Chapter 1

The Order Structure of Positive Operators

A linear operator between two ordered vector spaces that carries positive elements to positive elements is known in the literature as a positive operator. As we have mentioned in the preface, the main theme of this book is the study of positive operators. To obtain fruitful and useful results the domains and the ranges of positive operators will be taken to be Riesz spaces (vector lattices). For this reason, in order to make the material as self-sufficient as possible, the fundamental properties of Riesz spaces are discussed as they are needed.

Throughout this book the symbol $\mathbb{R}$ will denote the set of real numbers, $\mathbb{N}$ will denote the set of natural numbers, $\mathbb{Q}$ will denote the set of rational numbers, and $\mathbb{Z}$ will denote the set of integers.

1.1. Basic Properties of Positive Operators

A real vector space E is said to be an **ordered vector space** whenever it is equipped with an order relation $\geq$ (i.e., $\geq$ is a reflexive, antisymmetric, and transitive binary relation on E) that is compatible with the algebraic structure of E in the sense that it satisfies the following two axioms:

(1) If $x \geq y$, then $x + z \geq y + z$ holds for all $z \in E$.

(2) If $x \geq y$, than $\alpha x \geq \alpha y$ holds for all $\alpha \geq 0$.

An alternative notation for $x \geq y$ is $y \leq x$. A vector x in an ordered vector space E is called **positive** whenever $x \geq 0$ holds. The set of all positive vectors of E will be denoted by E^+, i.e., $E^+ := \{x \in E\colon\ x \geq 0\}$. The set E^+ of positive vectors is called the **positive cone** of E.

Definition 1.1. *An* ***operator*** *is a linear map between two vector spaces.*

That is, a mapping $T\colon E \to F$ between two vector spaces is called an operator if and only if $T(\alpha x + \beta y) = \alpha T(x) + \beta T(y)$ holds for all $x, y \in E$ and all $\alpha, \beta \in \mathbb{R}$. As usual, the value $T(x)$ will also be designated by Tx.

Definition 1.2. *An operator $T\colon E \to F$ between two ordered vector spaces is said to be* ***positive*** (*in symbols $T \geq 0$ or $0 \leq T$*) *if $T(x) \geq 0$ for all $x \geq 0$.*

Clearly, an operator $T\colon E \to F$ between two ordered vector spaces is positive if and only if $T(E^+) \subseteq F^+$ (and also if and only if $x \leq y$ implies $Tx \leq Ty$).

A **Riesz space** (or a **vector lattice**) is an ordered vector space E with the additional property that for each pair of vectors $x, y \in E$ the supremum and the infimum of the set $\{x, y\}$ both exist in E. Following the classical notation, we shall write

$$x \vee y := \sup\{x, y\} \quad \text{and} \quad x \wedge y := \inf\{x, y\}.$$

Typical examples of Riesz spaces are provided by the function spaces. A **function space** is a vector space E of real-valued functions on a set Ω such that for each pair $f, g \in E$ the functions

$$[f \vee g](\omega) := \max\{f(\omega), g(\omega)\} \quad \text{and} \quad [f \wedge g](\omega) := \min\{f(\omega), g(\omega)\}$$

both belong to E. Clearly, every function space E with the pointwise ordering (i.e., $f \leq g$ holds in E if and only if $f(\omega) \leq g(\omega)$ for all $\omega \in \Omega$) is a Riesz space. Here are some important examples of function spaces:

(a) $\mathbb{R}^\Omega$, all real-valued functions defined on a set Ω.

(b) $C(\Omega)$, all continuous real-valued functions on a topological space Ω.

(c) $C_b(\Omega)$, all bounded real-valued continuous functions on a topological space Ω.

(d) $\ell_\infty(\Omega)$, all bounded real-valued functions on a set Ω.

(e) ℓ_p $(0 < p < \infty)$, all real sequences $(x_1, x_2, \ldots)$ with $\sum_{n=1}^{\infty} |x_n|^p < \infty$.

The class of L_p-spaces is another important class of Riesz spaces. If (X, Σ, μ) is a measure space and $0 < p < \infty$, then $L_p(\mu)$ is the vector space of all real-valued μ-measurable functions f on X such that $\int_X |f|^p \, d\mu < \infty$. Also, $L_\infty(\mu)$ is the vector space of all real-valued μ-measurable functions f on X such that $\operatorname{esssup} |f| < \infty$. As usual, functions differing on a set of measure zero are treated as identical, i.e., $f = g$ in $L_p(\mu)$ means that $f(x) = g(x)$ for μ-almost all $x \in X$. (In other words, each $L_p(\mu)$-space consists of equivalence classes rather than functions.) It is easy to see that under the ordering $f \le g$ whenever $f(x) \le g(x)$ holds for μ-almost all $x \in X$, each $L_p(\mu)$ is a Riesz space.

There are several useful identities that are true in a Riesz space some of which are included in the next few results.

Theorem 1.3. *If x, y and z are elements in a Riesz space, then:*

(1) $x \vee y = -\big[(-x) \wedge (-y)\big]$ *and* $x \wedge y = -\big[(-x) \vee (-y)\big]$.

(2) $x + y = x \wedge y + x \vee y$.

(3) $x + (y \vee z) = (x+y) \vee (x+z)$ *and* $x + (y \wedge z) = (x+y) \wedge (x+z)$.

(4) $\alpha(x \vee y) = (\alpha x) \vee (\alpha y)$ *and* $\alpha(x \wedge y) = (\alpha x) \wedge (\alpha y)$ *for all* $\alpha \ge 0$.

Proof. (1) From $x \le x \vee y$ and $y \le x \vee y$ we get $-(x \vee y) \le -x$ and $-(x \vee y) \le -y$, and so $-(x \vee y) \le (-x) \wedge (-y)$. On the other hand, if $-x \ge z$ and $-y \ge z$, then $-z \ge x$ and $-z \ge y$, and hence $-z \ge x \vee y$. Thus, $-(x \vee y) \ge z$ holds and this shows that $-(x \vee y)$ is the infimum of the set $\{-x, -y\}$. That is, $(-x) \wedge (-y) = -(x \vee y)$. To get the identity for $x \wedge y$ replace x by $-x$ and y by $-y$ in the above proven identity.

(2) From $x \wedge y \le y$ it follows that $y - x \wedge y \ge 0$ and so $x \le x + y - x \wedge y$. Similarly, $y \le x + y - x \wedge y$. Consequently, we have $x \vee y \le x + y - x \wedge y$ or $x \wedge y + x \vee y \le x + y$. On the other hand, from $y \le x \vee y$ we see that $x + y - x \vee y \le x$, and similarly $x + y - x \vee y \le y$. Thus, $x + y - x \vee y \le x \wedge y$ so that $x + y \le x \wedge y + x \vee y$, and the desired identity follows.

(3) Clearly, $x + y \le x + y \vee z$ and $x + z \le x + y \vee z$, and therefore $(x+y) \vee (x+z) \le x + y \vee z$. On the other hand, we have $y = -x + (x+y) \le -x + (x+y) \vee (x+z)$, and likewise $z \le -x + (x+y) \vee (x+z)$, and so $y \vee z \le -x + (x+y) \vee (x+z)$. Therefore, $x + y \vee z \le (x+y) \vee (x+z)$ also holds, and thus $x + y \vee z = (x+y) \vee (x+z)$. The other identity can be proven in a similar manner.

(4) Fix $\alpha > 0$. Clearly, $(\alpha x) \vee (\alpha y) \le \alpha(x \vee y)$. If $\alpha x \le z$ and $\alpha y \le z$ are both true, then $x \le \frac{1}{\alpha} z$ and $y \le \frac{1}{\alpha} z$ also are true, and so $x \vee y \le \frac{1}{\alpha} z$. This implies $\alpha(x \vee y) \le z$, and this shows that $\alpha(x \vee y)$ is the supremum of the set $\{\alpha x, \alpha y\}$. Therefore, $(\alpha x) \vee (\alpha y) = \alpha(x \vee y)$. The other identity can be proven similarly. ■

The reader can establish in a similar manner the following general versions of the preceding formulas in (1), (3), and (4). If A is a nonempty subset of a Riesz space for which $\sup A$ exists, then:

(a) The infimum of the set $-A := \{-a\colon\ a \in A\}$ exists and
$$\inf(-A) = -\sup A\,.$$

(b) For each vector x the supremum of the set $x+A := \{x+a\colon\ a \in A\}$ exists and
$$\sup(x + A) = x + \sup A\,.$$
(c) For each $\alpha \geq 0$ the supremum of the set $\alpha A := \{\alpha a\colon\ a \in A\}$ exists and
$$\sup(\alpha A) = \alpha \sup A\,.$$

We have also the following useful inequality between positive vectors.

Lemma 1.4. *If $x, x_1, x_2, \ldots, x_n$ are positive elements in a Riesz space, then*
$$x \wedge (x_1 + x_2 + \cdots + x_n) \leq x \wedge x_1 + x \wedge x_2 + \cdots + x \wedge x_n\,.$$

Proof. Assume that x and x_1, x_2 are all positive vectors. For simplicity, let $y = x \wedge (x_1 + x_2)$. Then $y \leq x_1 + x_2$ and so $y - x_1 \leq x_2$. Also we have $y - x_1 \leq y \leq x$. Consequently $y - x_1 \leq x \wedge x_2$. This implies $y - x \wedge x_2 \leq x_1$ and since $y - x \wedge x_2 \leq y \leq x$, we infer that $y - x \wedge x_2 \leq x \wedge x_1$ or $y \leq x \wedge x_1 + x \wedge x_2$. The proof now can be completed by induction. ■

For any vector x in a Riesz space define
$$x^+ := x \vee 0\,, \quad x^- := (-x) \vee 0\,, \quad \text{and} \quad |x| := x \vee (-x)\,.$$
The element x^+ is called the **positive part**, x^- is called the **negative part**, and $|x|$ is called the **absolute value** of x. The vectors x^+, x^-, and $|x|$ satisfy the following important identities.

Theorem 1.5. *If x is an arbitrary vector in a Riesz space E, then:*

(1) $x = x^+ - x^-$.

(2) $|x| = x^+ + x^-$.

(3) $x^+ \wedge x^- = 0$.

Moreover, the decomposition in (1) *satisfies the following minimality and uniqueness properties.*

(a) *If $x = y - z$ with $y, z \in E^+$, then $y \geq x^+$ and $z \geq x^-$.*

(b) *If $x = y - z$ with $y \wedge z = 0$, then $y = x^+$ and $z = x^-$.*

Proof. (1) From Theorem 1.3 we see that
$$x = x + 0 = x \vee 0 + x \wedge 0 = x \vee 0 - (-x) \vee 0 = x^+ - x^-\,.$$

(2) Using Theorem 1.3 and (1), we get
$$\begin{aligned} |x| &= x \vee (-x) = (2x) \vee 0 - x = 2(x \vee 0) - x \\ &= 2x^+ - x = 2x^+ - (x^+ - x^-) = x^+ + x^-\,. \end{aligned}$$

(3) Note that

$$\begin{aligned} x^+ \wedge x^- &= (x^+ - x^-) \wedge 0 + x^- = x \wedge 0 + x^- \\ &= -[(-x) \vee 0] + x^- = -x^- + x^- = 0\,. \end{aligned}$$

(a) Assume that $x = y - z$ with $y \geq 0$ and $z \geq 0$. From $x = x^+ - x^-$, we get $x^+ = x^- + y - z \leq x^- + y$, and so from Lemma 1.4 we get

$$x^+ = x^+ \wedge x^+ \leq x^+ \wedge (x^- + y) \leq x^+ \wedge x^- + x^+ \wedge y = x^+ \wedge y \leq y\,.$$

Similarly, $x^- \leq z$.

(b) Let $x = y - z$ with $y \wedge z = 0$. Then, using Theorem 1.3, we see that $x^+ = (y - z) \vee 0 = y \vee z - z = (y + z - y \wedge z) - z = y$. Similarly, $x^- = z$. ■

We also have the following useful inequality regarding positive operators.

Lemma 1.6. *If $T: E \to F$ is a positive operator between two Riesz spaces, then for each $x \in E$ we have*

$$|Tx| \leq T|x|\,.$$

Proof. If $x \in E$, then $\pm x \leq |x|$ and the positivity of T yields $\pm Tx \leq T|x|$, which is equivalent to $|Tx| \leq T|x|$. ■

A few more useful lattice identities are included in the next result.

Theorem 1.7. *If x and y are elements in a Riesz space, then we have:*

(1) $x = (x - y)^+ + x \wedge y$.
(2) $x \vee y = \frac{1}{2}\big(x + y + |x - y|\big)$ *and* $x \wedge y = \frac{1}{2}\big(x + y - |x - y|\big)$.
(3) $|x - y| = x \vee y - x \wedge y$.
(4) $|x| \vee |y| = \frac{1}{2}\big(|x + y| + |x - y|\big)$.
(5) $|x| \wedge |y| = \frac{1}{2}\big||x + y| - |x - y|\big|$.
(6) $|x + y| \wedge |x - y| = \big||x| - |y|\big|$.
(7) $|x + y| \vee |x - y| = |x| + |y|$.

