed under inclusion $\subset$, and if $L_a \subset L_b$, the imbedding map $h_{b,a}$ of L_a into L_b is continuous.

6.3

Let L be a regularly ordered vector space over $\boldsymbol{R}$, and denote by H any subset of the positive cone C of L which is cofinal with C for $\leqq$. Then $(L, \mathfrak{T}_O)$ is the inductive limit $\varinjlim h_{b,a} L_a$ $(a, b \in H)$.

Proof. By (6.1), the assumption on L implies that $\mathfrak{T}_O$ is Hausdorff. In view of the preceding remarks and the definition of inductive limit (Chapter II, Section 6), it suffices to show that $\mathfrak{T}_O$ is the finest l.c. topology on L for which each of the imbedding maps $f_a \colon L_a \to L$ $(a \in H)$ is continuous. Since H is cofinal with C, each order interval $[x, y] \subset L$ is contained in a translate of some $[-a, a]$ where $a \in H$, and hence $[x, y]$ is bounded for the topology $\mathfrak{T}$ of the inductive limit; hence $\mathfrak{T}_O$ is finer than $\mathfrak{T}$. On the other hand, if W is a convex 0-neighborhood in $(L, \mathfrak{T}_O)$, then W absorbs all order intervals in L, which implies that $f_a^{-1}(W)$ is a 0-neighborhood in L_a $(a \in H)$, and hence $\mathfrak{T}$ is finer than $\mathfrak{T}_O$.

COROLLARY 1. *If the order of L is regular and each positive sequence of type l^1 in L is order summable, $(L, \mathfrak{T}_O)$ is barreled.*

Proof. In fact, the assumption implies by (6.2) that each of the spaces L_a ($a \in H$) is normable and complete and hence barreled; the result follows from (II, 7.2).

COROLLARY 2. *If the order of L is regular and the positive cone C satisfies condition* (D) *of* (1.1), *then C is normal for* $\mathfrak{T}_O$ (*hence the dual of* $(L, \mathfrak{T}_O)$ *is* L^+).

Proof. By definition of the topology of inductive limit, a 0-neighborhood base for $\mathfrak{T}_O$ is given by the family of all convex radial subsets U of L such that $V = U \cap (C - C)$ is of the form $V = \Gamma\{\rho_a[-a, a]: a \in H\}$ where $a \to \rho_a$ is any mapping of H into the set of real numbers > 0. We prove the normality of C via (3.1) (c) by showing that $x \in U$ and $y \in [0, x]$ imply $y \in U$. If $x \in U$ and $x \geqq 0$, then x is of the form $x = \sum_{i=1}^{n} \lambda_i z_i$, where $\sum_{i=1}^{n} |\lambda_i| \leqq 1$ ($\lambda_i \in \mathbf{R}$) and $z_i \in \rho_{a_i}[-a_i, a_i]$ ($i = 1, \ldots, n$). If $y \in [0, x]$ it follows that $y \leqq \sum_{i=1}^{n} |\lambda_i| \rho_{a_i} a_i$; by repeated application of (D), we obtain $y = \sum_{i=1}^{n} |\lambda_i| y_i$, where $y_i \in \rho_{a_i}[0, a_i]$ ($i = 1, \ldots, n$). Hence $y \in V \subset U$ as was to be shown.

REMARK. Since $L_a \subset L_b$ (where $a, b \in C$) is equivalent with $a \leqq \lambda b$ for a suitable scalar $\lambda > 0$, it suffices in (6.3) to require that the set of all positive scalar multiples of the elements $a \in H$ be cofinal with C (for $\leqq$); in particular, if L has an order unit e, it suffices to take $H = \{e\}$. Let us note also that the inductive limit of (6.3) is in general not strict (the topology induced by L_b on L_a ($b > a$) is, in general, not the order topology of L_a). For example, if L is the space $L^2(\mu)$ and a, b are the respective equivalence classes of two functions f, g such that $0 \leqq f \leqq g$, f is bounded and g μ-essentially unbounded, then the topology of L_a is strictly finer than the topology induced on L_a by L_b. (Cf. Exercise 12.)

We apply the preceding description of $\mathfrak{T}_O$ to the case where L is a vector lattice; the lattice structure compensates in part for the lack of an order unit and one obtains a characterization of $\mathfrak{T}_O$ that can be compared with (6.2).

6.4

Let L be a vector lattice whose order is regular and let $\mathfrak{T}$ *be a locally convex topology on L. These assertions are equivalent:*

(a) $\mathfrak{T}$ *is the order topology* $\mathfrak{T}_O$.
(b) $\mathfrak{T}$ *is the finest l.c. topology on L for which C is normal.*
(c) $\mathfrak{T}$ *is the Mackey topology with respect to* $\langle L, L^+\rangle$.

Proof. Let us note first that by (1.4), $L^b = L^+$ and that since the order of L is assumed to be regular, $\langle L, L^+\rangle$ is a duality. (a) $\Leftrightarrow$ (b): Since the positive cone of a vector lattice satisfies (D) of (1.1), this follows from the fact that by (3.1), Corollary 2, $\mathfrak{T}_O$ is finer than any l.c. topology for which C is normal, in view of (6.3), Corollary 2. (a) $\Leftrightarrow$ (c): Since $(L, \mathfrak{T}_0)' = L^b = L^+$, $\mathfrak{T}_0$ is consistent with the duality $\langle L, L^+\rangle$; since $(L, \mathfrak{T}_0)$ is bornological, $\mathfrak{T}_0$ is necessarily the Mackey topology with respect to $\langle L, L^+\rangle$.

The following corollary is now a substitute for (6.2), Corollary 2.

COROLLARY. *Let* $(E, \mathfrak{T})$ *be an ordered* (F)-*space* (*over* $\boldsymbol{R}$) *which is a vector lattice. Then* $\mathfrak{T} = \mathfrak{T}_0$ *if and only if the positive cone* C *of* E *is normal in* $(E, \mathfrak{T})$.

Proof. If $\mathfrak{T} = \mathfrak{T}_0$, then C is normal by (6.4). Conversely, if C is normal, then $E' = C' - C'$ by (3.3), and $C' - C' = E^+$ by (5.5) (for C is closed in $(E, \mathfrak{T})$ and $E = C - C$); since E is a Mackey space by (IV, 3.4), the assertion follows from (6.4) (c).

7. TOPOLOGICAL VECTOR LATTICES

Let L be a t.v.s. over $\boldsymbol{R}$ and a vector lattice, and consider the maps $x \to |x|$, $x \to x^+$, $x \to x^-$ of L into itself, and the maps $(x, y) \to \sup(x, y)$ and $(x, y) \to \inf(x, y)$ of $L \times L$ into L. By utilizing the identities (1), (2) and (3) of Section 1, it is not difficult to prove that the continuity of one of these maps implies the continuity (in fact, the uniform continuity) of all of them; in this case, we say that "the lattice operations are continuous" in L. Recall that a subset A of L is called solid if $x \in A$ and $|y| \leqq |x|$ imply that $y \in A$; we call L **locally solid** if the t.v.s. L possesses a 0-neighborhood base of solid sets.

7.1

Let L be a t.v.s. over $\boldsymbol{R}$ *and a vector lattice. The following assertions are equivalent*:

(a) *L is locally solid.*
(b) *The positive cone of L is normal, and the lattice operations are continuous.*

Proof. (a) $\Rightarrow$ (b): Let $\mathfrak{U}$ be a 0-neighborhood base in L consisting of solid sets; if $x \in U \in \mathfrak{U}$ and $0 \leqq y \leqq x$, then $y \in U$ and hence the positive cone C of L is normal by (3.1) (c). Moreover, if $x - x_0 \in U$, we conclude from (6) of (1.1) that $x^+ - x_0^+ \in U$ $(U \in \mathfrak{U})$ and hence the lattice operations are continuous.

(b) $\Rightarrow$ (a): Suppose that C is normal and the lattice operations are continuous. Let $\mathfrak{U}$ be a 0-neighborhood base in L consisting of circled C-saturated sets (Section 3). For a given $U \in \mathfrak{U}$, choose $V \in \mathfrak{U}$, $W \in \mathfrak{U}$ so that $V + V \subset U$ and that $x \in W$ implies $x^+ \in V$. Now if $x \in W$, then $-x \in W$, since W is circled; hence x^+ and $x^- = (-x)^+$ are in V and $|x| = x^+ + x^- \in U$. If $|y| \leqq |x|$, then $y \in [-|x|, |x|]$; hence, since U is C-saturated, it follows that $y \in U$. Therefore, the set $\{y$: *there exists* $x \in W$ *such that* $|y| \leqq |x|\}$ is a 0-neighborhood contained in U, and is obviously solid.