Proof. (1) Using Theorem 1.3 we see that

$$\begin{aligned} x &= x \vee y - y + x \wedge y = (x - y) \vee (y - y) + x \wedge y \\ &= (x - y) \vee 0 + x \wedge y = (x - y)^+ + x \wedge y\,. \end{aligned}$$

(2) For the first identity note that

$$\begin{aligned} x + y + |x - y| &= x + y + (x - y) \vee (y - x) \\ &= \big[(x + y) + (x - y)\big] \vee \big[(x + y) + (y - x)\big] \\ &= (2x) \vee (2y) = 2(x \vee y)\,. \end{aligned}$$

(3) Subtract the two identities in (2).

(4) Using (2) above, we see that

$$\begin{aligned}
|x+y|+|x-y| &= (x+y)\vee(-x-y)+|x-y| \\
&= \big(x+y+|x-y|\big)\vee\big(-x-y+|x-y|\big) \\
&= 2\big([x\vee y]\vee[(-x)\vee(-y)]\big) \\
&= 2\big([x\vee(-x)]\vee[y\vee(-y)]\big) \\
&= 2\big(|x|\vee|y|\big)\,.
\end{aligned}$$

(5) Using (2) and (4) above we get

$$\begin{aligned}
\big||x+y|-|x-y|\big| &= 2\big(|x+y|\vee|x-y|\big)-\big(|x+y|+|x-y|\big) \\
&= 2\big(|x|+|y|\big)-2\big(|x|\vee|y|\big) \\
&= 2\big(|x|\wedge|y|\big)\,.
\end{aligned}$$

(6) Notice that

$$\begin{aligned}
&|x+y|\wedge|x-y| \\
&\quad= \big[(x+y)\vee(-x-y)\big]\wedge\big[(x-y)\vee(y-x)\big] \\
&\quad= \big\{\big[(x+y)\vee(-x-y)\big]\wedge(x-y)\big\}\vee\big\{\big[(x+y)\vee(-x-y)\big]\wedge(y-x)\big\} \\
&\quad= \big[(x+y)\wedge(x-y)\big]\vee\big[(-x-y)\wedge(x-y)\big]\vee\cdots \\
&\qquad\quad \cdots\vee\big[(x+y)\wedge(y-x)\big]\vee\big[(-x-y)\wedge(y-x)\big] \\
&\quad= \big[x+y\wedge(-y)\big]\vee\big[-y+(-x)\wedge x\big]\vee\cdots \\
&\qquad\quad \cdots\vee\big[y+x\wedge(-x)\big]\vee\big[-x+(-y)\wedge y\big] \\
&\quad= \Big\{\big[x+y\wedge(-y)\big]\vee\big[-x+y\wedge(-y)\big]\Big\}\vee\cdots \\
&\qquad\quad \cdots\vee\Big\{\big[-y+(-x)\wedge x\big]\vee\big[y+x\wedge(-x)\big]\Big\} \\
&\quad= \big[x\vee(-x)+y\wedge(-y)\big]\vee\big[(-y)\vee y+x\wedge(-x)\big] \\
&\quad= \big[|x|-|y|\big]\vee\big[|y|-|x|\big]=\big||x|-|y|\big|\,.
\end{aligned}$$

(7) Using (3) and (5) we get

$$\begin{aligned}
|x+y|\vee|x-y| &= \big||x+y|-|x-y|\big|+|x+y|\wedge|x-y| \\
&= 2\big(|x|\wedge|y|\big)+\big||x|-|y|\big| \\
&= 2\big(|x|\wedge|y|\big)+\big(|x|\vee|y|-|x|\wedge|y|\big) \\
&= |x|\wedge|y|+|x|\vee|y|=|x|+|y|\,,
\end{aligned}$$

and the proof is finished. ■

It should be noted that the identities in (2) above show that an ordered vector space is a Riesz space if and only if the absolute value $|x| = x \vee (-x)$ exists for each vector x.

In a Riesz space, two elements x and y are said to be **disjoint** (in symbols $x \perp y$) whenever $|x| \wedge |y| = 0$ holds. Note that according to part (5) of Theorem 1.7 we have $x \perp y$ if and only if $|x+y| = |x-y|$. Two subsets A and B of a Riesz space are called **disjoint** (denoted $A \perp B$) if $a \perp b$ holds for all $a \in A$ and all $b \in B$.

If A is a nonempty subset of a Riesz space E, then its **disjoint complement** A^{d} is defined by

$$A^{\mathrm{d}} := \big\{x \in E\colon\ x \perp y \ \text{ for all } \ y \in A\big\}.$$

We write A^{dd} for $(A^{\mathrm{d}})^{\mathrm{d}}$. Note that $A \cap A^{\mathrm{d}} = \{0\}$.

If A and B are subsets of a Riesz space, then we shall employ in this book the following self-explanatory notation:

$$\begin{aligned}
|A| &:= \big\{|a|\colon\ a \in A\big\}\\
A^+ &:= \big\{a^+\colon\ a \in A\big\}\\
A^- &:= \big\{a^-\colon\ a \in A\big\}\\
A \vee B &:= \big\{a \vee b\colon\ a \in A \text{ and } b \in B\big\}\\
A \wedge B &:= \big\{a \wedge b\colon\ a \in A \text{ and } b \in B\big\}\\
x \vee A &:= \big\{x \vee a\colon\ a \in A\big\}\\
x \wedge A &:= \big\{x \wedge a\colon\ a \in A\big\}
\end{aligned}$$

The next theorem tells us that every Riesz space satisfies the infinite distributive law.

Theorem 1.8 (The Infinite Distributive Law)**.** *Let A be a nonempty subset of a Riesz space. If* $\sup A$ *exists, then for each vector x the supremum of the set $x \wedge A$ exists and*

$$\sup(x \wedge A) = x \wedge \sup A\,.$$

Similarly, if $\inf A$ *exists, then* $\inf(x \vee A)$ *exists for each vector x and*

$$\inf(x \vee A) = x \vee \inf A\,.$$

Proof. Assume that $\sup A$ exists. Let $y = \sup A$ and fix some vector x. Clearly, for each $a \in A$ we have $x \wedge a \leq x \wedge y$, i.e., $x \wedge y$ is an upper bound of the set $x \wedge A$. To see that $x \wedge y$ is the least upper bound of the set $x \wedge A$, assume that some vector z satisfies $x \wedge a \leq z$ for all $a \in A$. Since for each $a \in A$ we have $a = x \wedge a + x \vee a - x \leq z + x \vee y - x$, it follows that $y \leq z + x \vee y - x$. This implies $x \wedge y = x + y - x \vee y \leq z$, and from this we see that $\sup(x \wedge A)$ exists and that $\sup(x \wedge A) = x \wedge \sup A$ holds. The other formula can be proven in a similar manner. ∎

The next result includes most of the major inequalities that are used extensively in estimations.

Theorem 1.9. *For arbitrary elements x, y, and z in a Riesz space we have the following inequalities.*

(1) $\big||x| - |y|\big| \leq |x + y| \leq |x| + |y|$ (*the triangle inequality*).

(2) $|x \vee z - y \vee z| \leq |x - y|$ *and* $|x \wedge z - y \wedge z| \leq |x - y|$ (*Birkhoff's inequalities*).

Proof. (1) Clearly, $x + y \leq |x| + |y|$ and $-x - y \leq |x| + |y|$ both hold. Thus, $|x + y| = (x + y) \vee (-x - y) \leq |x| + |y|$.

Now observe that the inequality $|x| = \big|(x + y) - y\big| \leq |x + y| + |y|$ implies $|x| - |y| \leq |x + y|$. Similarly, $|y| - |x| \leq |x + y|$, and hence $\big||x| - |y|\big| \leq |x + y|$ is also true.

(2) Note that

$$\begin{aligned} x \vee z - y \vee z &= \big[(x - z) \vee 0 + z\big] - \big[(y - z) \vee 0 + z\big] \\ &= (x - z)^+ - (y - z)^+ \\ &= \big[(x - y) + (y - z)\big]^+ - (y - z)^+ \\ &\leq \big[(x - y)^+ + (y - z)^+\big] - (y - z)^+ \\ &= (x - y)^+ \leq |x - y| \,. \end{aligned}$$

Similarly, $y \vee z - x \vee z \leq |x - y|$, and so $|x \vee z - y \vee z| \leq |x - y|$. The other inequality can be proven in a similar manner. ∎

In particular, note that in any Riesz space we have

$$\big|x^+ - y^+\big| \leq |x - y| \quad \text{and} \quad \big|x^- - y^-\big| \leq |x - y| \,.$$

These inequalities will be employed quite often in our discussions.

A net $\{x_\alpha\}$ in a Riesz space is said to be **decreasing** (in symbols $x_\alpha \downarrow$) whenever $\alpha \succeq \beta$ implies $x_\alpha \leq x_\beta$. The notation $x_\alpha \downarrow x$ means that $x_\alpha \downarrow$ and $\inf\{x_\alpha\} = x$ both hold. The meanings of $x_\alpha \uparrow$ and $x_\alpha \uparrow x$ are analogous.

The Archimedean property states that for each real number $x > 0$ the sequence $\{nx\}$ is unbounded above in $\mathbb{R}$. This is, of course, equivalent to saying that $\frac{1}{n}x \downarrow 0$ holds in $\mathbb{R}$ for each $x > 0$. Motivated by this property, a Riesz space (and in general an ordered vector space) E is called **Archimedean** whenever $\frac{1}{n}x \downarrow 0$ holds in E for each $x \in E^+$. All classical spaces of functional analysis (notably the function spaces and L_p-spaces) are Archimedean. For this reason, the focus of our work will be on the study of positive operators between Archimedean Riesz spaces. Accordingly:

- *Unless otherwise stated, throughout this book all Riesz spaces will be assumed to be Archimedean.*

The starting point in the theory of positive operators is a fundamental extension theorem of L. V. Kantorovich [**91**]. The importance of the result lies in the fact that in order for a mapping $T\colon E^+ \to F^+$ to be the restriction of a (unique) positive operator from E to F it is necessary and sufficient to be additive on E^+. The details follow.

Theorem 1.10 (Kantorovich)**.** *Suppose that E and F are two Riesz spaces with F Archimedean. Assume also that $T\colon E^+ \to F^+$ is an* ***additive mapping****, that is, $T(x+y) = T(x) + T(y)$ holds for all $x, y \in E^+$. Then T has a unique extension to a positive operator from E to F. Moreover, the extension* (*denoted by T again*) *is given by*

$$T(x) = T(x^+) - T(x^-)$$

for all $x \in E$.

Proof. Let $T\colon E^+ \to F^+$ be an additive mapping. Consider the mapping $S\colon E \to F$ defined by

$$S(x) = T(x^+) - T(x^-)\,.$$

Clearly, $S(x) = T(x)$ for each $x \in E^+$. So, the mapping S extends T to all of E. Since $x = x^+ - x^-$ for each $x \in E$, it follows that S is the only possible linear extension of T to all of E. Therefore, in order to complete the proof, we must show that S is linear. That is, we must prove that S is additive and homogeneous.

For the additivity of S start by observing that if any vector $x \in E$ can be written as a difference of two positive vectors, say $x = x_1 - x_2$ with $x_1, x_2 \in E^+$, then $S(x) = T(x_1) - T(x_2)$ holds. To see this, fix any $x \in E$ and assume that $x = x^+ - x^- = x_1 - x_2$, where $x_1, x_2 \in E^+$. Then $x^+ + x_2 = x_1 + x^-$, and so the additivity of T on E^+ yields

$$T(x^+) + T(x_2) = T(x^+ + x_2) = T(x_1 + x^-) = T(x_1) + T(x^-)$$

or $S(x) = T(x^+) - T(x^-) = T(x_1) - T(x_2)$. From this property, we can easily establish that S is additive. Indeed, if $x, y \in E$, then note that

$$\begin{aligned} S(x+y) &= S\big(x^+ + y^+ - (x^- + y^-)\big) \\ &= T(x^+ + y^+) - T(x^- + y^-) \\ &= T(x^+) + T(y^+) - T(x^-) - T(y^-) \\ &= \big[T(x^+) - T(x^-)\big] + \big[T(y^+) - T(y^-)\big] \\ &= S(x) + S(y)\,. \end{aligned}$$

In particular, the additivity of S implies that $S(rx) = rS(x)$ holds for all $x \in E$ and all rational numbers r.

It remains to show that S is homogeneous. For this, we need to prove first that S is monotone. That is, $x \geq y$ in E implies $S(x) \geq S(y)$ in F. Indeed, if $x \geq y$, then $x - y \in E^+$, and so by the additivity of S we get

$$S(x) = S\big((x-y)+y\big) = S(x-y) + S(y) = T(x-y) + S(y) \geq S(y)\,.$$

Now fix $x \in E^+$ and let $\lambda \geq 0$. Pick two sequences of non-negative rational numbers $\{r_n\}$ and $\{t_n\}$ such that $r_n \uparrow \lambda$ and $t_n \downarrow \lambda$. The inequalities $r_n x \leq \lambda x \leq t_n x$ and the monotonicity of S imply

$$r_n S(x) = S(r_n x) \leq S(\lambda x) \leq S(t_n x) = t_n S(x)$$

for each n. Using that F is Archimedean, we easily get $\lambda S(x) = S(\lambda x)$. Finally, if $\lambda \in \mathbb{R}$ and $x \in E$, then

$$\begin{aligned} S(\lambda x) &= S(\lambda x^+ + (-\lambda)x^-) = S\big(\lambda x^+\big) + S\big((-\lambda)x^-\big) \\ &= \lambda S(x^+) - \lambda S(x^-) = \lambda\big[T(x^+) - T(x^-)\big] = \lambda S(x)\,. \end{aligned}$$

So, S is also homogeneous, and the proof is finished. ■

The preceding lemma is not true if F is not Archimedean.

Example 1.11. Let $\phi\colon \mathbb{R} \to \mathbb{R}$ be an additive function that is not linear, i.e., not of the form $\phi(x) = cx$, and let F be the lexicographic plane. Consider the mapping $T\colon \mathbb{R}^+ \to F^+$ defined by $T(x) = \big(x, \phi(x)\big)$ for each x in $\mathbb{R}^+$. Note that T is additive and that if T could be extended to an operator from $\mathbb{R}$ to F, then ϕ should be linear. ■

Thus, a mapping $T\colon E^+ \to F^+$ extends to a (unique) positive operator from E to F if and only if T is additive on E^+. In other words, a positive operator is determined completely by its action on the positive cone of its domain. In the sequel, the expression "**the mapping** $T\colon E^+ \to F^+$ **defines a positive operator**" will simply mean that T is additive on E^+ (and hence extendable by Theorem 1.10 to a unique positive operator).

The (real) vector space of all operators from E to F will be denoted by $\mathcal{L}(E,F)$. It is not difficult to see that $\mathcal{L}(E,F)$ under the ordering $T \geq S$

whenever $T - S$ is a positive operator (i.e., whenever $T(x) \geq S(x)$ holds for all $x \in E^+$) is an ordered vector space.

Definition 1.12. *For an operator $T\colon E \to F$ between two Riesz spaces we shall say that its modulus $|T|$ exists (or that T possesses a **modulus**) whenever*

$$|T| := T \vee (-T)$$

exists—in the sense that $|T|$ is the supremum of the set $\{-T, T\}$ in $\mathcal{L}(E, F)$.

In order to study the elementary properties of the modulus, we need a decomposition property of Riesz spaces.

Theorem 1.13 (The Decomposition Property)**.** *If $|x| \leq |y_1 + \cdots + y_n|$ holds in a Riesz space, then there exist $x_1, \ldots, x_n$ satisfying $x = x_1 + \cdots + x_n$ and $|x_i| \leq |y_i|$ for each $i = 1, \ldots, n$. Moreover, if x is positive, then the x_i also can be chosen to be positive.*

Proof. By using induction it is enough to establish the result when $n = 2$. So, let $|x| \leq |y_1 + y_2|$.

Put $x_1 = \big[x \vee (-|y_1|)\big] \wedge |y_1|$, and observe that $|x_1| \leq |y_1|$ (and that $0 \leq x_1 \leq x$ holds if x is positive). Now put $x_2 = x - x_1$ and observe that

$$x_2 = x - \big[x \vee (-|y_1|)\big] \wedge |y_1| = \big[0 \wedge (x + |y_1|)\big] \vee (x - |y_1|)\,.$$

On the other hand, $|x| \leq |y_1| + |y_2|$ implies $-|y_1| - |y_2| \leq x \leq |y_1| + |y_2|$, from which it follows that

$$-|y_2| = (-|y_2|) \wedge 0 \leq \big(x + |y_1|\big) \wedge 0 \leq x_2 \leq 0 \vee \big(x - |y_1|\big) \leq |y_2|\,.$$

Thus, $|x_2| \leq |y_2|$ also holds, and the proof is finished. ■

An important case for the modulus to exist is described next.

Theorem 1.14. *Let $T\colon E \to F$ be an operator between two Riesz spaces such that $\sup\big\{|Ty|\colon\ |y| \leq x\big\}$ exists in F for each $x \in E^+$. Then the modulus of T exists and*

$$|T|(x) = \sup\big\{|Ty|\colon\ |y| \leq x\big\}$$

holds for all $x \in E^+$.

Proof. Define $S\colon E^+ \to F^+$ by $S(x) = \sup\big\{|Ty|\colon\ |y| \leq x\big\}$ for each x in E^+. Since $|y| \leq x$ implies $|\pm y| = |y| \leq x$, it easily follows that we have $S(x) = \sup\big\{Ty\colon\ |y| \leq x\big\}$ for each $x \in E^+$. We claim that S is additive.

To see this, let $u, v \in E^+$. If $|y| \leq u$ and $|z| \leq v$, then $|y + z| \leq |y| + |z| \leq u + v$, and so it follows from $T(y) + T(z) = T(y + z) \leq S(u + v)$ that $S(u) + S(v) \leq S(u + v)$. On the other hand, if $|y| \leq u + v$, then by Theorem 1.13 there exist y_1 and y_2 with $|y_1| \leq u$, $|y_2| \leq v$, and $y = y_1 + y_2$.

Then $T(y) = T(y_1) + T(y_2) \leq S(u) + S(v)$ holds, from which it follows that $S(u+v) \leq S(u) + S(v)$. Therefore, $S(u+v) = S(u) + S(v)$ holds. By Theorem 1.10 the mapping S defines a positive operator from E to F.

To see that S is the supremum of $\{-T, T\}$, note first that $T \leq S$ and $-T \leq S$ hold trivially in $\mathcal{L}(E,F)$. Now assume that $\pm T \leq R$ in $\mathcal{L}(E,F)$. Clearly, R is a positive operator. Fix $x \in E^+$. If $|y| \leq x$, then note that

$$Ty = Ty^+ - Ty^- \leq Ry^+ + Ry^- = R|y| \leq Rx\,.$$

Therefore, $S(x) \leq R(x)$ holds for each $x \in E^+$, and so $S = T \vee (-T)$ holds in $\mathcal{L}(E,F)$. ■

It is easy to check, but important to observe, that if the modulus of an operator $T\colon E \to F$ exists, then

$$\big|T(x)\big| \leq |T|\big(|x|\big)$$

holds for all $x \in E$.

If x and y are two vectors in a Riesz space E with $x \leq y$, then the **order interval** $[x,y]$ is the subset of E defined by

$$[x,y] := \big\{z \in E\colon\ x \leq z \leq y\big\}\,.$$

A subset A of a Riesz space is said to be **bounded above** whenever there exists some x satisfying $y \leq x$ for all $y \in A$. Similarly, a set A of a Riesz space is **bounded below** whenever there exists some x satisfying $y \geq x$ for all $y \in A$. Finally, a subset in a Riesz space is called **order bounded** if it is bounded both above and below (or, equivalently, if it is included in an order interval).

Besides $\mathcal{L}(E,F)$, a number of other important vector subspaces of $\mathcal{L}(E,F)$ will be considered. The vector subspace $\mathcal{L}_{\mathrm{b}}(E,F)$ of all order bounded operators from E to F will be of fundamental importance.

Definition 1.15. *An operator $T\colon E \to F$ between two Riesz spaces is said to be **order bounded** if it maps order bounded subsets of E to order bounded subsets of F.*

The vector space of all order bounded operators from E to F will be denoted $\mathcal{L}_{\mathrm{b}}(E,F)$.

An operator $T\colon E \to F$ between two Riesz spaces is said to be **regular** if it can be written as a difference of two positive operators. Of course, this is equivalent to saying that there exists a positive operator $S\colon E \to F$ satisfying $T \leq S$.

Every positive operator is order bounded. Therefore, every regular operator is likewise order bounded. Thus, if $\mathcal{L}_{\mathrm{r}}(E,F)$ denotes the vector space of all regular operators (which is the same as the vector subspace generated by the positive operators), then the following vector subspace inclusions hold:

$$\mathcal{L}_\mathrm{r}(E,F) \subseteq \mathcal{L}_\mathrm{b}(E,F) \subseteq \mathcal{L}(E,F)\,.$$

Of course, $\mathcal{L}_\mathrm{r}(E,F)$ and $\mathcal{L}_\mathrm{b}(E,F)$ with the ordering inherited from $\mathcal{L}(E,F)$ are both ordered vector spaces. For brevity $\mathcal{L}(E,E)$, $\mathcal{L}_\mathrm{b}(E,E)$, and $\mathcal{L}_\mathrm{r}(E,E)$ will be denoted by $\mathcal{L}(E)$, $\mathcal{L}_\mathrm{b}(E)$ and $\mathcal{L}_\mathrm{r}(E)$, respectively.

The inclusion $\mathcal{L}_\mathrm{r}(E,F) \subseteq \mathcal{L}_\mathrm{b}(E,F)$ can be proper, as the next example of H. P. Lotz (oral communication) shows.

Example 1.16 (Lotz)**.** Consider the operator $T\colon C[-1,1] \to C[-1,1]$ defined for each $f \in C[-1,1]$ by

$$Tf(t) = f\big(\sin\tfrac{1}{t}\big) - f\big(\sin\big(t+\tfrac{1}{t}\big)\big)$$

if $0 < |t| \le 1$ and $Tf(0) = 0$. Note that the uniform continuity of f, coupled with the inequality $\big|\sin(\frac{1}{t}) - \sin(t+\frac{1}{t})\big| \le |t|$, shows that Tf is indeed continuous at zero, and so indeed $Tf \in C[-1,1]$ for each $f \in C[-1,1]$.

Next, observe that $T[-\mathbf{1},\mathbf{1}] \subseteq 2[-\mathbf{1},\mathbf{1}]$ holds, where $\mathbf{1}$ denotes the constant function one on $[-1,1]$. Since for every $f \in C[-1,1]$ there exists some $\lambda > 0$ with $|f| \le \lambda\mathbf{1}$, it easily follows that T is an order bounded operator.

However, we claim that T is not a regular operator. To see this, assume by way of contradiction that some positive operator $S\colon C[-1,1] \to C[-1,1]$ satisfies $T \le S$. We claim that for each $0 \le f \in C[-1,1]$ we have

$$[Sf](0) \ge f(t) \ \text{ for all } \ t \in [-1,1]\,. \qquad (\star)$$

To establish this, fix $0 < f \in C[-1,1]$, and let $0 < c < 2\pi$. Also, for each $n \in \mathbb{N}$ let $t_n = \frac{1}{c+2n\pi}$ and note that $t_n \to 0$. Next pick some $g_n \in C[-1,1]$ with $0 \le g_n \le f$ such that $g_n(\sin c) = f(\sin c)$ and $g_n\big(\sin(c+t_n)\big) = 0$. Therefore,

$$[Sf](t_n) \ge [Sg_n](t_n) \ge [Tg_n](t_n) = f(\sin c)$$

for all n, and so $[Sf](0) \ge f(\sin c)$ for all $0 < c < 2\pi$, i.e., $[Sf](0) \ge f(t)$ for all $t \in [-1,1]$.

Now for each n, let $P_n = \{a_0, a_1, \ldots, a_n\}$ be a partition of $[-1,1]$ into n subintervals. For each $1 \le i \le n$ pick some $f_i \in C[-1,1]$ such that $0 \le f_i \le \mathbf{1}$, f_i is zero outside the interval (a_{i-1}, a_i) and $f_i\big(\frac{a_{i-1}+a_i}{2}\big) = 1$. Taking into account that $\sum_{i=1}^n f_i \le \mathbf{1}$, it follows from $(\star)$ that

$$[S\mathbf{1}](0) \ge \Big[S\Big(\sum_{i=1}^n f_i\Big)\Big](0) = \sum_{i=1}^n [Sf_i](0) \ge n$$

holds for each n, which is impossible. Thus, T is not a regular operator. ■

Not every regular operator has a modulus. The next example of S. Kaplan [**94**] clarifies the situation.

Example 1.17 (Kaplan)**.** Let c be the Riesz space of all convergent (real) sequences, i.e., $c = \{(x_1, x_2, \ldots)\colon\ \lim x_n \text{ exists in } \mathbb{R}\}$. Consider the two positive operators $S, T\colon c \to c$ defined by

$$S(x_1, x_2, \ldots) = \big(x_2, x_1, x_4, x_3, x_6, x_5, \ldots\big)$$

and

$$T(x_1, x_2, \ldots) = \big(x_1, x_1, x_3, x_3, x_5, x_5, \ldots\big)\,.$$

We claim the modulus of the regular operator $R = S - T$ does not exist.