It is plausible that for t.v.s. that are vector lattices, just as for more general types of ordered t.v.s., the axiom (*LTO*) (closedness of the positive cone) by itself is too weak to produce useful results. We define a **topological vector lattice** to be a vector lattice and a Hausdorff t.v.s. over $\boldsymbol{R}$ that is locally solid; it will be seen from (7.2) below that in these circumstances, the positive cone of L is automatically closed, and hence every topological vector lattice is an ordered t.v.s. over $\boldsymbol{R}$. A **locally convex vector lattice** (abbreviated **l.c.v.l.**) is a topological vector lattice whose topology is locally convex. Every solid set is circled (with respect to $\boldsymbol{R}$, cf. (4) of (1.1)); hence a topological vector lattice possesses a base of circled solid 0-neighborhoods. Since the convex hull of a solid set is solid (hence also circled), a l.c.v.l. possesses a 0-neighborhood base of convex solid sets. The gauge function p of a radial, convex solid set is characterized by being a semi-norm such that $|y| \leqq |x|$ implies $p(y) \leqq p(x)$, and is called a **lattice semi-norm** on L. Therefore, the topology of a l.c.v.l. can be generated by a family of lattice semi-norms (for example, by the family of all continuous lattice semi-norms). A **Fréchet lattice** is a l.c.v.l. which is an (F)-space; a **normed lattice** is a normed space (over $\boldsymbol{R}$) whose unit ball $\{x\colon \|x\| \leqq 1\}$ is solid. By utilizing (I, 1.5) and the uniform continuity of the lattice operations, it is easy to see that with respect to the continuous extension of the lattice operations, the completion of a topological vector lattice is a topological vector lattice; in particular, the completion of a normed lattice is a complete normed lattice with respect to the continuous extension of its norm. A complete normed lattice is called a **Banach lattice**. Let us record the following elementary consequences of the definition of a topological vector lattice.

7.2

In every topological vector lattice L, the positive cone C is closed, normal, and a strict $\mathfrak{B}$-cone; if L is order complete, every band is closed in L.

Proof. C is normal cone by (7.1) and, since $C = \{x\colon x^- = 0\}$, C is closed, since the topology of L is Hausdorff and $x \to x^-$ is continuous. To show that C is a strict $\mathfrak{B}$-cone, recall that if B is a circled, bounded set, then $B^+ = B^-$, and hence $B \subset B^+ - B^+$. It suffices, therefore, to show that B^+ is bounded if B is bounded. If B is bounded and U is a given solid 0-neighborhood in L, there exists $\lambda > 0$ such that $B \subset \lambda U$; since λU is evidently solid, it follows that $B^+ \subset \lambda U$ hence B^+ is bounded. Finally, if A is a band in L, then $A = A^{\perp\perp}$ by (1.3), Corollary 1. Now each set $\{a\}^\perp = \{x \in L\colon \inf(|x|, |a|) = 0\}$ is closed, since L is Hausdorff and $x \to \inf(|x|, |a|)$ is continuous, and we have $A = \bigcap \{\{a\}^\perp\colon a \in A^\perp\}$.

Examples

1. The Banach spaces (over $\boldsymbol{R}$) $L^p(\mu)$ (Chapter II, Section 2, Example 2) are Banach lattices under their canonical orderings; it will be seen below that these are order complete for $p < \infty$, and the spaces $L^1(\mu)$ and $L^\infty(\mu)$ will be important concrete examples for the discussion in Section

8. The corresponding spaces over C can be included in the discussion, as they are complexifications (Chapter I, Section 7) of their real counterparts.

2. Let λ be a subspace of ω_d such that $\lambda = \lambda^{\times\times}$ (Chapter IV, Section 1, Example 4); λ is a perfect space in the sense of Köthe [5]). Under the normal topology (Köthe [5], Peressini [2]) λ is a l.c.v.l. when endowed with its canonical ordering as a subspace of ω_d. The normal topology is the topology of uniform convergence on all order intervals of $\lambda^\times$, and the coarsest topology consistent with $\langle \lambda, \lambda^\times \rangle$ such that the lattice operations are continuous. (Cf. Exercise 20.)

3. Let X be a locally compact (Hausdorff) space and let E be the space of all real-valued functions with compact support in X, endowed with its inductive limit topology (Chapter II, Section 6, Example 3). The topology of E is the order topology $\mathfrak{T}_O$ (Section 6), so that E is a locally convex vector lattice (see (7.3)); E is, in general, not order complete. The dual of $(E, \mathfrak{T}_O)$ is the order dual E^+ of E (the space of all real Radon measures on X); under its canonical order, E^+ is an order complete vector lattice by (1.4), Corollary, and a l.c.v.l. for its strong topology $\beta(E^+, E)$ ((7.4) below). Of particular interest are the spaces $E = \mathscr{C}(X)$ when X is compact (Section 8).

We now supplement the results on the order topology $\mathfrak{T}_O$ obtained in the previous section.

7.3

Let E be a regularly ordered vector lattice. Then the order topology $\mathfrak{T}_O$ is the finest topology $\mathfrak{T}$ on E such that $(E, \mathfrak{T})$ is a l.c.v.l. Moreover, if E is order complete, then $(E, \mathfrak{T}_O)$ is barreled, and every band decomposition of E is a topological direct sum for $\mathfrak{T}_O$.

Proof. In view of (6.1) (and $E^+ = E^b$, (1.4)), the regularity of the order of E is sufficient (and necessary, cf. Exercise 19) for $\mathfrak{T}_O$ to be a Hausdorff topology. By (6.3), $(E, \mathfrak{T}_O)$ is the inductive limit of the normed spaces L_a $(a \geqq 0)$ that are normed lattices in the present circumstances; one shows, as in the proof of (6.3), Corollary 2, that the convex circled hull of any family $\{\rho_a[-a, a]: a \geqq 0\}$ is solid, and hence that $(E, \mathfrak{T}_O)$ is locally solid. The fact that $\mathfrak{T}_O$ is the finest topology $\mathfrak{T}$ such that $(E, \mathfrak{T})$ is a l.c.v.l. then follows from (6.4) (b), since the positive cone is normal for all these topologies, (7.1). If E is order complete, then clearly every positive sequence of type l^1 is order summable; hence $(E, \mathfrak{T}_O)$ is barreled by (6.3), Corollary 1. The last assertion is clear from the corollary of (6.1), since $\mathfrak{T}_O$ induces on each band $B \subset E$ the order topology of B. (Exercise 12.)

COROLLARY 1. *If the order of the vector lattice E is regular, then $(E, \mathfrak{T}_O)$ is a l.c.v.l. whose topology is generated by the family of all lattice semi-norms on E.*

From the corollary of (6.4), we obtain:

COROLLARY 2. *If E is a vector lattice and an ordered* (F)-*space in which the positive cone is normal, the lattice operations are continuous in E.*

It is interesting that the strong dual of a l.c.v.l. E reflects the properties of E in a strengthened form; in addition, E'_β is complete when E is barreled. (As has been pointed out in Chapter IV, Section 6, the strong dual of a barreled l.c.s. is in general not complete.)

7.4

Theorem. *Let E be a l.c.v.l. Then the strong dual E'_β is an order complete l.c.v.l. under its canonical order, and a solid subspace of E^+; moreover, if E is barreled, then E' is a band in E^+, and E'_β is a complete l.c.s.*

Proof. Since the positive cone C of E is normal in $(E, \mathfrak{T})$ by (7.2), it follows from (3.3) that $E' = C' - C' \subset C^* - C^* = E^+$.

It follows from the corollary of (1.5) that the polar U° of every solid 0-neighborhood U in E is a solid subset of E^+. Since E' is the union of these polars, as U runs through a base of solid 0-neighborhoods, E' is a solid subspace and therefore a sublattice of E^+. In particular, it follows that E' is an order complete sublattice of E^+. To see that E' is a l.c.v.l. for the strong topology $\beta(E', E)$, it suffices to observe that the family of all solid bounded subsets of E is a fundamental family of bounded sets; by the corollary of (1.5) the polars B° (with respect to $\langle E, E'\rangle$) of these sets B form a 0-neighborhood base for $\beta(E', E)$ that consists of solid subsets of E'.