To this end, assume by way of contradiction that the modulus $|R|$ exists. Let $P_n\colon c \to c$ be the positive operator defined by

$$P_n(x_1, \ldots, x_{n-1}, x_n, x_{n+1}, \ldots) = (x_1, \ldots, x_{n-1}, 0, x_{n+1}, \ldots)\,.$$

Then $\pm R \le |R|P_{2n} \le |R|$ holds, and so $|R|P_{2n} = |R|$ holds for each n. This means that the image under $|R|$ of every element of c has its even components zero. On the other hand, if e_n is the sequence whose n^{th} component is one and every other zero and $e = (1, 1, 1, \ldots)$, then it follows from the inequalities

$$-R(e_n) \le |R|e_n \le |R|e$$

that the odd components of $|R|e$ are greater than or equal to one, and hence $|R|e \notin c$. Therefore, $|R|$ does not exist, as claimed. ■

A Riesz space is called **Dedekind complete** whenever every nonempty bounded above subset has a supremum (or, equivalently, whenever every nonempty bounded below subset has an infimum). A Riesz space E is Dedekind complete if and only if $0 \le x_\alpha \uparrow\le x$ implies the existence of $\sup\{x_\alpha\}$. Similarly, a Riesz space is said to be **Dedekind σ-complete** if every countable subset that is bounded above has a supremum (or, equivalently, whenever $0 \le x_n \uparrow\le x$ implies the existence of $\sup\{x_n\}$. The L_p-spaces are examples of Dedekind complete Riesz spaces.

When F is Dedekind complete, the ordered vector space $\mathcal{L}_{\mathrm{b}}(E, F)$ has the structure of a Riesz space. This important result was established first by F. Riesz [**166**] for the special case $F = \mathbb{R}$, and later L. V. Kantorovich [**90, 91**] extended it to the general setting.

Theorem 1.18 (F. Riesz–Kantorovich)**.** *If E and F are Riesz spaces with F Dedekind complete, then the ordered vector space $\mathcal{L}_{\mathrm{b}}(E, F)$ is a Dedekind complete Riesz space. Its lattice operations satisfy*

$$\begin{aligned} |T|(x) &= \sup\big\{|Ty|\colon\ |y| \le x\big\}\,, \\ [S \vee T](x) &= \sup\big\{S(y) + T(z)\colon\ y, z \in E^+ \text{ and } y + z = x\big\}\,, \text{ and} \\ [S \wedge T](x) &= \inf\big\{S(y) + T(z)\colon\ y, z \in E^+ \text{ and } y + z = x\big\} \end{aligned}$$

for all $S, T \in \mathcal{L}_{\mathrm{b}}(E, F)$ and $x \in E^+$.

In addition, $T_\alpha \downarrow 0$ *in* $\mathcal{L}_b(E,F)$ *if and only if* $T_\alpha(x) \downarrow 0$ *in* F *for each* $x \in E^+$.

Proof. Fix $T \in \mathcal{L}_b(E,F)$. Since T is order bounded,

$$\sup\{|Ty|: \ |y| \le x\} = \sup\{Ty: \ |y| \le x\} = \sup T[-x,x]$$

exists in F for each $x \in E^+$, and so by Theorem 1.14 the modulus of T exists, and moreover

$$|T|(x) = \sup\{Ty: \ |y| \le x\}.$$

From Theorem 1.7 we see that $\mathcal{L}_b(E,F)$ is a Riesz space.

Now let $S,T \in \mathcal{L}_b(E,F)$ and $x \in E^+$. By observing that $y,z \in E^+$ satisfy $y+z=x$ if and only if there exists some $|u| \le x$ with $y = \frac{1}{2}(x+u)$ and $z = \frac{1}{2}(x-u)$, it follows from Theorem 1.7 that

$$\begin{aligned}
[S \vee T](x) &= \tfrac{1}{2}\big(Sx + Tx + |S-T|x\big)\\
&= \tfrac{1}{2}\big(Sx + Tx + \sup\{(S-T)u: \ |u| \le x\}\big)\\
&= \tfrac{1}{2}\sup\{Sx + Su + Tx - Tu: \ |u| \le x\}\\
&= \sup\{S\big(\tfrac{1}{2}(x+u)\big) + T\big(\tfrac{1}{2}(x-u)\big): \ |u| \le x\}\\
&= \sup\{S(y) + T(z): \ y,z \in E^+ \text{ and } y+z=x\}.
\end{aligned}$$

The formula for $S \wedge T$ can be proven in a similar manner.

Finally, we establish that $\mathcal{L}_b(E,F)$ is Dedekind complete. To this end, assume that $0 \le T_\alpha \uparrow \le T$ holds in $\mathcal{L}_b(E,F)$. For each $x \in E^+$ let $S(x) = \sup\{T_\alpha(x)\}$ and note that $T_\alpha(x) \uparrow S(x)$. From $T_\alpha(x+y) = T_\alpha(x) + T_\alpha(y)$, it follows (by taking order limits) that the mapping $S\colon E^+ \to F^+$ is additive, and so S defines a positive operator from E to F. Clearly, $T_\alpha \uparrow S$ holds in $\mathcal{L}_b(E,F)$, proving that $\mathcal{L}_b(E,F)$ is a Dedekind complete Riesz space. ■

From the preceding discussion it follows that when E and F are Riesz spaces with F Dedekind complete, then each order bounded operator $T\colon E \to F$ satisfies

$$\begin{aligned}
T^+(x) &= \sup\{Ty: \ 0 \le y \le x\}, \text{ and}\\
T^-(x) &= \sup\{-Ty: \ 0 \le y \le x\}
\end{aligned}$$

for each $x \in E^+$. From $T = T^+ - T^-$, it follows that $\mathcal{L}_b(E,F)$ coincides with the vector subspace generated by the positive operators in $\mathcal{L}(E,F)$. In other words, when F is Dedekind complete we have $\mathcal{L}_r(E,F) = \mathcal{L}_b(E,F)$.

Recall that a subset D of a Riesz space is said to be **directed upward** (in symbols $D\uparrow$) whenever for each pair $x,y \in D$ there exists some $z \in D$ with $x \le z$ and $y \le z$. The symbol $D \uparrow x$ means that D is directed upward and $x = \sup D$ holds. The meanings of $D\downarrow$ and $D \downarrow x$ are analogous. Also, the symbol $D \leq x$ means that $y \leq x$ holds for all $y \in D$.

The existence of the supremum of an upward directed subset of $\mathcal{L}_b(E,F)$ is characterized as follows.

Theorem 1.19. *Let E and F be two Riesz spaces with F Dedekind complete, and let D be a nonempty subset of $\mathcal{L}_b(E,F)$ satisfying $D\uparrow$. Then $\sup D$ exists in $\mathcal{L}_b(E,F)$ if and only if the set $\{T(x)\colon\ T \in D\}$ is bounded above in F for each $x \in E^+$. In this case,*

$$[\sup D](x) = \sup\bigl\{T(x)\colon\ T \in D\bigr\}$$

holds for all $x \in E^+$.

Proof. the "only if" part is trivial. The "if" part needs proof. So, assume that $D\uparrow$ holds in $\mathcal{L}_b(E,F)$ and that the set $\{T(x)\colon\ T \in D\}$ is bounded above in F for each $x \in E^+$. It is easy to see that without loss of generality we can assume that $D \subseteq \mathcal{L}_b^+(E,F)$. Define $S\colon E^+ \to F^+$ by

$$S(x) = \sup\bigl\{T(x)\colon\ T \in D\bigr\}\,,$$

and we claim that S is additive. To see this, let $x, y \in E^+$. Since for each $T \in D$ we have $T(x+y) = T(x) + T(y) \leq S(x) + S(y)$, we see that $S(x+y) \leq S(x) + S(y)$ holds. On the other hand, if $T_1, T_2 \in D$, then pick $T_3 \in D$ satisfying $T_1 \leq T_3$ and $T_2 \leq T_3$, and note that

$$T_1(x) + T_2(y) \leq T_3(x) + T_3(y) = T_3(x+y) \leq S(x+y)$$

implies $S(x) + S(y) \leq S(x+y)$. Therefore, $S(x+y) = S(x) + S(y)$ holds, and so S is additive. By Theorem 1.10 the mapping S defines a positive operator from E to F, and a routine argument shows that $S = \sup D$ holds in $\mathcal{L}_b(E,F)$. ■

Our next objective is to describe the lattice operations of $\mathcal{L}_b(E,F)$ in terms of directed sets. To do this, we need a result from the theory of Riesz spaces known as the **Riesz Decomposition Property**; it is due to F. Riesz [**167**].

Theorem 1.20 (The Riesz Decomposition Property)**.** *Let $x_1, \ldots, x_n$ and $y_1, \ldots, y_m$ be positive vectors in a Riesz space. If*

$$\sum_{i=1}^{n} x_i = \sum_{j=1}^{m} y_j$$

holds, then there exists a finite subset $\{z_{ij}\colon\ i = 1, \ldots, n;\ j = 1, \ldots, m\}$ of positive vectors such that

$$x_i = \sum_{j=1}^{m} z_{ij}\,, \ \ \textit{for each}\ \ i = 1, \ldots, n\,,$$

and

$$y_j = \sum_{i=1}^{n} z_{ij}\,, \quad \textit{for each } j = 1, \ldots, m\,.$$

Proof. We shall use induction on m. For $m = 1$ the desired conclusion follows from Theorem 1.13. Thus, assume the result to be true for some m and all $n = 1, 2, \ldots$. Let

$$\sum_{i=1}^{n} x_i = \sum_{j=1}^{m+1} y_j\,,$$

where the vectors x_i and the y_j are all positive. Since $\sum_{j=1}^{m} y_j \le \sum_{i=1}^{n} x_i$ holds, it follows from Theorem 1.13 that there exist vectors $u_1, \ldots, u_n$ satisfying $0 \le u_i \le x_i$ for each $i = 1, \ldots, n$ and $\sum_{i=1}^{n} u_i = \sum_{j=1}^{m} y_j$. Therefore, from our induction hypothesis, there exists a set of positive vectors $\{z_{ij}\colon\ i = 1, \ldots, n;\ j = 1, \ldots, m\}$ such that:

$$u_i = \sum_{j=1}^{m} z_{ij} \ \text{ for } i = 1, \ldots, n \quad \text{and} \quad y_j = \sum_{i=1}^{n} z_{ij} \ \text{ for } j = 1, \ldots, m\,.$$

For each $i = 1, \ldots, n$ put $z_{i,m+1} = x_i - u_i \ge 0$ and note that the collection of positive vectors $\bigl\{z_{ij}\colon\ i = 1, \ldots, n;\ j = 1, \ldots, m+1\bigr\}$ satisfies

$$x_i = \sum_{j=1}^{m+1} z_{ij} \ \text{ for } i = 1, \ldots, n \quad \text{and} \quad y_j = \sum_{i=1}^{n} z_{ij} \ \text{ for } j = 1, \ldots, m+1\,.$$

Thus, the conclusion is valid for $m + 1$ and all $n = 1, 2, \ldots$, and the proof is finished. ■

We are now in a position to express the lattice operations of $\mathcal{L}_{\mathrm{b}}(E, F)$ in terms of directed sets.

Theorem 1.21. *If E and F are two Riesz spaces with F Dedekind complete, then for all $S, T \in \mathcal{L}_{\mathrm{b}}(E, F)$ and each $x \in E^+$ we have:*

$$(1) \quad \Bigl\{\sum_{i=1}^{n} S(x_i) \vee T(x_i)\colon\ x_i \in E^+ \ \textit{and} \ \sum_{i=1}^{n} x_i = x\Bigr\} \uparrow [S \vee T](x)\,.$$

$$(2) \quad \Bigl\{\sum_{i=1}^{n} S(x_i) \wedge T(x_i)\colon\ x_i \in E^+ \ \textit{and} \ \sum_{i=1}^{n} x_i = x\Bigr\} \downarrow [S \wedge T](x)\,.$$

$$(3) \quad \Bigl\{\sum_{i=1}^{n} |T(x_i)|\colon\ x_i \in E^+ \ \textit{and} \ \sum_{i=1}^{n} x_i = x\Bigr\} \uparrow |T|(x)\,.$$

Proof. (1) Consider the set

$$D = \Big\{ \sum_{i=1}^{n} S(x_i) \vee T(x_i) \colon \; x_i \in E^+ \text{ for each } i \text{ and } \sum_{i=1}^{n} x_i = x \Big\} .$$

Since $\sum_{i=1}^{n} x_i = x$ with each $x_i \in E^+$ implies

$$\sum_{i=1}^{n} S(x_i) \vee T(x_i) \le \sum_{i=1}^{n} \big[(S \vee T)x_i\big] \vee \big[(S \vee T)x_i\big] = [S \vee T](x) ,$$

we see that $D \le [S \vee T](x)$. On the other hand, if $D \le u$ holds, then for each $y, z \in E^+$ with $y + z = x$ we have

$$S(y) + T(z) \le S(y) \vee T(y) + S(z) \vee T(z) \le u ,$$

and consequently

$$[S \vee T](x) = \sup \big\{ S(y) + T(z) \colon \; y, z \in E^+ \text{ and } y + z = x \big\} \le u .$$

Thus, $\sup D = [S \vee T](x)$, and it remains to be shown that D is directed upward.

To this end, let $x = \sum_{i=1}^{n} x_i = \sum_{j=1}^{m} y_j$ with all the x_i and y_j in E^+. By Theorem 1.20 there exists a finite collection $\{z_{ij} \colon \; i = 1, \ldots, n; \; j = 1, \ldots, m\}$ of positive vectors such that

$$x_i = \sum_{j=1}^{m} z_{ij} , \;\text{ for each }\; i = 1, \ldots, n ,$$

and

$$y_j = \sum_{i=1}^{n} z_{ij} , \;\text{ for each }\; j = 1, \ldots, m .$$

In particular, we have $\sum_{i=1}^n \sum_{j=1}^m z_{ij} = x$. On the other hand, using the lattice identity $x \vee y = \frac{1}{2}(x + y + |x - y|)$, we see that

$$\begin{aligned}
&\sum_{i=1}^n S(x_i) \vee T(x_i) \\
&\qquad = \tfrac{1}{2} \sum_{i=1}^n \big[S(x_i) + T(x_i) + |S(x_i) - T(x_i)|\big] \\
&\qquad = \tfrac{1}{2} \sum_{i=1}^n \Big[\sum_{j=1}^m S(z_{ij}) + \sum_{j=1}^m T(z_{ij}) + \Big|\sum_{j=1}^m \big\{S(z_{ij}) - T(z_{ij})\big\}\Big|\Big] \\
&\qquad \le \tfrac{1}{2} \sum_{i=1}^n \Big[\sum_{j=1}^m \big\{S(z_{ij}) + T(z_{ij}) + |S(z_{ij}) - T(z_{ij})|\big\}\Big] \\
&\qquad = \sum_{i=1}^n \sum_{j=1}^m S(z_{ij}) \vee T(z_{ij})\,.
\end{aligned}$$

Similarly,

$$\sum_{j=1}^m S(y_j) \vee T(y_j) \le \sum_{i=1}^n \sum_{j=1}^m S(z_{ij}) \vee T(z_{ij})$$

holds, and so D is directed upward.

(2) Use (1) in conjunction with the identity $T \wedge S = -\big[(-S) \vee (-T)\big]$.

(3) Use (1) and the identity $|T| = T \vee (-T)$. ■

The next result presents an interesting local approximation property of positive operators.

Theorem 1.22. *Let $T\colon E \to F$ be a positive operator between two Riesz spaces with F Dedekind σ-complete. Then for each $x \in E^+$ there exists a positive operator $S\colon E \to F$ such that:*

(1) $0 \le S \le T$.

(2) $S(x) = T(x)$.

(3) $S(y) = 0$ *for all* $y \perp x$.

Proof. Let $x \in E^+$ be fixed and define $S\colon E^+ \to F^+$ by

$$S(y) = \sup\big\{T(y \wedge nx)\colon\ n = 1, 2, \dots\big\}\,.$$

(The supremum exists since F is Dedekind σ-complete and the sequence $\{T(y \wedge nx)\}$ is bounded above in F by Ty.) We claim that S is additive.

To see this, let $y, z \in E^+$. From $(y + z) \wedge nx \le y \wedge nx + z \wedge nx$ we get

$$T\big((y + z) \wedge nx\big) \le T(y \wedge nx) + T(z \wedge nx) \le S(y) + S(z)\,,$$

and so $S(y+z) \le S(y)+S(z)$. On the other hand, for each m and n we have $y \wedge nx + z \wedge mx \le (y+z) \wedge (n+m)x$, and thus

$$T(y \wedge nx) + T(z \wedge mx) \le T\big((y+z) \wedge (n+m)x\big) \le S(y+z)$$

holds for all n and m. This implies $S(y)+S(z) \le S(y+z)$, and hence $S(y+z) = S(y)+S(z)$, so that S is additive.

By Theorem 1.10 the mapping S extends uniquely to all of E as a positive operator. Now it is a routine matter to verify that the operator S satisfies the desired properties. ■

As an application of the preceding result let us derive some formulas that are in some sense the "dual" formulas to those stated after Theorem 1.18.

Theorem 1.23. *If $T\colon E \to F$ is a positive operator between two Riesz spaces with F Dedekind σ-complete, then for each $x \in E$ we have:*

$$\begin{aligned} T(x^+) &= \max\big\{S(x)\colon\ S \in \mathcal{L}(E,F) \ \textit{and}\ 0 \le S \le T\big\}\,. \\ T(x^-) &= \max\big\{-S(x)\colon\ S \in \mathcal{L}(E,F) \ \textit{and}\ 0 \le S \le T\big\}\,. \\ T(|x|) &= \max\big\{S(x)\colon\ S \in \mathcal{L}(E,F) \ \textit{and}\ -T \le S \le T\big\}\,. \end{aligned}$$

Proof. (1) Let $x \in E$ be fixed. By Theorem 1.22 there exists a positive operator $R\colon E \to F$ such that $0 \le R \le T$, $R(x^+) = T(x^+)$, and $R(x^-) = 0$. Therefore, $T(x^+) = R(x)$. On the other hand, if $S \in \mathcal{L}(E,F)$ satisfies $0 \le S \le T$, then we have $S(x) \le S(x^+) \le T(x^+)$, and the conclusion follows.

(2) Apply (1) to the identity $x^- = (-x)^+$.

(3) If the operator $S\colon E \to F$ satisfies $-T \le S \le T$, then

$$S(x) = S(x^+) - S(x^-) \le T(x^+) + T(x^-) = T(|x|)$$

holds. On the other hand, according to Theorem 1.22, there exist two positive operators $R_1, R_2\colon E \to F$ bounded by T such that:

(a) $R_1(x^+) = T(x^+)$ and $R_1(x^-) = 0$.
(b) $R_2(x^-) = T(x^-)$ and $R_2(x^+) = 0$.

Then the operator $S = R_1 - R_2$ satisfies $-T \le S \le T$ and $T(|x|) = S(x)$, and the desired conclusion follows. ■

Now let $\{E_i\colon\ i \in I\}$ be a family of Riesz spaces. Then it is not difficult to check that the **Cartesian product** ΠE_i, under the ordering $\{x_i\} \ge \{y_i\}$ whenever $x_i \ge y_i$ holds in E_i for each $i \in I$, is a Riesz space. Clearly, if $x = \{x_i\}$ and $y = \{y_i\}$ are vectors of ΠE_i, then

$$x \vee y = \{x_i \vee y_i\} \quad \text{and} \quad x \wedge y = \{x_i \wedge y_i\}\,.$$

The **direct sum** $\Sigma\oplus E_i$ (or more formally $\Sigma_{i\in I}\oplus E_i$) is the vector subspace of ΠE_i consisting of all vectors $x=\{x_i\}$ for which $x_i=0$ holds for all but a finite number of indices i. With the pointwise algebraic and lattice operations $\Sigma\oplus E_i$ is a Riesz subspace of ΠE_i (and hence a Riesz space in its own right). Note that if, in addition, each E_i is Dedekind complete, then ΠE_i and $\Sigma\oplus E_i$ are likewise both Dedekind complete Riesz spaces.

It is not difficult to see that every operator $T\colon \Sigma\oplus E_i\to\Sigma\oplus F_j$ between two direct sums of families of Riesz spaces can be represented by a matrix $T=[T_{ji}]$, where $T_{ji}\colon E_i\to F_j$ are operators defined appropriately. Sometimes it pays to know that the algebraic and lattice operations represented by matrices are the pointwise ones. The next result (whose easy proof is left for the reader) clarifies the situation.

Theorem 1.24. *Let $\{E_i\colon\ i\in I\}$ and $\{F_j\colon\ j\in J\}$ be two families of Riesz spaces with each F_j Dedekind complete. If $S=[S_{ji}]$ and $T=[T_{ji}]$ are order bounded operators from $\Sigma\oplus E_i$ to $\Sigma\oplus F_j$, then*

(1) $S+T=[S_{ji}+T_{ji}]$ *and* $\lambda S=[\lambda S_{ji}]$, *and*

(2) $S\vee T=[S_{ji}\vee T_{ji}]$ *and* $S\wedge T=[S_{ji}\wedge T_{ji}]$

hold in $\mathcal{L}_{\rm b}(\Sigma\oplus E_i,\Sigma\oplus F_j)$.

Exercises

1. Let E be an Archimedean Riesz space and let $A\subseteq\mathbb{R}$ be nonempty and bounded above. Show that for each $x\in E^+$ the supremum of the set $Ax:=\{\alpha x\colon\ \alpha\in A\}$ exists and $\sup(Ax)=(\sup A)x$.

2. Show that in a Riesz space $x\perp y$ implies
(a) $\alpha x\perp\beta y$ for all $\alpha,\beta\in\mathbb{R}$, and
(b) $|x+y|=|x|+|y|$.

Use the conclusion in (b) to establish that if in a Riesz space the nonzero vectors $x_1,\dots,x_n$ are pairwise disjoint, then $x_1,\dots,x_n$ are linearly independent. [*Hint*: If $|x|\wedge|y|=0$, then

$$\begin{aligned}|x+y| &\ge \big||x|-|y|\big| = |x|\vee|y|-|x|\wedge|y|\\ &= |x|\vee|y|+|x|\wedge|y| = |x|+|y|\ge|x+y|\,.]\end{aligned}$$

3. In this exercise we ask you to complete the missing details in Example 1.11. Let G be the **lexicographic plane**. (That is, we consider $G=\mathbb{R}^2$ as a Riesz space under the **lexicographic ordering** $(x_1,x_2)\ge(y_1,y_2)$ whenever either $x_1>y_1$ or else $x_1=y_1$ and $x_2\ge y_2$.) Also, let $\phi\colon\mathbb{R}\to\mathbb{R}$ be an additive function that is not linear (i.e., not of the form $\phi(x)=cx$).

Show that the mapping $T\colon\mathbb{R}^+\to G^+$ defined by

$$T(x)=(x,\phi(x))$$

is additive but that it cannot be extended to a positive operator from $\mathbb{R}$ to G. Why does this not contradict Theorem 1.10?

4. Let E and F be two Riesz spaces with F Dedekind complete, and let $\mathcal{A}$ be a nonempty subset of $\mathcal{L}_b(E,F)$. Show that $\sup \mathcal{A}$ exists in $\mathcal{L}_b(E,F)$ if and only if for each $x \in E^+$ the set $\bigl\{(\bigvee_{i=1}^n T_i)x\colon\ T_1,\dots,T_n \in \mathcal{A}\bigr\}$ is bounded above in F.

5. Consider the positive operators $S, T\colon L_1[0,1] \to L_1[0,1]$ defined by

$$S(f) = f \quad \text{and} \quad T(f) = \Bigl[\int_0^1 f(x)\,dx\Bigr] \cdot \mathbf{1}\,,$$

where $\mathbf{1}$ is the constant function one. Show that $S \wedge T = 0$.

6. Let E and F be two Riesz spaces with F Dedekind complete. Then for each $T \in \mathcal{L}_b(E,F)$ and each $x \in E^+$ show that:

$$\begin{aligned} T^+(x) &= \sup\bigl\{(Ty)^+\colon\ 0 \le y \le x\bigr\}\,. \\ T^-(x) &= \sup\bigl\{(Ty)^-\colon\ 0 \le y \le x\bigr\}\,. \end{aligned}$$

7. Let $T\colon E \to F$ be a positive operator between two Riesz spaces with F Dedekind complete. If $x, y \in E$, then show that:
(a) $T(x \vee y) = \max\bigl\{Rx + Sy\colon\ R, S \in \mathcal{L}_b^+(E,F) \text{ and } R + S = T\bigr\}$.
(b) $T(x \wedge y) = \min\bigl\{Rx + Sy\colon\ R, S \in \mathcal{L}_b^+(E,F) \text{ and } R + S = T\bigr\}$.

8. If $0 < p < 1$, then show that the only positive operator from $L_p[0,1]$ to $C[0,1]$ is the zero operator.

9. Consider the continuous function $g\colon [0,1] \to [0,1]$ defined by $g(x) = x$ if $0 \le x \le \frac{1}{2}$ and $g(x) = \frac{1}{2}$ if $\frac{1}{2} < x \le 1$. Now define the operator $T\colon C[0,1] \to C[0,1]$ by $[Tf](x) = f\bigl(g(x)\bigr) - f\bigl(\frac{1}{2}\bigr)$.
Show that T is a regular operator whose modulus does not exist.