If $(E, \mathfrak{T})$ is barreled and S is a directed ($\leqq$) subset of the dual cone C' such that S is majorized in E^+, then each section of S is bounded for $\sigma(E^+, E)$, hence for $\sigma(E', E)$ and, consequently, $\sigma(E', E)$-relatively compact, (IV, 5.2). Thus the section filter of S converges weakly to some $f \in C'$, and it is clear from the definition of the order of E' that $f = \sup S$ (cf. (4.2), which is, however, not needed for the conclusion). Since we have shown before that E' is a solid sublattice of E^+, it is now clear that E' is a band in E^+.

There remains to show that if $(E, \mathfrak{T})$ is barreled, then $(E', \beta(E'E))$ is complete. Let us note first that E^+, which is the dual of $(E, \mathfrak{T}_O)$ by (6.1) (note that $E^+ = E^b$ by (1.4)), is complete under $\beta(E^+, E)$ by (IV, 6.1), for it is the strong dual of a bornological space. Hence by the preceding results and (7.3), $(E^+, \beta(E^+, E))$ is a l.c.v.l. (7.2) shows that E', being a band in E^+, is closed in $(E^+, \beta(E^+, E))$ and hence complete for the topology induced by $\beta(E^+, E)$. On the other hand, this latter topology is coarser than $\beta(E', E)$, since $\mathfrak{T}$ is coarser than $\mathfrak{T}_O$; hence if $\mathfrak{F}$ is a $\beta(E', E)$-Cauchy filter in E', $\mathfrak{F}$ has a unique $\beta(E^+, E)$-limit $g \in E'$. Clearly, $\mathfrak{F}$ converges to g pointwise on E, and (since $\mathfrak{F}$ is a Cauchy filter for $\beta(E', E)$), it follows from a simple argument that $\lim \mathfrak{F} = g$ for $\beta(E', E)$. This completes the proof.

COROLLARY 1. *Every reflexive locally convex vector lattice is order complete, and a complete l.c.s.*

In fact, the strong dual of E is a l.c.v.l. which is reflexive by (IV, 5.6), Corollary 1, and hence barreled, and E can be identified (under evaluation) with the strong dual of E'_β. More generally, if E is a l.c.v.l. that is semi-reflexive, then E is order complete and $(E, \beta(E, E'))$ is complete (cf. Corollary 2 of (7.5) below).

COROLLARY 2. *If E is a normed lattice, its strong dual E' is a Banach lattice with respect to dual norm and canonical order. If, in addition, E is a Banach space then $E' = E^+$.*

Proof. The first assertion is clear, since the unit ball of E' is solid by the corollary of (1.5). The second assertion is a consequence of (5.5) and (7.2).

The following result is the topological counterpart of (1.6).

COROLLARY 3. *If E is an infrabarreled l.c.v.l., then E can be identified, under evaluation, with a topological vector sublattice of its strong bidual E'' (which is an order complete l.c.v.l. under its canonical order).*

Proof. The assumption that E is infrabarreled (Chapter IV, Section 5) means precisely that the evaluation map $x \to \tilde{x}$ is a homeomorphism of E into E''; the remainder follows from (1.6), since E' is a solid subspace of E^+.

It would, however, be a grave error to infer from the foregoing corollary that for an *infinite* subset $S \subset E$ such that $x = \sup S$ exists in E, one has necessarily $\tilde{x} = \sup \tilde{S}$. Thus even if E is order complete, E can, in general, not be identified (under evaluation) with an order complete sublattice of E''. For example, let $E = l^\infty$ be endowed with its usual norm and order; E is an order complete Banach lattice (in fact, E can be identified with the strong dual of the Banach lattice l^1). Denote by x_n $(n \in N)$ the vector in E whose n first coordinates are 1, the remaining ones being 0; $\{x_n: n \in N\}$ is a monotone sequence in E such that $\sup_n x_n = e$, where $e = (1, 1, 1, \ldots)$. Let $z = \sup_n \tilde{x}_n$ in E'' $(= E^{++}$ by virtue of (5.5)); we assert that $z \neq \tilde{e}$. In view of $E' = C' - C'$, $\{x_n\}$ is a weak Cauchy sequence in E and $z(f) = \sup_n f(x_n)$ for each $f \in C'$; if we had $z = \tilde{e}$, the sequence $\{x_n\}$ would be weakly convergent to e in E, and hence norm convergent by (4.3). On the other hand, one has $\|x_{n+p} - x_n\| = 1$ for all $n \in N$, $p \in N$, which is contradictory, and it follows that $z < \tilde{e}$.

Our next objective is a characterization of those l.c. vector lattices that can be identified (under evaluation) with order complete sublattices of their bidual E''; this will yield, in particular, a characterization of order complete vector lattices of minimal type (Section 1). A filter $\mathfrak{F}$ in an order complete vector lattice is called **order convergent** if $\mathfrak{F}$ contains an order bounded set Y (hence an order interval), and if

$$\sup_Y(\inf Y) = \inf_Y(\sup Y),$$

where Y runs through all order bounded sets $Y \in \mathfrak{F}$. The common value of the right- and left-hand terms is called the **order limit** of $\mathfrak{F}$. Let us note also that if E is a l.c.v.l., then the bidual E'' of E is a l.c.v.l. under its natural topology (the topology of uniform convergence on the equicontinuous subsets of E', Chapter IV, Section 5); in fact, the polar of every solid 0-neighborhood in E is a solid subset of E' by the corollary of (1.5), and hence the family of all solid equicontinuous subsets of E' is a fundamental family of equicontinuous sets. Hence their respective polars (in E'') form a 0-neighborhood base for the natural topology, consisting of solid sets.

7.5

Let $(E, \mathfrak{T})$ be an order complete l.c.v.l., and let E'' be endowed with its natural topology and canonical order (under which it is an order complete l.c.v.l.). The following assertions are equivalent:

(a) *Under evaluation, E is isomorphic with an order complete sublattice of E''.*
(b) *For every majorized, directed ($\leqq$) subset S of E, the section filter of S $\mathfrak{T}$-converges to* sup S.
(c) *Every order convergent filter in E $\mathfrak{T}$-converges to its order limit.*

REMARK. The equivalences remain valid when "to sup S" and "to its order limit" are dropped in (*b*) and (*c*), respectively; if the corresponding filters converge for $\mathfrak{T}$, they converge automatically to the limits indicated, by (4.2).

Proof of (7.5). (a) ⇒ (b): Let S be a directed ($\leqq$) subset of E such that $x_0 = \sup S$; identifying E with its canonical image in E'', we obtain (by definition of the canonical order of E'') $f(x_0) = \sup\{f(x): x \in S\}$ for every continuous, positive linear form on E. It follows that the section filter of S converges weakly to x_0, and hence for $\mathfrak{T}$ by (4.3), since the positive cone C is normal in E.

(b) ⇒ (c): Let $\mathfrak{T}$ be an order convergent filter in E with order limit x_0 and let $\mathfrak{G}$ be the base of $\mathfrak{F}$ consisting of all order bounded subsets $Y \in \mathfrak{F}$. Let $a(Y) = \inf Y$ ($Y \in \mathfrak{G}$); the family $\{a(Y): Y \in \mathfrak{G}\}$ is directed ($\leqq$) with least upper bound x_0; hence by hypothesis its section filter converges to x_0 for $\mathfrak{T}$. Likewise, if $b(Y) = \sup Y$, the family $\{b(Y): Y \in \mathfrak{G}\}$ is directed ($\geqq$) with greatest lower bound x_0, and hence its section filter $\mathfrak{T}$-converges to x_0. Let U be any C-saturated 0-neighborhood in E; there exists a set $Y_0 \in \mathfrak{G}$ such that $a(Y_0) \in x_0 + U$ and $b(Y_0) \in x_0 + U$, and this implies that $Y_0 \subset x_0 + U$. Since (C being normal) the family of all C-saturated 0-neighborhoods is a base at 0, it follows that $\mathfrak{F}$ converges to x_0 for $\mathfrak{T}$.

(c) ⇒ (a): Let S be a directed ($\leq$) subset of E such that $x_0 = \sup S$. It is clear that the section filter of S is order convergent with order limit x_0, and hence it $\mathfrak{T}$-converges to x_0 by assumption. It follows that $f(x_0) = \sup\{f(x): x \in S\}$ for every $f \in C'$; hence from the definition of order in E'' it follows that $\tilde{x}_0 = \sup \tilde{S}$, where $x \to \tilde{x}$ is the evaluation map of E into E''. The proof is complete.

COROLLARY 1. *Let E be an order complete vector lattice whose order is regular. The following assertions are equivalent*:

(a) *E is of minimal type.*
(b) *For every majorized, directed ($\leqq$) subset S of E, the section filter converges to* sup S *for $\mathfrak{T}_O$.*
(c) *Every order convergent filter in E converges for $\mathfrak{T}_O$.*

Moreover, if E is minimal, then $\mathfrak{T}_O$ is the finest l.c. topology on E for which every order convergent filter converges.