10. Let $T\colon C[0,1] \to C[0,1]$ be the regular operator defined by

$$[Tf](x) = f(\sin x) - f(\cos x)\,.$$

Show that T^+ and T^- both exist and that

$$[T^+f](x) = f(\sin x) \quad \text{and} \quad [T^-f](x) = f(\cos x)\,.$$

11. For each $n \ge 2$ fix a continuous function $e_n\colon [0,1] \to [0,1]$ such that:
(a) $0 \le e_n \le \mathbf{1}$.
(b) $e_n = 0$ outside $\bigl[\frac{1}{2} + \frac{1}{n+1}, \frac{1}{2} + \frac{1}{n}\bigr]$.
(c) $e_n(x) = 1$ for some $x \in \bigl[\frac{1}{2} + \frac{1}{n+1}, \frac{1}{2} + \frac{1}{n}\bigr]$.
Now define the operator $T\colon C[0,1] \to C[0,1]$ by

$$Tf = \sum_{n=2}^{\infty} \Bigl[\int_0^1 f(x)\sin(n\pi x)\,dx\Bigr]\,e_n\,.$$

Show that T is indeed an operator from $C[0,1]$ to $C[0,1]$, that T is a regular operator, and that its modulus does not exist.

12. Prove Theorem 1.24.

1.2. Extensions of Positive Operators

In this section we shall gather some basic extension theorems for operators, and, in particular, for positive operators.

A function $p\colon G \to F$, where G is a (real) vector space and F is an ordered vector space, is called **sublinear** whenever

(a) $p(x+y) \le p(x) + p(y)$ for all $x, y \in G$, and
(b) $p(\lambda x) = \lambda p(x)$ for all $x \in G$ and all $\lambda \ge 0$.

The next result is the most general version of the classical Hahn–Banach extension theorem. This theorem plays a fundamental role in modern analysis and without any doubt it will be of great importance to us here. It is due to H. Hahn [**74**] and S. Banach [**30**].

Theorem 1.25 (Hahn–Banach)**.** *Let G be a (real) vector space, F a Dedekind complete Riesz space, and let $p\colon G \to F$ be a sublinear function. If H is a vector subspace of G and $S\colon H \to F$ is an operator satisfying $S(x) \le p(x)$ for all $x \in H$, then there exists some operator $T\colon G \to F$ such that:*

(1) *$T = S$ on H, i.e., T is a linear extension of S to all of G.*
(2) *$T(x) \le p(x)$ holds for all $x \in G$.*

Proof. The critical step is to show that S has a linear extension satisfying (2) on an arbitrary vector subspace generated by H and one extra vector. If this is done, then an application of Zorn's lemma guarantees the existence of an extension of S to all of G with the desired properties.

To this end, let $x \notin H$, and let $V = \{y + \lambda x\colon\ y \in H \text{ and } \lambda \in \mathbb{R}\}$. If $T\colon V \to F$ is a linear extension of S, then

$$T(y + \lambda x) = S(y) + \lambda T(x)$$

must hold true for all $y \in H$ and all $\lambda \in \mathbb{R}$. Put $z = T(x)$. To complete the proof, we must establish the existence of some $z \in F$ such that

$$S(y) + \lambda z \le p(y + \lambda x) \qquad (\star)$$

holds for all $y \in H$ and $\lambda \in \mathbb{R}$. For $\lambda > 0$, $(\star)$ is equivalent to

$$S(y) + z \le p(y + x)$$

for all $y \in H$, while for $\lambda < 0$ the inequality $(\star)$ is equivalent to

$$S(y) - z \le p(y - x)$$

for all $y \in H$. The last two inequalities certainly will be satisfied by a choice of z for which

$$S(y) - p(y - x) \le z \le p(u + x) - S(u) \qquad (\star\star)$$

holds for all $y, u \in H$.

To see that there exists some $z \in F$ satisfying $(\star\star)$, start by observing that for each $y, u \in H$ we have

$$\begin{aligned} S(y) + S(u) &= S(y+u) \le p(y+u) = p\big(y - x + (u+x)\big) \\ &\le p(y-x) + p(u+x)\,, \end{aligned}$$

and so

$$S(y) - p(y-x) \le p(u+x) - S(u)$$

holds for all $y, u \in H$. This inequality in conjunction with the Dedekind completeness of F guarantees that both suprema

$$s = \sup\big\{S(y) - p(y-x)\colon\ y \in H\big\} \quad \text{and} \quad t = \inf\big\{p(u+x) - S(u)\colon\ u \in H\big\}$$

exist in F, and satisfy $s \le t$. Now any $z \in F$ satisfying $s \le z \le t$ (for instance $z = s$) satisfies $(\star\star)$, and hence $(\star)$. This complete the proof of the theorem. ■

Recall that a vector subspace G of a Riesz space E is said to be a **Riesz subspace** (or a **vector sublattice**) whenever G is closed under the lattice operations of E, i.e., whenever for each pair $x, y \in G$ the vector $x \vee y$ (taken in E) belongs to G.

As a first application of the Hahn–Banach extension theorem we present the following useful extension property of positive operators.

Theorem 1.26. *Let $T\colon E \to F$ be a positive operator between two Riesz spaces with F Dedekind complete. Assume also that G is a Riesz subspace of E and that $S\colon G \to F$ is an operator satisfying $0 \le Sx \le Tx$ for all $x \in G^+$. Then S can be extended to a positive operator from E to F such that $0 \le S \le T$ holds in $\mathcal{L}(E,F)$.*

Proof. Define $p\colon E \to F$ by $p(x) = T(x^+)$, and note that p is sublinear and satisfies $S(x) \le p(x)$ for all $x \in G$. By Theorem 1.25 there exists a linear extension of S to all of E (which we denote by S again) satisfying $S(x) \le p(x)$ for all $x \in E$. Now if $x \in E^+$, then

$$-S(x) = S(-x) \le p(-x) = T\big((-x)^+\big) = T(0) = 0\,,$$

and so $0 \le S(x) \le p(x) = T(x)$ holds, as desired. ■

The rest of the section is devoted to extension properties of positive operators. The first result of this kind informs us that a positive operator whose domain is a Riesz subspace extends to a positive operator if and only if it is dominated by a monotone sublinear mapping. As usual, a mapping $f\colon E \to F$ between two ordered vector spaces is called **monotone** whenever $x \le y$ in E implies $f(x) \le f(y)$ in F.

Theorem 1.27. *Let E and F be Riesz spaces with F Dedekind complete. If G is a Riesz subspace of E and $T\colon G \to F$ is a positive operator, then the following statements are equivalent.*

(1) *T extends to a positive operator from E to F.*

(2) *T extends to an order bounded operator from E to F.*

(3) *There exists a monotone sublinear mapping $p\colon E \to F$ satisfying $T(x) \le p(x)$ for all $x \in G$.*

Proof. (1) $\Longrightarrow$ (2) Obvious.

(2) $\Longrightarrow$ (3) Let $S \in \mathcal{L}_{\rm b}(E,F)$ satisfy $S(x) = T(x)$ for all $x \in G$. Then the mapping $p\colon E \to F$ defined by $p(x) = |S|(x^+)$ is monotone, sublinear and satisfies

$$T(x) \le T(x^+) = S(x^+) \le |S|(x^+) = p(x)$$

for all $x \in G$.

(3) $\Longrightarrow$ (1) Let $p\colon E \to F$ be a monotone sublinear mapping satisfying $T(x) \le p(x)$ for all $x \in G$. Then the formula $q(x) = p(x^+)$ defines a sublinear mapping from E to F such that

$$T(x) \le T(x^+) \le p(x^+) = q(x)$$

holds for all $x \in G$. Thus, by the Hahn–Banach Extension Theorem 1.25 there exists an extension $R \in \mathcal{L}(E,F)$ of T satisfying $R(x) \le q(x)$ for all $x \in E$. In particular, if $x \in E^+$, then the relation

$$-R(x) = R(-x) \le q(-x) = p\big((-x)^+\big) = p(0) = 0$$

implies $R(x) \ge 0$. That is, R is a positive linear extension of T to all of E, and the proof is finished. ■

A subset A of a Riesz space is called **solid** whenever $|x| \le |y|$ and $y \in A$ imply $x \in A$. A solid vector subspace of a Riesz space is referred to as an **ideal**. From the lattice identity $x \vee y = \frac{1}{2}(x + y + |x - y|)$, it follows immediately that every ideal is a Riesz subspace.

The next result deals with restrictions of positive operators to ideals.

Theorem 1.28. *If $T\colon E \to F$ is a positive operator between two Riesz spaces with F Dedekind complete, then for every ideal A of E the formula*

$$T_A(x) = \sup\big\{T(y)\colon\ y \in A \ \text{and}\ 0 \le y \le x\big\}\,,\ \ x \in E^+\,,$$

defines a positive operator from E to F. Moreover, we have:

(a) *$0 \le T_A \le T$.*

(b) *$T_A = T$ on A and $T_A = 0$ on $A^{\rm d}$.*

(c) *If B is another ideal with $A \subseteq B$, then $T_A \le T_B$ holds.*

Proof. Note first that

$$T_A(x) = \sup\bigl\{T(x \wedge y)\colon\ y \in A^+\bigr\}$$

holds for all $x \in E^+$. According to Theorem 1.10 it suffices to show that T_A is additive on E^+.

To this end, let $x, y \in E^+$. If $z \in A^+$, then the inequality

$$(x+y) \wedge z \le x \wedge z + y \wedge z$$

implies that $T\bigl((x+y) \wedge z\bigr) \le T(x \wedge z) + T(y \wedge z) \le T_A(x) + T_A(y)$, and hence

$$T_A(x+y) \le T_A(x) + T_A(y)\,.$$

On the other hand, the inequality $x \wedge u + y \wedge v \le (x+y) \wedge (u+v)$ implies

$$T_A(x) + T_A(y) \le T_A(x+y)\,.$$

Therefore, $T_A(x+y) = T_A(x) + T_A(y)$ holds, so that T_A is additive on E^+.

Properties (1)–(3) are now easy consequences of the formula defining the operator T_A. ■

As mentioned before, if G is a vector subspace of an ordered vector space and F is another ordered vector space, then it is standard to call an operator $T\colon G \to F$ **positive** whenever $0 \le x \in G$ implies $0 \le T(x) \in F$.

Now consider a positive operator $T\colon G \to F$, where G is a vector subspace of an ordered vector space E and F is a Dedekind complete Riesz space. We shall denote by $\mathcal{E}(T)$ the collection of all positive extensions of T to all of E. That is,

$$\mathcal{E}(T) := \bigl\{S \in \mathcal{L}(E,F)\colon\ S \ge 0 \ \text{ and } \ S = T \ \text{ on } \ G\bigr\}\,.$$

The set $\mathcal{E}(T)$ is always a convex subset of $\mathcal{L}(E,F)$, i.e. $\lambda S + (1-\lambda)R \in \mathcal{E}(T)$ holds for all $S, R \in \mathcal{E}(T)$ and all $0 \le \lambda \le 1$. The set $\mathcal{E}(T)$ might happen to be empty. The next example presents such a case.

Example 1.29. Let $E = L_p[0,1]$ with $0 < p < 1$ and let $G = L_1[0,1]$. Clearly, $G \subseteq E$ and G is an ideal of E. (Here $f \ge g$ means that $f(x) \ge g(x)$ holds for almost all x with respect to the Lebesgue measure.)

Now consider the operator $T\colon G \to \mathbb{R}$ defined by

$$T(f) = \int_0^1 f(x)\,dx\,.$$

We claim that T does not have a positive linear extension to all of E. To see this, assume by way of contradiction that T is extendable to a positive operator from E to $\mathbb{R}$. In particular, this implies that if $f \in E$ is defined by $f(x) = \frac{1}{x}$, then the set of real numbers

$$D = \bigl\{T(g)\colon\ g \in G \ \text{ and } \ 0 \le g \le f\bigr\}$$

is bounded. on the other hand, if $g_n = f\chi_{(\frac{1}{n},1)}$, then $T(g_n) = \ln n \in D$ holds for each n. Therefore, D must be unbounded, a contradiction. Consequently, in this case we have $\mathcal{E}(T) = \emptyset$. ■

A positive operator $T\colon G \to F$ (where G is a vector subspace of an ordered vector space E) is said to have a **smallest extension** whenever there exists some $S \in \mathcal{E}(T)$ satisfying $S \le R$ for all $R \in \mathcal{E}(T)$, in which case S is called the **smallest extension** of T. In other words, T has a smallest extension if and only if $\min \mathcal{E}(T)$ exists in $\mathcal{L}(E, F)$.

It turns out that an extendable positive operator whose domain is an ideal always has a smallest extension.

Theorem 1.30. *Let E and F be two Riesz spaces with F Dedekind complete, let A be an ideal of E, and let $T\colon A \to F$ be a positive operator. If $\mathcal{E}(T) \ne \emptyset$, then T has a smallest extension. Moreover, if in this case $S = \min \mathcal{E}(T)$, then*

$$S(x) = \sup\{Ty\colon\ y \in A \ \text{ and } \ 0 \le y \le x\}$$

holds for all $x \in E^+$.

Proof. Since T has (at least) one positive extension, the formula

$$T_A(x) = \sup\{T(y)\colon\ y \in A \ \text{ and } \ 0 \le y \le x\}\,, \ \ x \in E^+\,,$$

defines a positive operator from E to F satisfying $T_A = T$ on A, and so $T_A \in \mathcal{E}(T)$. (See the proof of Theorem 1.28.)

Now if $S \in \mathcal{E}(T)$, then $S = T$ holds on A, and hence $T_A = S_A \le S$. Therefore, $T_A = \min \mathcal{E}(T)$ holds, as desired. ■

For a positive operator $T\colon E \to F$ with F Dedekind complete, Theorem 1.30 implies that for each ideal A of E the positive operator T_A is the smallest extension of the restriction of T to A.

Among the important points of a convex set are its extreme points. Recall that a vector e of a convex set C is said to be an **extreme point** of C whenever the expression $e = \lambda x + (1 - \lambda)y$ with $x, y \in C$ and $0 < \lambda < 1$ implies $x = y = e$.

The extreme points of the convex set $\mathcal{E}(T)$ have been characterized by Z. Lipecki, D. Plachky, and W. Thomsen [**116**] as follows.

Theorem 1.31 (Lipecki–Plachky–Thomsen)**.** *Let E and F be two Riesz spaces with F a Dedekind complete. If G is a vector subspace of E and $T\colon G \to F$ is a positive operator, then for an operator $S \in \mathcal{E}(T)$ the following statements are equivalent:*

(1) *S is an extreme point of $\mathcal{E}(T)$.*

(2) *For each $x \in E$ we have $\inf\{S(|x - y|)\colon\ y \in G\} = 0$.*

Proof. (1) $\Longrightarrow$ (2) Let S be an extreme point of $\mathcal{E}(T)$. Define the mapping $p\colon E \to F$ for each $x \in E$ by

$$p(x) = \inf\bigl\{S(|x-y|)\colon\ y \in G\bigr\}.$$

Clearly, p is a sublinear mapping that satisfies $0 \le p(x) = p(-x) \le S|x|$ for all $x \in E$, and $p(y) = 0$ for each $y \in G$.

Next, we claim that $p(x) = 0$ holds for all $x \in E$. To see this, assume by way of contradiction that $p(x) > 0$ holds for some $x \in E$. Define the operator $R\colon \{\lambda x\colon\ \lambda \in \mathbb{R}\} \to F$ by $R(\lambda x) = \lambda p(x)$, and note that $R(\lambda x) \le p(\lambda x)$ holds. So, by the Hahn–Banach Extension Theorem 1.25, the operator R has a linear extension to all of E (which we shall denote by R again) such that $R(z) \le p(z)$ holds for all $z \in E$; clearly, $R \ne 0$. It is easy to see that $|R(z)| \le p(z)$ for all $z \in E$, and so $R(y) = 0$ for all $y \in G$. Since for each $z \ge 0$ we have $R(z) \le p(z) \le S(|z|) = S(z)$ and

$$-R(z) = R(-z) \le p(-z) \le S(|-z|) = S(z),$$

it easily follows that $S - R \ge 0$ and $S + R \ge 0$ both hold. Thus, $S - R$ and $S + R$ both belong to $\mathcal{E}(T)$. Now the identity

$$S = \tfrac{1}{2}(S - R) + \tfrac{1}{2}(S + R),$$

in conjunction with $S - R \ne S$ and $S + R \ne S$, shows that S is not an extreme point of $\mathcal{E}(T)$, a contradiction. Thus, $p(x) = 0$ holds for each $x \in E$, as desired.

(2) $\Longrightarrow$ (1) Let S satisfy (2) and assume that $S = \lambda Q + (1-\lambda)R$ with $Q, R \in \mathcal{E}(T)$ and $0 < \lambda < 1$. Then for each $x, y \in E$ we have

$$\bigl|Q(x) - Q(y)\bigr| \le Q|x-y| = \bigl(\tfrac{1}{\lambda}S - \tfrac{1-\lambda}{\lambda}R\bigr)|x-y| \le \tfrac{1}{\lambda}S|x-y|.$$

In particular, if $x \in E$ and $y \in G$, then from $S(y) = Q(y) = T(y)$ it follows that

$$\bigl|S(x) - Q(x)\bigr| \le \bigl|S(x) - S(y)\bigr| + \bigl|Q(y) - Q(x)\bigr| \le \bigl(1 + \tfrac{1}{\lambda}\bigr)S|x-y|.$$

Taking into account our hypothesis, the last inequality yields $S(x) = Q(x)$ for each $x \in E$, and this shows that S is an extreme point of $\mathcal{E}(T)$. ■

Let us say that a vector subspace G of an ordered vector space E is **majorizing** E whenever for each $x \in E$ there exists some $y \in G$ with $x \le y$ (or, equivalently, if for each $x \in E$ there exists some $y \in G$ with $y \le x$).

It is important to know that every positive operator whose domain is a majorizing vector subspace and whose values are in a Dedekind complete Riesz space always has a positive extension. This is a classical result due to L. V. Kantorovich [**90**].

Theorem 1.32 (Kantorovich). *Let E and F be two ordered vector spaces with F a Dedekind complete Riesz space. If G is a majorizing vector subspace of E and $T\colon G \to F$ is a positive operator, then T has a positive linear extension to all of E.*

Proof. Fix $x \in E$ and let $y \in G$ satisfy $x \le y$. Since G is majorizing there exists a vector $u \in G$ with $u \le x$. Hence, $u \le y$ and the positivity of T implies $T(u) \le T(y)$ for all $y \in G$ with $x \le y$. In particular, it follows that $\inf\{T(y)\colon\ y \in G \text{ and } x \le y\}$ exists in F for each $x \in E$. Thus, a mapping $p\colon E \to F$ can be defined via the formula

$$p(x) = \inf\{T(y)\colon\ y \in G \text{ and } x \le y\}\,.$$

Clearly, $T(x) = p(x)$ holds for each $x \in G$ and an easy argument shows that p is also sublinear.

Now, by the Hahn–Banach Extension Theorem 1.25, the operator T has a linear extension S to all of E satisfying $S(z) \le p(z)$ for each $z \in E$. If $z \in E^+$, then $-z \le 0$, and so from

$$-S(z) = S(-z) \le p(-z) \le T(0) = 0\,,$$

we see that $S(z) \ge 0$. This shows that S is a positive linear extension of T to all of E. ■

It is a remarkable fact that in case the domain of a positive operator T is a majorizing vector subspace, then the convex set $\mathcal{E}(T)$ is not merely nonempty but it also has extreme points. This result is due to Z. Lipecki [**115**].

Theorem 1.33 (Lipecki). *Let E and F be two Riesz spaces with F Dedekind complete. If G is a majorizing vector subspace of E and $T\colon G \to F$ is a positive operator, then the nonempty convex set $\mathcal{E}(T)$ has an extreme point.*

Proof. According to Theorem 1.31 we must establish the existence of some $S \in \mathcal{E}(T)$ satisfying

$$\inf\{S(|x-y|)\colon\ y \in G\} = 0$$

for all $x \in E$.

Start by considering pairs (H, S) where H is a vector subspace majorizing E and $S\colon H \to F$ is a positive operator. For every such pair (H, S) define $p_{H,S}\colon E \to F$ by

$$p_{H,S}(x) = \inf\{S(y)\colon\ y \in H \text{ and } x \le y\}\,.$$

It should be clear that $p_{H,S}$ is a sublinear mapping satisfying $p_{H,S}(y) = S(y)$ for every $y \in H$. In addition, if (H_1, S_1) and (H_2, S_2) satisfy $H_1 \subseteq H_2$ and $S_2 = S_1$ on H_1, then $p_{H_2,S_2}(x) \le p_{H_1,S_1}(x)$ holds for all $x \in E$.

Now let $\mathcal{C}$ be the collection of all pairs (H, S) such that:

(1) H is a vector subspace of E with $G \subseteq H$ (and so H majorizes E).

(2) $S\colon H \to F$ is a positive operator with $S = T$ on G.

(3) $\inf\{p_{H,S}(|x-y|)\colon\ y \in G\} = 0$ holds in F for all $x \in H$.

In view of $(G,T) \in \mathcal{C}$, the set $\mathcal{C}$ is nonempty. Moreover, if we define a binary relation $\geq$ on $\mathcal{C}$ by letting $(H_2,S_2) \geq (H_1,S_1)$ whenever $H_2 \supseteq H_1$ and $S_2 = S_1$ on H_1, then $\geq$ is an order relation on $\mathcal{C}$. By a routine argument we can verify that every chain of $\mathcal{C}$ has an upper bound in $\mathcal{C}$. Therefore, by Zorn's lemma the collection $\mathcal{C}$ has a maximal element, say (M,R). The rest of the proof is devoted to proving that $M = E$. (If this is done, then $R = p_{M,R}$ must be the case, which by Theorem 1.31 shows that R must be an extreme point of $\mathcal{E}(T)$.)

To this end, assume by way of contradiction that there exists some vector x that does not belong to M. Consider $H = \{u + \lambda x\colon\ u \in M \text{ and } \lambda \in \mathbb{R}\}$, and then define $S\colon H \to F$ by $S(u+\lambda x) = R(u) + \lambda p_{M,R}(x)$. Clearly, M is a proper subspace of H, $S = R$ holds on M, and $S\colon H \to F$ is a positive operator. (For the positivity of S let $u + \lambda x \geq 0$ with $u \in M$. For $\lambda > 0$ the inequality $x \leq -\frac{u}{\lambda}$ implies $p_{M,R}(x) \geq -R(\frac{u}{\lambda})$, and consequently $S(u+\lambda x) = R(u) + \lambda p_{M,R}(x) \geq 0$. The case $\lambda < 0$ is similar, while the case $\lambda = 0$ is trivial.) Finally, we verify that (H,S) satisfies (3). First, observe that by the sublinearity of $p_{H,S}$ the set

$$V = \{y \in E \mid\ \inf\{p_{H,S}(|y-z|)\colon\ z \in M\} = 0\}$$

is a vector subspace of E satisfying $M \subseteq V$. Also, from

$$\begin{aligned} 0 &\leq \inf\{p_{H,S}(|x-z|)\colon\ z \in M\} \\ &\leq \inf\{p_{H,S}(z-x)\colon\ z \in M \text{ and } x \leq z\} \\ &= \inf\{R(z) - p_{M,R}(x)\colon\ z \in M \text{ and } x \leq z\} \\ &= \inf\{R(z)\colon\ z \in M \text{ and } x \leq z\} - p_{M,R}(x) = 0\,, \end{aligned}$$

we see that $x \in V$, and hence $H \subseteq V$. Now for arbitrary $u \in H$, $z \in M$, and $v \in G$ we have

$$\begin{aligned} p_{H,S}(|u-v|) &\leq p_{H,S}(|u-z|) + p_{H,S}(|v-z|) \\ &\leq p_{H,S}(|u-z|) + p_{M,R}(|v-z|)\,, \end{aligned}$$

and so from $(M,R) \in \mathcal{C}$ and $u \in H \subseteq V$, it follows that

$$\inf\{p_{H,S}(|u-v|)\colon\ v \in G\} = 0$$

holds for all $u \in H$.

Thus, $(H,S) \in \mathcal{C}$. However, $(H,S) \geq (M,R)$ and $(H,S) \neq (M,R)$ contradict the maximality of (M,R). Therefore, $M = E$ must be true, as required. ∎

Exercises

1. Let E and F be two Riesz spaces with F Dedekind complete, and let A be an ideal of E. For each $T \in \mathcal{L}_b(E,F)$ let $\mathcal{R}(T)$ denote the restriction of T to A. Show that the positive operator $\mathcal{R}\colon \mathcal{L}_b(E,F) \to \mathcal{L}_b(A,F)$ satisfies

$$\mathcal{R}(S \vee T) = \mathcal{R}(S) \vee \mathcal{R}(T) \quad \text{and} \quad \mathcal{R}(S \wedge T) = \mathcal{R}(S) \wedge \mathcal{R}(T)$$

for all $S, T \in \mathcal{L}_b(E,F)$.[1]

2. For two arbitrary solid sets A and B of a Riesz space show that:
(a) $A + B$ is a solid set.
(b) If $0 \le c \in A + B$ holds, then there exist $0 \le a \in A$ and $0 \le b \in B$ with $c = a + b$.

3. Let $T\colon E \to F$ be a positive operator between two Riesz spaces with F Dedekind complete. If two ideals A and B of E satisfy $A \perp B$, then show that:
(a) $T_A \wedge T_B = 0$.
(b) The ideal $A + B$ satisfies $T_{A+B} = T_A + T_B = T_A \vee T_B$.

4. As usual, ℓ_∞ denotes the Riesz space of all bounded real sequence, and c the Riesz subspace of ℓ_∞ consisting of all convergent sequences. If $\phi\colon c \to \mathbb{R}$ is the positive operator defined by

$$\phi(x_1, x_2, \ldots) = \lim_{n\to\infty} x_n\,,$$

then show that ϕ has a positive linear extension to all of ℓ_∞.