Proof. Applying (7.5) to $(E, \mathfrak{T}_O)$ we see that $E' = E^+$, and $E'' = (E^+, \beta(E^+, E))'$ is a solid subspace of E^{++} by (7.4). Hence E is minimal (that is, isomorphic with an order complete sublattice of E^{++} under evaluation) if and only if E is isomorphic with an order complete sublattice of E'', which proves the first assertion.

For the second assertion there remains, in view of (c), only to show that every l.c. topology $\mathfrak{T}$ on E for which every order convergent filter converges is coarser than $\mathfrak{T}_O$. Hence let $\mathfrak{T}$ be such a topology, and let $a \in C$ be fixed. Now $\{\varepsilon[-a, a]: \varepsilon > 0\}$ $(\varepsilon \in \boldsymbol{R})$ is a filter base in E, and it is immediate that the corresponding filter is order convergent with order limit 0 (E is regular, hence Archimedean ordered); thus if U is a convex 0-neighborhood for $\mathfrak{T}$, it follows that there exists $\varepsilon > 0$ such that $\varepsilon[-a, a] \subset U$. Therefore, U absorbs arbitrary order intervals in E, which shows that $\mathfrak{T}$ is coarser than $\mathfrak{T}_O$.

COROLLARY 2. *Let E be a l.c.v.l. which is semi-reflexive; then E is order complete. If, in addition, every positive linear form on E is continuous, then E is of minimal type, $\tau(E, E') = \mathfrak{T}_O$, and $(E, \mathfrak{T}_O)$ is reflexive.*

Proof. The first assertion follows at once from (4.3), Corollary 2. If every positive linear form on E is continuous, then $E' = E^+$, and the equality $\tau(E, E') = \mathfrak{T}_O$ follows from (6.4) in view of the fact that the order of E is regular, (4.1), Corollary 2. Hence E is minimal by Corollary 1, for the section filter of every majorized, directed ($\leqq$) subset S of E converges weakly to sup S, so it converges for $\mathfrak{T}_O$ by (4.3). Finally, $(E, \mathfrak{T}_O)$ is reflexive, since it is semi-reflexive and (by (7.3)) barreled.

Examples

4. Each of the Banach lattices $L^p(\mu)$, $1 < p < +\infty$ (Chapter II, Section 2, Example 2; take $K = \boldsymbol{R}$) is order complete and of minimal type; in particular, the norm topology is the finest l.c. topology for which every order convergent filter converges.

5. The Banach lattice $L^1(\mu)$ is order complete and of minimal type. In fact, if S is a directed ($\leqq$) subset of the positive cone C and majorized by h, then for any subset $\{f_1, \ldots, f_n\}$ of S such that $f_1 \leqq \cdots \leqq f_n$ one obtains

$$\|h - f_1\| = \|h - f_n\| + \|f_n - f_{n-1}\| + \cdots + \|f_2 - f_1\|,$$

since the norm of $L^1(\mu)$ is additive on C; this shows that the section filter of S is a Cauchy filter for the norm topology, and hence convergent. Since the latter topology is $\mathfrak{T}_O$, it follows that $L^1(\mu)$ is of minimal type. Obviously these conclusions apply to any Banach lattice whose norm is additive on the positive cone; these lattices are called abstract (L)-spaces (cf. Kakutani [1] and Section 8 below).

6. Suppose μ to be totally σ-finite. As strong duals of $L^1(\mu)$, the spaces $L^\infty(\mu)$ are order complete Banach lattices by (7.4); in general, these spaces are not of minimal type as the example preceding (7.5) shows and hence (in contrast with $L^1(\mu)$), in general, not bands in their respective order biduals.

7. Each perfect space (Example 2 above) is order complete and, if each order interval is $\sigma(\lambda, \lambda^+)$-compact, of minimal type.

As we have observed earlier, ordered vector spaces possessing an order unit are comparatively rare; it will be shown in Section 8 below that every Banach lattice with an order unit is isomorphic (as an ordered t.v.s.) with $\mathscr{C}_{\mathbf{R}}(X)$ for a suitable compact space X. A weaker notion that can act as a substitute was introduced by Freudenthal [1]; an element $x \geqq 0$ of a vector lattice L is called a **weak order unit** if $\inf(x, |y|) = 0$ implies $y = 0$ for each $y \in L$. A corresponding topological notion is the following: If L is an ordered t.v.s., an element $x \geqq 0$ is called a **quasi-interior point** of the positive cone C of L if the order interval $[0, x]$ is a total subset of L. The remainder of this section is devoted to some results on weak order units and their relationship with quasi-interior points of C.

7.6

Let E be an ordered l.c.s. over $\mathbf{R}$ which is metrizable and separable, and suppose that the positive cone C of E is a complete, total subset of E. Then the set Q of quasi-interior points of C is dense in C.

Proof. Since C is separable, there exists a subset $\{x_n: n \in N\}$ which is dense in C; denote by $\{p_n: n \in N\}$ an increasing sequence of semi-norms that generate the topology of E. Since C is complete, $x_0 = \sum_1^\infty 2^{-n} x_n / p_n(x_n)$ is an element of C. Now the linear hull of $[0, x_0]$ contains each x_n $(n \in N)$, and hence is dense in $C - C$ and, therefore, in E; that is, $x_0 \in Q$. It is obvious that $C_1 = \{0\} \cup Q$ is a subcone of C, and that $\bar{Q} = \bar{C}_1$. Suppose that $\bar{Q} \neq C$. There exists, by (II, 9.2), a linear form $f \in E'$ such that $f(x) \geqq 0$ when $x \in \bar{Q}$, and a point $y \in C$ such that $f(y) = -1$. Consequently, there exists $\lambda > 0$ such that $f(x_0 + \lambda y) < 0$, which conflicts with $x_0 + \lambda y \in Q$.

COROLLARY. *Let E be a Fréchet lattice which is separable. Then the set of weak order units is dense in the positive cone of E.*

Proof. It suffices to show that each quasi-interior point of C is a weak order unit. But if x is quasi-interior to C, then $y \perp x$ implies that y is disjoint from the linear hull of $[0, x]$ which is dense in E; hence $y = 0$ since the lattice operations are continuous.

REMARK. The assumptions that E be metrizable and separable are not dispensable in (7.6); if E is, for example, either the l.c. direct sum of infinitely many copies of $\boldsymbol{R}_0$ or the Hilbert direct sum of uncountably many copies of l^2 (under their respective canonical orderings), then the set of quasi-interior points of C (equivalently, by (7.7), the set of weak order units) is empty.

7.7

Let E be an order complete vector lattice of minimal type. For each $x > 0$, the following assertions are equivalent:

(a) *x is weak order unit.*
(b) *For each positive linear form $f \neq 0$ on E, $f(x) > 0$.*
(c) *For each topology $\mathfrak{T}$ on E such that $(E, \mathfrak{T})$ is a l.c.v.l., x is a quasi-interior point of the positive cone.*

Proof. If B_x denotes the band in E generated by $\{x\}$, then x is a weak order unit if and only if $B_x = E$, by virtue of (1.3). Now if f is a positive linear form on E, then, since E is minimal, $f(x) = 0$ is equivalent with $f(B_x) = \{0\}$; this shows that (a) $\Leftrightarrow$ (b). Moreover, E being minimal, $B_x = E$ is equivalent with the assertion that the linear hull of $[0, x]$ is dense in $(E, \mathfrak{T}_O)$ (for the closure of each solid subspace G in $(E, \mathfrak{T}_O)$ contains the band generated by G); hence (a) $\Rightarrow$ (c), since the topologies mentioned in (c) are necessarily coarser than $\mathfrak{T}_O$ by (7.3). (c) $\Rightarrow$ (a) is clear in view of the continuity of the lattice operations in $(E, \mathfrak{T})$ (cf. proof of (7.6), Corollary).

For example, in the spaces $L^p(\mu)$ $(1 \leqq p < +\infty)$ the weak order units ($=$ quasi-interior points of C) are those classes containing a function which is > 0 a.e. (μ). By contrast, a point in $L^\infty(\mu)$ is quasi-interior to C exactly when it is interior to C; the classes containing a function which is > 0 a.e. (μ) are weak order units, but not necessarily quasi-interior to C. Hence the minimality assumption is not dispensable in (7.7).

8. CONTINUOUS FUNCTIONS ON A COMPACT SPACE. THEOREMS OF STONE-WEIERSTRASS AND KAKUTANI

This final section is devoted to several theorems on Banach lattices of type $\mathscr{C}(X)$, where X is a compact space, in particular, the order theoretic and algebraic versions of the Stone-Weierstrass theorem and representation theorems for (AM)-spaces with unit and for (AL)-spaces. For a detailed account of this circle of ideas, which is closely related to the Krein-Milman theorem, we refer to Day [2]; the present section is mainly intended to serve as an illustration for the general theory of ordered vector spaces and lattices developed earlier. Let us point out that with only minor modifications most of the following results are applicable to spaces $\mathscr{C}_0(X)$ (continuous functions on a locally compact space X that vanish at infinity); for $\mathscr{C}_0(X)$ can be viewed as a solid sublattice of codimension 1 in $\mathscr{C}(\dot{X})$, where $\dot{X}$ denotes the one-point compactification of X.

With one exception (see (8.3) below) we consider in this section only vector spaces over the real field $\boldsymbol{R}$; *mutatis mutandis*, many of the results can be generalized without difficulty to the complex case, since $\mathscr{C}(X)$ over $\boldsymbol{C}$ is the complexification (Chapter I, Section 7) of $\mathscr{C}(X)$ over $\boldsymbol{R}$; (8.3) is an example for this type of generalization. To avoid ambiguity we shall denote by $\mathscr{C}_{\boldsymbol{R}}(X)$ the Banach lattice of real-valued continuous functions on X, and by $\mathscr{C}_{\boldsymbol{C}}(X)$ the (B)-space of complex-valued continuous functions on X.

Let us recall some elementary facts on the Banach lattice $\mathscr{C}_{\boldsymbol{R}}(X)$, where $X \neq \varnothing$ is any compact space. $\mathscr{C}_{\boldsymbol{R}}(X)$ possesses order units; $f \in \mathscr{C}_{\boldsymbol{R}}(X)$ is an order unit if and only if $\inf\{f(t): t \in X\} > 0$. Thus the order units of $\mathscr{C}_{\boldsymbol{R}}(X)$ are exactly the functions f that are interior to the positive cone C. Distinguished among these is the constantly-one function e; in fact, the norm $f \to \|f\| = \sup\{|f(t)|: t \in X\}$ is the gauge function p_e of $[-e, e]$ and, of course, the topology of $\mathscr{C}_{\boldsymbol{R}}(X)$ is the order topology $\mathfrak{T}_O$ (Section 6). We begin with the following classical result, the order theoretic form of the Stone-Weierstrass theorem.

8.1

Theorem. *If F is a vector sublattice of $\mathscr{C}_{\boldsymbol{R}}(X)$ that contains e and separates points in X, then F is dense in $\mathscr{C}_{\boldsymbol{R}}(X)$.*

REMARK. The subsequent proof will show that a subset $F \subset \mathscr{C}_{\boldsymbol{R}}(X)$ is dense if it satisfies the following condition: F is a (not necessarily linear) sublattice of the lattice $\mathscr{C}_{\boldsymbol{R}}(X)$, and for every $\varepsilon > 0$ and quadruple $(s, t; \alpha, \beta) \in X^2 \times \boldsymbol{R}^2$ such that $\alpha = \beta$ whenever $s = t$, there exists $f \in F$ satisfying $|f(s) - \alpha| < \varepsilon$ and $|f(t) - \beta| < \varepsilon$.

Proof of (8.1). Let s, t be given points of X and α, β given real numbers such that $\alpha = \beta$ if $s = t$; the hypothesis implies the existence of $f \in F$ such that $f(s) = \alpha$, $f(t) = \beta$. This is clear if $s = t$, since $e \in F$; if $s \neq t$, there exists $g \in F$ such that $g(s) \neq g(t)$, and a suitable linear combination of e and g will satisfy the requirement.

Now let $h \in \mathscr{C}_{\boldsymbol{R}}(X)$ and $\varepsilon > 0$ be preassigned and let s be any fixed element of X. Then for each $t \in X$, there exists an $f_t \in F$ such that $f_t(s) = h(s)$ and $f_t(t) = h(t)$. The set $U_t = \{r \in X: f_t(r) > h(r) - \varepsilon\}$ is open and contains t; hence $X = \bigcup_{t \in X} U_t$, and the compactness of X implies the existence of a finite set $\{t_1, \ldots, t_n\}$ such that $X = \bigcup_{\nu=1}^{n} U_{t_\nu}$. Using the lattice property of F, form the function $g_s = \sup\{f_{t_1}, \ldots, f_{t_n}\}$; it is clear that $g_s(t) > h(t) - \varepsilon$ for all $t \in X$, since each t is contained in at least one U_{t_ν}. Moreover, $g_s(s) = h(s)$.

Now consider this procedure applied to each $s \in X$; we obtain a family

$\{g_s: s \in X\}$ in F such that $g_s(s) = h(s)$ for all $s \in X$, and $g_s(t) > h(t) - \varepsilon$ for all $t \in X$ and $s \in X$. The set $V_s = \{r \in X: g_s(r) < h(r) + \varepsilon\}$ is open and contains s; hence $X = \bigcup_{s \in X} V_s$, and the compactness of X implies the existence of a finite set $\{s_1, \dots, s_m\}$ such that $X = \bigcup_{\mu=1}^{m} V_{s_\mu}$. Let $g = \inf\{g_{s_1}, \dots, g_{s_m}\}$; then $g \in F$ and $h(r) - \varepsilon < g(r) < h(r) + \varepsilon$ for all $r \in X$; hence $\|h - g\| < \varepsilon$, and the proof is complete.

The algebraic form of the Stone-Weierstrass theorem replaces the hypothesis that F be a sublattice of $\mathscr{C}_{\mathbf{R}}(X)$ by assuming that F be a subalgebra (that is, a subspace of $\mathscr{C}_{\mathbf{R}}(X)$ invariant under multiplication). Our proof follows de Branges [1], but does not involve Borel measures. The proof is an interesting application of the Krein-Milman theorem and provides an opportunity to apply the concept of Radon measure that has been utilized earlier (Chapter IV, Sections 9 and 10).

The space $\mathscr{M}_{\mathbf{R}}(X)$ of (real) Radon measures on X is, by definition, the dual of $\mathscr{C}_{\mathbf{R}}(X)$ (Chapter II, Section 2, Example 3); since E is a Banach lattice, $\mathscr{M}_{\mathbf{R}}(X)$ is a Banach lattice under its dual norm and canonical order by (7.4), Corollary 2. Thus $\|\mu\| = \||\mu|\|$ for each μ; if $\mu \geqq 0$, then $\|\mu\| = \sup\{\mu(f): f \in [-e, e]\} = \mu(e)$, and this implies that $\|\mu\| = \mu^+(e) + \mu^-(e)$ for all $\mu \in \mathscr{M}_{\mathbf{R}}(X)$. If $g \in \mathscr{C}_{\mathbf{R}}(X)$ is fixed, then $f \to gf$ (pointwise multiplication) is a continuous linear map u of $\mathscr{C}_{\mathbf{R}}(X)$ into itself; the image of $\mu \in \mathscr{M}_{\mathbf{R}}(X)$ under the adjoint u' is a Radon measure denoted by $g \,.\, \mu$. Obviously $|g \,.\, \mu| \leqq \|g\||\mu|$ and hence u' leaves each band in $\mathscr{M}_{\mathbf{R}}(X)$ invariant; in particular, if $g \geqq 0$, then $g \,.\, \mu = g \,.\, \mu^+ - g \,.\, \mu^-$, where $\inf(g \,.\, \mu^+, g \,.\, \mu^-) = 0$. It follows from (1.1) that $(g \,.\, \mu)^+ = g \,.\, \mu^+$, $(g \,.\, \mu)^- = g \,.\, \mu^-$ in this case, and that $|g \,.\, \mu| = g \,.\, |\mu|$.

The **support** of $f \in \mathscr{C}_{\mathbf{R}}(X)$ is the closure S_f of $\{t \in X: f(t) \neq 0\}$ in X; we define the **support** S_μ of $\mu \in \mathscr{M}_{\mathbf{R}}(X)$ to be the complement (in X) of the largest open set U such that $S_f \subset U$ implies $\mu(f) = 0$ (equivalently, such that $S_f \subset U$ implies $|\mu|(f) = 0$). An application of Urysohn's theorem (cf. Prerequisites) shows that if $f \geqq 0$ and $\mu \geqq 0$, then $\mu(f) = 0$ if and only if $f(t) = 0$ whenever $t \in S_\mu$. Notice a particular consequence of this: if μ is such that $S_\mu = \{t_0\}$, then μ is of the form $\mu(f) = \mu(e)f(t_0)$ (hence, up to a factor $\mu(e) \neq 0$, evaluation at t_0). For $S_\mu = \{t_0\}$ implies that $|\mu(f - f(t_0)e)| \leqq |\mu|(|f - f(t_0)e|) = 0$, which is the assertion. Finally, $\mu = 0$ if and only if $S_\mu = \varnothing$. The following lemma is now the key to the proof of (8.2).

LEMMA. *Let F be a subspace of $\mathscr{C}_{\mathbf{R}}(X)$, and suppose that the Radon measure μ is an extreme point of $F^\circ \cap [-e, e]^\circ \subset \mathscr{M}_{\mathbf{R}}(X)$. If $g \in \mathscr{C}_{\mathbf{R}}(X)$ is such that $g \,.\, \mu \in F^\circ$, then g is constant on S_μ.*

Proof. If $\mu = 0$, there is nothing to prove. Otherwise, it can be arranged (by adding a suitable scalar multiple of e and subsequent normalization) that $g \geqq 0$ and $|\mu|(g) = 1$. Suppose, for the moment, that $g \leqq e$; since μ is an extreme point of $F^\circ \cap [-e, e]^\circ$, we have $\|\mu\| = 1$ and it follows that $|\mu|(e - g) = \|\mu\| - |\mu|(g) = 0$, which implies that $1 = e(t) = g(t)$ for all $t \in S_\mu$, in view of the remarks preceding the lemma. We complete the proof by showing that $\|g\| > 1$ is impossible. In fact, assume that $\|g\| > 1$, let $\beta = \|g\|^{-1}$, and define the Radon measures μ_1, μ_2 by $\mu_1 = g_1 . \mu$ and $\mu_2 = g . \mu$, where $g_1 = (e - \beta g)/(1 - \beta)$. We observe that $\mu_1 \in F^\circ$, $\mu_2 \in F^\circ$ and that $|\mu_2| = g . |\mu|$; hence $\|\mu_2\| = g . |\mu|(e) = 1$. Moreover, $\mu_1^+ = g_1 . \mu^+$ and $\mu_1^- = g_1 . \mu^-$, since $g_1 \geqq 0$, and in view of $\|\mu_1\| = \mu_1^+(e) + \mu_1^-(e)$, it follows from a short computation that $\|\mu_1\| = 1$. On the other hand, it is easy to see that $\mu = (1 - \beta)\mu_1 + \beta\mu_2$, which conflicts with the hypothesis that μ be an extreme point of $F^\circ \cap [-e, e]^\circ$.

The following is the algebraic form of the Stone-Weierstrass theorem.

8.2

Theorem. *If F is a subalgebra of $\mathscr{C}_{\boldsymbol{R}}(X)$ that contains e and separates points in X, then F is dense in $\mathscr{C}_{\boldsymbol{R}}(X)$.*

Proof. The set $F^\circ \cap [-e, e]^\circ$ is a convex, circled, weakly compact subset of $\mathscr{M}_{\boldsymbol{R}}(X)$; hence by the Krein-Milman theorem (II, 10.4) there exists an extreme point μ of $F^\circ \cap [-e, e]^\circ$. Since F is a subalgebra of $\mathscr{C}_{\boldsymbol{R}}(X)$, each $f \in F$ satisfies the hypothesis of the lemma with respect to μ; hence each $f \in F$ is constant on the support S_μ of μ. This is clearly impossible if S_μ contains at least two points, since F separates points in X; on the other hand, if $S_\mu = \{t_0\}$, then $\mu(f) = \mu(e)f(t_0)$, and it follows that each $f \in F$ vanishes at t_0, which is impossible since $e \in F$. Hence S_μ is empty which implies $\mu = 0$ and, therefore, $F^\circ = \{0\}$; consequently, F is dense in $\mathscr{C}_{\boldsymbol{R}}(X)$ by the bipolar theorem (IV, 1.5).

The preceding theorem is essentially a theorem on real algebras $\mathscr{C}(X)$; for instance, if X is the unit disk in the complex plane and F is the algebra of all complex polynomials (restricted to X), then F separates points in X and $e \in F$, but F is not dense in $\mathscr{C}_{\boldsymbol{C}}(X)$ (for each $f \in \bar{F}$ is holomorphic in the interior of X). One can, nevertheless, derive results for the complex case from (8.1) and (8.2) by making appeal to the fact that $\mathscr{C}_{\boldsymbol{C}}(X)$ is the complexification of $\mathscr{C}_{\boldsymbol{R}}(X)$; we say that a subset F of the complex algebra $\mathscr{C}_{\boldsymbol{C}}(X)$ is **conjugation-invariant** if $f \in F$ implies $f^* \in F$ (where $f^*(t) = f(t)^*$, $t \in X$). We consider $\mathscr{C}_{\boldsymbol{C}}(X)$ as ordered by the cone of real functions $\geqq 0$ (Section 2).

8.3

COMPLEX STONE-WEIERSTRASS THEOREM. *Let F be a vector subspace of the complex Banach space $\mathscr{C}_{\boldsymbol{C}}(X)$ such that $e \in F$ and F separates points in X and is conjugation-invariant. Then either of the following assumptions implies that F is dense in $\mathscr{C}_{\boldsymbol{C}}(X)$:*

(i) *F is lattice ordered* (Section 2)
(ii) *F is a subalgebra of $\mathscr{C}_{\boldsymbol{C}}(X)$.*

Proof. If F_1 denotes the subset of F whose elements are the real-valued functions contained in F, then $F = F_1 + iF_1$ by the conjugation-invariance of the subspace F; clearly, $e \in F_1$ and F_1 separates points in X, since F does. Thus if F is lattice ordered, then F_1 is a vector lattice (Section 2), and (8.1) shows that F_1 is dense in $\mathscr{C}_{\mathbf{R}}(X)$; by (8.2) the same conclusion holds if F is a subalgebra of $\mathscr{C}_C(X)$, for then F_1 is a subalgebra of $\mathscr{C}_{\mathbf{R}}(X)$. This completes the proof.

It is customary to call a Banach lattice E an (AL)-**space** (abstract L-space) if the norm of E is additive on the positive cone C. The reason for this terminology is that every $L^1(\mu)$ (over $\mathbf{R}$) possesses this property and that, conversely, every (AL)-space is isomorphic (as a Banach lattice) with a suitable space $L^1(\mu)$ (Kakutani [1]; cf. Exercise 22). A Banach lattice E is called an (AM)-**space** (abstract (m)-space) if the norm of E satisfies $\|\sup(x, y)\| = \sup(\|x\|, \|y\|)$ for all x, y in the positive cone C; E is called an (AM)-**space with unit** u if, in addition, there exists $u \in C$ such that $[-u, u]$ is the unit ball of E. (Clearly, such u is unique and an order unit of E.) It is immediate that every Banach lattice $\mathscr{C}_{\mathbf{R}}(X)$ is an (AM)-space with unit (the unit being the constantly-one function e); we will show that this property characterizes the spaces $\mathscr{C}(X)$ over $\mathbf{R}$ among Banach lattices. More generally, every (AM)-space is isomorphic with a closed vector sublattice of a suitable $\mathscr{C}_{\mathbf{R}}(X)$ (Kakutani [2]). Let us record first the following elementary facts on (AL)- and (AM)-spaces; by the strong dual of a Banach lattice E, we understand the dual E' ($= E^+$) under its natural norm and canonical order.

8.4

The strong dual of an (AM)-*space with unit is an* (AL)-*space, and the strong dual of an* (AL)-*space is an* (AM)-*space with unit. Moreover, if E is an Archimedean ordered vector lattice, u an order unit of E, and p_u the gauge function of* $[-u, u]$, *then the completion of* (E, p_u) *is an* (AM)-*space with unit u.*

Proof. Let E be an (AM)-space with unit u; the strong dual E' is a Banach lattice by (7.4), Corollary 2. If $x' \in C'$ then $\|x'\| = \sup\{|\langle x, x'\rangle| : x \in [-u, u]\} = \langle u, x'\rangle$; hence the norm of E' is additive on the dual cone C'.

If F is an (AL)-space, the norm of F is an additive, positive homogeneous real function on C, and hence defines a (unique) linear form f_0 on F such that $f_0(x) = \|x\|$ for all $x \in C$; evidently we have $0 \leqq f_0 \in F'$. It follows that $g \in F'$ satisfies $\|g\| \leqq 1$ if and only if $g \in [-f_0, f_0]$, and hence the norm of the strong dual F' is the gauge function of $[-f_0, f_0]$. Now if $g \geqq 0$, $h \geqq 0$ are elements of F' such that $\|g\| = \lambda_1$, $\|h\| = \lambda_2$, then $g \leqq \lambda_1 f_0$ and $h \leqq \lambda_2 f_0$, since the order of F' is Archimedean. Consequently, $\|\sup(g, h)\| \leqq \sup(\lambda_1, \lambda_2)$, and here equality must hold or else both the relations $\|g\| = \lambda_1$, $\|h\| = \lambda_2$ could not be valid. Therefore, under its canonical order, F' is an (AM)-space with unit f_0.

To prove the third assertion, we observe that if E is an Archimedean ordered vector lattice and u is an order unit of E, then p_u is a norm on E, and even a lattice norm, since $[-u, u]$ is clearly solid. The completion $(\tilde{E}, \tilde{p}_u)$

of (E, p_u) is a Banach lattice (with respect to the continuous extension of the lattice operations) whose unit ball is the set $\{x \in \tilde{E}: -f(u) \leqq f(x) \leqq f(u), f \in C'\}$, and hence the order interval $[-u, u]$ in $\tilde{E}$. As in the preceding paragraph, it follows that $(\tilde{E}, p_u)$ is an (AM)-space with unit u. This ends the proof.

Let $E \neq \{0\}$ be an (AM)-space with unit u; the intersection of the hyperplane $H = \{x': \langle u, x' \rangle = 1\}$ with the dual cone C' is a convex, $\sigma(E', E)$-closed subset H_0 of the dual unit ball $[-u, u]^\circ$. It follows that H_0, which is called the **positive face** of $[-u, u]^\circ$, is $\sigma(E', E)$-compact; hence C' is a cone with weakly compact base, and $t \in H_0$ is an extreme point of H_0 if and only if $\{\lambda t: \lambda \geqq 0\}$ is an extreme ray of C' (Chapter II, Exercise 30).

Now we can prove the representation theorem of Kakutani [2] for (AM)-spaces with unit.

8.5

Theorem. *Let $E \neq \{0\}$ be an* (AM)*-space with unit and let X be the set of extreme points of the positive face of the dual unit ball. Then X is non-empty and $\sigma(E', E)$-compact, and the evaluation map $x \to f$ (where $f(t) = \langle x, t \rangle$, $t \in X$) is an isomorphism of the* (AM)*-space E onto $\mathscr{C}_{\boldsymbol{R}}(X)$.*

Proof. Let u be the unit of E. Since the positive face H_0 of $[-u, u]^\circ$ is convex and $\sigma(E', E)$-compact, the Krein-Milman theorem (II, 10.4) implies that the set X of extreme points of H_0 is non-empty. Since H_0 is a base of C', it follows from (1.7) that $t \in X$ if and only if t is a lattice homomorphism of E onto $\boldsymbol{R}$ such that $t(u) = 1$. It is clear from this that X is closed, hence compact for $\sigma(E', E)$. The mapping $x \to f$ is clearly a linear map of E into $\mathscr{C}_{\boldsymbol{R}}(X)$ that preserves the lattice operations, since each $t \in X$ is a lattice homomorphism; to show that $x \to f$ is a norm isomorphism, it suffices (since E and $\mathscr{C}_{\boldsymbol{R}}(X)$ are Banach lattices) that $\|f\| = \|x\|$ when $x \geqq 0$. For $x \geqq 0$ we have $\|x\| = \sup\{\langle x, x' \rangle: \|x'\| \leqq 1\} = \sup\{\langle x, x' \rangle: x' \in H_0\}$; since H_0 is the $\sigma(E', E)$-closed convex hull of X and each $x \in E$ is linear and $\sigma(E', E)$-continuous, it follows that $(x \geqq 0)$ $\sup\{\langle x, x' \rangle: x' \in H_0\} = \sup\{\langle x, t \rangle: t \in X\} = \|f\|$. Thus $x \to f$ is an isomorphism of E onto a vector sublattice F of $\mathscr{C}_{\boldsymbol{R}}(X)$ that is complete and contains e (the image of u); since E separates points in E' and a fortiori in X, it follows from (8.1) that $F = \mathscr{C}_{\boldsymbol{R}}(X)$, which completes the proof.

We conclude this section with two applications of the preceding result; the first of these gives us some more information on the structure of (AL)-spaces, the second on more general locally convex vector lattices.

From (8.4) we know that the strong dual $E'(= E^+)$ of an (AL)-space E is an (AM)-space with unit; hence by (8.5), E' can be identified with a space $\mathscr{C}_{\boldsymbol{R}}(X)$, where X is the set of extreme points of the positive face of the unit ball in E''. By (7.4), the Banach lattice E' is order complete, which has the interesting consequence that X is extremally disconnected (that is, the closure of every open set in X is open). In fact, let $G \subset X$ be open and denote by S

the family of all $f \in \mathscr{C}_{\mathbf{R}}(X)$ such that $f \in [0, e]$ and the support S_f is contained in G. S is directed ($\leqq$) and majorized by e; hence $f_0 = \sup S$ exists. Since G is open, it follows from Urysohn's theorem that $f_0(s) = 1$ whenever $s \in G$, and that $f_0(t) = 0$ whenever $t \notin \bar{G}$. Thus f_0 is necessarily the characteristic function of $\bar{G}$ since f_0 is continuous, and this implies that $\bar{G}$ is open.

Therefore, if E is an (AL)-space, then E' can be identified with a space $\mathscr{C}_{\mathbf{R}}(X)$, where X is compact and extremally disconnected, and it follows that E itself can be identified with a closed subspace of the Banach lattice $\mathscr{M}_{\mathbf{R}}(X)$ which is the strong bidual of E. For a characterization of E within $\mathscr{M}_{\mathbf{R}}(X)$, let us consider the subset $B \subset \mathscr{M}_{\mathbf{R}}(X)$ such that $\mu \in B$ if and only if for each directed ($\leqq$), majorized subset $S \subset \mathscr{C}_{\mathbf{R}}(X)$ it is true that $\lim \mu(f) = \mu(\sup S)$, the limit being taken along the section filter of S. It is not difficult to verify that B is a vector sublattice of $\mathscr{M}_{\mathbf{R}}(X)$; in fact, if S is directed ($\leqq$) and $f_0 = \sup S$, and if $f_0 \geqq 0$ (which is no restriction of generality) then there exists, for given $\mu \in B$ and $\varepsilon > 0$, a decomposition $f_0 = g_0 + h_0 (g_0 \geqq 0, h_0 \geqq 0)$ such that $\mu^+(h_0) < \varepsilon$ and $\mu^-(g_0) < \varepsilon$ ((1.5), formula (7)). Using that $\mu \in B$, we obtain after a short computation that $\mu^+(f_0) < \sup\{\mu^+(f): f \in S\} + 3\varepsilon$, which proves that $\mu^+ \in B$. Thus B is a sublattice of $\mathscr{M}_{\mathbf{R}}(X)$ which is clearly solid; it is another straightforward matter to prove that B is a band in $\mathscr{M}_{\mathbf{R}}(X)$. The only assertion in the following representation theorem that remains to be proved is the assertion that $B = E$.

8.6

Theorem. *Let E be an* (AL)*-space. The Banach lattice E' $(= E^+)$ can be identified with $\mathscr{C}_{\mathbf{R}}(X)$, where X is a compact, extremally disconnected space. Moreover, under evaluation, E is isomorphic with the band of all* (*real*) *Radon measures μ on X such that*

$$\lim_{f \in S} \mu(f) = \mu(\sup S)$$

for every majorized, directed ($\leqq$) *subset S of $\mathscr{C}_{\mathbf{R}}(X)$.*

Proof. It is easy to see that (identifying E' with $\mathscr{C}_{\mathbf{R}}(X)$ and E with its canonical image in $E'' = \mathscr{M}_{\mathbf{R}}(X)$) we have $E \subset B$ (see the preceding paragraph for notation). For if S is majorized and directed ($\leqq$), every section of S is $\sigma(E', E)$-bounded and hence the section filter $\sigma(E', E)$-converges to $\sup S$; the assertion follows since $\mu \in E$ is $\sigma(E', E)$-continuous.

To prove the reverse inclusion, let $0 \leqq \nu \in B$ and let $\mu_0 = \sup\, [0, \nu] \cap E$. Then the section filter of $[0, \nu] \cap E$ is a Cauchy filter for the norm topology, since E is an (AL)-space (Section 7, Example 5), and hence $\mu_0 \in E$, since E is norm complete. Now $\mu_1 = \nu - \mu_0$ is an element of B lattice disjoint from E; it will be shown that this implies $\mu_1 = 0$, and hence $B = E$ by (1.3).

Denote by T_1 the support of μ_1; if $T = X \sim T_1$, then T is open and $\mu_1(f) = 0$ for each f whose support T_f is contained in T. The family of all $f \in [0, e]$ such that $T_f \subset T$ is directed, and its least upper bound f_0 is necessarily the characteristic function of the closure $\overline{T}$. Since $\mu_1 \in B$, it follows that $\mu_1(f_0) = 0$; hence $\overline{T} \cap T_1 = \varnothing$, which shows that T_1 is open and closed. If $T_1 = \varnothing$ the proof is complete; hence assume that T_1 is non-empty. Since T_1 is open, there exist elements $\mu \in E$ whose support intersects T_1 (otherwise E would not distinguish points in $E' = \mathscr{C}_{\mathbf{R}}(X)$). There exists, consequently, a positive $\mu \in E$ such that $\|\mu\| = 1$ and whose support is contained in T_1 (it suffices to take a positive $\lambda \in E$ for which $\lambda(g_0) > 0$, where $g_0 = e - f_0$, and to consider $g_0 . \lambda$). The proof will now be completed by showing that this last statement is false.

Let $\varepsilon_n = 2^{-n}$ $(n \in N)$. By formula (7) of (1.5) there exist (since $\inf(\mu, \mu_1) = 0$) decompositions $g_0 = f_n + f_n'$, where $f_n \geqq 0$, $f_n' \geqq 0$ and such that $\mu_1(f_n) < \varepsilon_n^2$, $\mu(f_n') < \varepsilon_n^2$, so that $\mu(f_n) > 1 - \varepsilon_n^2$ $(n \in N)$. Let $G_n = \{t : f_n(t) > \varepsilon_n\}$ for all n, then G_n is open and $\overline{G}_n$ is closed and open. If we write $\mu(A)$ in place of $\mu(\chi_A)$ whenever $A \subset X$ is a subset whose characteristic function χ_A is continuous, we obtain $\mu_1(\overline{G}_n) < \varepsilon_n$; in fact, $\mu_1(\overline{G}_n) \geqq \varepsilon_n$ would imply that $\mu_1(f_n) \geqq \varepsilon_n \mu_1(\overline{G}_n) \geqq \varepsilon_n^2$, which is contradictory. Now let $H_k = \bigcup \{G_n : n \geqq k + 1\}$; then $\overline{H}_k$ is closed and open, and it follows from $\mu_1 \in B$ that $\mu_1(\overline{H}_k) < \varepsilon_k$, since the characteristic function of $\overline{H}_k$ is the least upper bound of the characteristic functions of the sets $\overline{G}_n (n \geqq k + 1)$. Now define g_n by $g_n = \sup\{f_\nu : \nu \geqq n\}$ $(n \in N)$; then $\{g_n : n \in N\}$ is a monotone $(\geqq)$ sequence; let $h = \inf\{g_n : n \in N\}$. In the complement of $\overline{H}_k$ one has $f_\nu(t) \leqq \varepsilon_\nu$ whenever $\nu \geqq k + 1$, and hence $g_n(t) \leqq \varepsilon_n$ whenever $n \geqq k + 1$; in view of $\mu_1(\overline{H}_k) < \varepsilon_k$, it is clear that $\mu_1(h) \leqq \varepsilon_k$. This implies $\mu_1(h) = 0$ and thus $h = 0$, for the support of h is contained in the support T_1 of μ_1. On the other hand, since $\mu \in E \subset B$, we have $\lim_n \mu(g_n) = \mu(h) = 0$, which conflicts with $\mu(f_n) > 1 - \varepsilon_n^2$, since $0 \leqq f_n \leqq g_n$ for all n. This completes the proof of (8.6).

COROLLARY 1. *In an* (AL)-*space E each order interval is weakly compact.*

Proof. Since E is a band in $E''(= E^{++})$, E is a solid subspace of E''; thus if $x, y \in E$, we have $[x, y] = (x + C) \cap (y - C) = (x + C'') \cap (y - C'')$ where C, C'' denote the positive cones of E, E'' respectively. Since C'' is $\sigma(E'', E')$-closed, it follows that $[x, y]$ is $\sigma(E'', E')$-closed and hence $\sigma(E'', E')$-compact.

COROLLARY 2. *Every* (AL)-*space E is an order complete vector lattice of minimal type; by contrast, its order dual E^+ is not of minimal type, unless E is of finite dimension.*

Proof. Since E can be identified with a band in $E'' = E^{++}$, it is clearly of minimal type (Section 7). If, on the other hand, E^+ (which can be identified with $\mathscr{C}_{\mathbf{R}}(X)$) is of minimal type, then by (7.5), Corollary 1, the section filter of each directed $(\leqq)$, majorized set S converges to $\sup S$ pointwise (even uniformly) on X, which implies that each open subset of X is closed, and hence that the topology of X is discrete. Since X is compact, X is finite, and hence E^+ and E are finite dimensional.

Our second application of (8.5) is the following result.

8.7

Let $(E, \mathfrak{T})$ be a l.c.v.l. which is bornological and sequentially complete. There exists a family of compact spaces $X_\alpha (\alpha \in \mathrm{A})$ and a family of vector lattice isomorphisms f_α of $\mathscr{C}_{\mathbf{R}}(X_\alpha)$ into $E\,(\alpha \in \mathrm{A})$ such that $\mathfrak{T}$ is the finest l.c. topology on E for which each f_α is continuous.

Proof. In view of (5.5) and (6.4), the assumption that $(E, \mathfrak{T})$ be bornological implies that $\mathfrak{T}$ is the order topology $\mathfrak{T}_O$. Hence by (6.3), $(E, \mathfrak{T})$ is the inductive limit of the subspaces (E_α, p_α) $(\alpha \in \mathrm{A})$ where $E_\alpha = \bigcup_{n=1}^{\infty} n[-a_\alpha, a_\alpha]$, p_α is the gauge of $[-a_\alpha, a_\alpha]$ on E_α, and $\{a_\alpha : \alpha \in \mathrm{A}\}$ is a directed subset of the positive cone of E such that $\bigcup_\alpha E_\alpha = E$. By (6.2) each (p_α, E_α) is a Banach lattice, and by (8.4) even an (AM)-space with unit a_α. Hence by (8.5), (E_α, p_α) can be identified with $\mathscr{C}_{\mathbf{R}}(X_\alpha)$ for a suitable compact space X_α, and the assertion follows from the definition of inductive topologies (Chapter II, Section 6).

EXERCISES

1. A reflexive, transitive binary relation "$\prec$" on a set S is called a **pre-order** on S. A pre-order on a vector space L over $\mathbf{R}$ is said to be compatible (with the vector structure of L) if $x \prec y$ implies $x + z \prec y + z$ and $\lambda x \prec \lambda y$ for all $z \in L$ and all scalars $\lambda > 0$.

(a) If (X, Σ, μ) is a measure space (Chapter II, Section 2, Example 2), the relation "$f(t) \leqq g(t)$ almost everywhere (μ)" defines a compatible pre-order on the vector space (over $\mathbf{R}$) of all real-valued Σ-measurable functions on X.

(b) If "$\prec$" is a compatible pre-order, the relation "$x \prec y$ *and* $y \prec x$" is an equivalence relation on L, the subset N of elements equivalent to 0 is a subspace of L, and L/N is an ordered vector space under the relation "$\hat{x} \leqq \hat{y}$ if there exist elements $x \in \hat{x}$, $y \in \hat{y}$ satisfying $x \prec y$".

(c) The family of all compatible pre-orders of a vector space L over $\mathbf{R}$ is in one–to–one correspondence with the family of all convex cones in L that contain their vertex 0.

2. The family of all total vector orderings (total orderings satisfying $(LO)_1$ and $(LO)_2$, Section 1) of a vector space L is in one–to–one correspondence with the family of all proper cones that are maximal (under set inclusion). Deduce from this that for each vector ordering R of L, there exists a total vector ordering of L that is coarser than R. (Use Zorn's lemma.) Show that a total vector ordering cannot be Archimedean if the real dimension of L is >1.

3. Let L be an ordered vector space with positive cone C. Let N be a subspace of L, and denote by $\hat{C}$ the canonical image of C in L/N.

(a) If N is C-saturated, then $\hat{C}$ defines the canonical order of L/N.

(b) If L is a t.v.s. and if for each 0-neighborhood V in L there exists a 0-neighborhood U such that $[(U + N) \cap C] \subset V + N$, then $\hat{C}$ is normal for the quotient topology. (Compare the proof of (3.1).)