1.3. Order Projections

In this section we shall study a special class of positive operators known as order (or band) projections. Before starting our discussion, let us review a few properties of order dense Riesz subspaces. Recall that a Riesz subspace G of a Riesz space E is said to be **order dense** in E whenever for each $0 < x \in E$ (i.e., $0 \le x$ and $x \ne 0$) there exists some $y \in G$ with $0 < y \le x$.

The following characterization of order dense Riesz subspaces in Archimedean Riesz spaces will be used freely in this book.

Theorem 1.34. *A Riesz subspace G of an Archimedean Riesz space E is order dense in E if and only if for each $x \in E^+$ we have*

$$\{y \in G\colon\ 0 \le y \le x\} \uparrow x\,.$$

Proof. If $\sup\{y \in G\colon\ 0 \le y \le x\} = x$ holds in E for each $x \in E^+$, then G is clearly order dense in E. For the converse, assume that G is order dense

[1] An operator between spaces of operators is referred to as a **transformer**. So, the operator $\mathcal{R}$ is an example of a transformer.

in E, and let $x \in E^+$. Assume by way of contradiction that some $z \in E$ satisfies $z < x$ and $y \leq z$ for each $y \in G$ with $0 \leq y \leq x$. Then, by the order denseness of G in E, there exists some $u \in G$ with $0 < u \leq x - z$. From $0 \leq u \leq x$ we see that $u \leq z$, and therefore $0 < 2u = u + u \leq (x-z)+z = x$. By induction, $0 < nu \leq x$ holds for each n, contradicting the Archimedean property of E. Thus, $\{y \in G\colon\ 0 \leq y \leq x\} \uparrow x$ holds in E, and the proof is finished. ■

Consider an order dense Riesz subspace G of a Riesz space E. It is useful to know that the embedding of G into E preserves arbitrary suprema and infima. The result (whose straightforward proof is left for the reader) is stated next.

Theorem 1.35. *Let G be either an ideal or an order dense Riesz subspace of a Riesz space E, and let $D \subseteq G^+$ satisfy $D\downarrow$. Then $D \downarrow 0$ holds in G if and only if $D \downarrow 0$ holds in E.*

Recall that a subset A of a Riesz space is called *solid* whenever $|x| \leq |y|$ and $y \in A$ imply $x \in A$. A solid vector subspace is called an *ideal*. From Theorem 1.13 it readily follows that if A and B are solid subsets of a Riesz space, then their **algebraic sum**

$$A + B := \big\{a + b\colon\ a \in A \text{ and } b \in B\big\}$$

is likewise a solid set. In particular, the algebraic sum of two ideals also is an ideal.

The next theorem describes the basic properties of order dense ideals. Keep in mind that the disjoint complement of an arbitrary nonempty set of a Riesz space is always an ideal.

Theorem 1.36. *For an ideal A of a Riesz space E we have the following.*

(1) *The ideal A is order dense in E if and only if $A^{\mathrm{d}} = \{0\}$.*

(2) *The ideal $A \oplus A^{\mathrm{d}}$ is order dense in E.*

(3) *The ideal A is order dense in A^{dd}.*

Proof. (1) Let A be order dense in E and let $x \in A^{\mathrm{d}}$. If $x \neq 0$ holds, then there exists some $y \in A$ with $0 < y \leq |x|$. This implies $y \in A \cap A^{\mathrm{d}} = \{0\}$, a contradiction. Thus, $A^{\mathrm{d}} = \{0\}$ holds.

For the converse, assume that $A^{\mathrm{d}} = \{0\}$ holds and let $0 < x \in E$. If $y \wedge x = 0$ holds for all $y \in A^+$, then $x \in A^{\mathrm{d}} = \{0\}$ also must be the case. Thus, $y \wedge x > 0$ must be true for some $y \in A^+$. But then $y \wedge x \in A$ and $0 < y \wedge x \leq x$ show that A is order dense in E.

(2) If $x \perp A \oplus A^{\mathrm{d}}$, then $x \perp A$ and $x \perp A^{\mathrm{d}}$ both hold. Therefore, $x \in A^{\mathrm{d}} \cap A^{\mathrm{dd}} = \{0\}$. This shows that $(A \oplus A^{\mathrm{d}})^{\mathrm{d}} = \{0\}$. By part (1) the ideal $A \oplus A^{\mathrm{d}}$ is order dense in E.

(3) This follows immediately from part (1). ■

A net $\{x_\alpha\}$ of a Riesz space is said to be **order convergent** to a vector x (in symbols $x_\alpha \xrightarrow{o} x$) whenever there exists another net $\{y_\alpha\}$ with the same index set satisfying $y_\alpha \downarrow 0$ and $|x_\alpha - x| \le y_\alpha$ for all indices α (abbreviated as $|x_\alpha - x| \le y_\alpha \downarrow 0$). A subset A of a Riesz space is said to be **order closed** whenever $\{x_\alpha\} \subseteq A$ and $x_\alpha \xrightarrow{o} x$ imply $x \in A$.

Lemma 1.37. *A solid subset A of a Riesz space is order closed if and only if $\{x_\alpha\} \subseteq A$ and $0 \le x_\alpha \uparrow x$ imply $x \in A$.*

Proof. Assume that a solid set A of a Riesz space has the stated property and let a net $\{x_\alpha\} \subseteq A$ satisfy $x_\alpha \xrightarrow{o} x$. Pick a net $\{y_\alpha\}$ with the same index net satisfying $y_\alpha \downarrow 0$ and $|x_\alpha - x| \le y_\alpha$ for each α. Now note that we have $(|x| - y_\alpha)^+ \le |x_\alpha|$ for each α and $0 \le (|x| - y_\alpha)^+ \uparrow |x|$, and from this it follows that $x \in A$. That is, A is order closed. ■

An order closed ideal is referred to as a **band**. Thus, according to Lemma 1.37 an ideal A is a band if and only if $\{x_\alpha\} \subseteq A$ and $0 \le x_\alpha \uparrow x$ imply $x \in A$ (or, equivalently, if and only if $D \subseteq A^+$ and $D \uparrow x$ imply $x \in A$). In the early developments of Riesz spaces a band was called a normal subspace (G. Birkhoff [**36**], S. Bochner and R. S. Phillips [**39**]), while F. Riesz was calling a band a *famille complète*.

Let A be a nonempty subset of a Riesz space E. Then the **ideal generated** by A is the smallest (with respect to inclusion) ideal that includes A. A moment's thought reveals that this ideal is

$$E_A = \Big\{x \in E\colon\ \exists\, x_1, \ldots, x_n \in A \ \text{ and } \ \lambda \in \mathbb{R}^+ \ \text{ with } \ |x| \le \lambda \sum_{i=1}^{n} |x_i|\Big\}.$$

The ideal generated by a vector $x \in E$ will be denoted by E_x. By the preceding discussion we have

$$E_x = \big\{y \in E\colon\ \exists\, \lambda > 0 \ \text{ with } \ |y| \le \lambda |x|\big\}.$$

Every ideal of the form E_x is referred to as a **principal ideal**.

Similarly, the **band generated** by a set A is the smallest band that includes the set A. Such a band always exists (since it is the intersection of the family of all bands that include A, and E is one of them.) Clearly, the band generated by A coincides with the band generated by the ideal generated by A. The band generated by a vector x is called the **principal band** generated by x and is denoted by B_x.

The band generated by an ideal is described as follows.

Theorem 1.38. *If A is an ideal of a Riesz space E, then the band generated by A is precisely the vector subspace:*

$$\big\{x \in E\colon\ \exists\,\{x_\alpha\} \subseteq A^+ \ \text{with}\ 0 \le x_\alpha \uparrow |x|\big\}\,.$$

In particular, every ideal is order dense in the band it generates.

Moreover, the principal band B_x generated by a vector x is given by

$$B_x = \big\{y \in E\colon\ |y| \wedge n|x| \uparrow |y|\big\}\,.$$

Proof. Let $B = \big\{x \in E\colon\ \exists\,\{x_\alpha\} \subseteq A^+ \text{ with } 0 \le x_\alpha \uparrow |x|\big\}$. Clearly, every band containing A must include B. Thus, in order to establish our result it is enough to show that B is a band.

To this end, let $x, y \in B$. Pick two nets $\{x_\alpha\} \subseteq A^+$ and $\{y_\beta\} \subseteq A^+$ with $0 \le x_\alpha \uparrow |x|$ and $0 \le y_\beta \uparrow |y|$. From

$$|x+y| \wedge (x_\alpha + y_\beta) \uparrow_{(\alpha,\beta)} |x+y| \wedge (|x|+|y|) = |x+y|$$

and

$$|\lambda| x_\alpha \uparrow |\lambda x|\,,$$

we see that B is a vector subspace. Also, if $|z| \le |x|$ holds, then from $\big\{|z| \wedge x_\alpha\big\} \subseteq A$ and $0 \le |z| \wedge x_\alpha \uparrow |z| \wedge |x| = |z|$, it follows that $z \in B$. Hence, B is an ideal. Finally, to see that B is a band, let $\{x_\alpha\} \subseteq B$ satisfy $0 \le x_\alpha \uparrow x$. Put $D = \{y \in A\colon\ \exists\,\alpha \text{ with } 0 \le y \le x_\alpha\}$. Then $D \subseteq A^+$ and $D \uparrow x$ hold. Therefore, $x \in B$ and so B is a band.

To establish the identity for B_x, let $y \in B_x$. By the above, there exists a net $\{x_\alpha\} \subseteq E_x$ with $0 \le x_\alpha \uparrow |y|$. Now given an index α there exists some n with $x_\alpha \le n|x|$, and so $x_\alpha \le |y| \wedge n|x| \le |y|$ holds. This easily implies $|y| \wedge n|x| \uparrow |y|$, and our conclusion follows. ■

From Theorem 1.8 it follows that A^{d} is always a band. It is important to know that the band generated by a set A is precisely A^{dd}.

Theorem 1.39. *The band generated by a nonempty subset A of an Archimedean Riesz space is precisely A^{dd} (and hence if A is a band, then $A = A^{\mathrm{dd}}$ holds).*

Proof. We mentioned before that the band generated by A is the same as the band generated by the ideal generated by A. Therefore, we can assume that A is an ideal. By part (3) of Theorem 1.36 we know that A is order dense in A^{dd}, and hence (by Theorem 1.34) for each $x \in A^{\mathrm{dd}}$ there exists a net $\{x_\alpha\} \subseteq A$ with $0 \le x_\alpha \uparrow |x|$. This easily implies that A^{dd} is the smallest band including A. ■

A useful condition under which an ideal is necessarily a band is presented next.

Theorem 1.40. *Let A and B be two ideals in a Riesz space E such that $E = A \oplus B$. Then A and B are both bands satisfying $A = B^{\mathrm{d}}$ and $B = A^{\mathrm{d}}$ (and hence $A = A^{\mathrm{dd}}$ and $B = B^{\mathrm{dd}}$ both hold).*

Proof. Note first that for each $a \in A$ and $b \in B$ we have

$$|a| \wedge |b| \in A \cap B = \{0\},$$

and so $A \perp B$. In particular, $A \subseteq B^{\mathrm{d}}$.

On the other hand, if $x \in B^{\mathrm{d}}$, then write $x = a+b$ with $a \in A$ and $b \in B$, and note that $b = x - a \in B \cap B^{\mathrm{d}} = \{0\}$ implies $x = a \in A$. Thus, $B^{\mathrm{d}} \subseteq A$, and so $A = B^{\mathrm{d}}$ holds. This shows that A is a band. By the symmetry of the situation $B = A^{\mathrm{d}}$ also holds. ■

A band B in a Riesz space E that satisfies $E = B \oplus B^{\mathrm{d}}$ is referred to as a **projection band**. The next result characterizes the ideals that are projection bands.

Theorem 1.41. *For an ideal B in a Riesz space E the following statements are equivalent.*

(1) *B is a projection band, i.e., $E = B \oplus B^{\mathrm{d}}$ holds.*

(2) *For each $x \in E^+$ the supremum of the set $B^+ \cap [0, x]$ exists in E and belongs to B.*

(3) *There exists an ideal A of E such that $E = B \oplus A$ holds.*

Proof. (1) $\Longrightarrow$ (2) Let $x \in E^+$. Choose the (unique) vectors $0 \le y \in B$ and $0 \le z \in B^{\mathrm{d}}$ such that $x = y + z$. If $u \in B^+$ satisfies $u \le x = y + z$, then it follows from $0 \le (u - y)^+ \le z \in B^{\mathrm{d}}$ and $(u - y)^+ \in B$ that $(u - y)^+ = 0$. Thus, $u \le y$, and so y is an upper bound of the set $B^+ \cap [0, x]$. Since $y \in B \cap [0, x]$, we see that $y = \sup\{u \in B^+ \colon\ u \le x\} = \sup B \cap [0, x]$ in E.

(2) $\Longrightarrow$ (3) Fix some $x \in E^+$, and let $u = \sup B \cap [0, x]$. Clearly, u belongs to B. Put $y = x - u \ge 0$. If $0 \le w \in B$, then $0 \le y \wedge w \in B$, and moreover from $0 \le u + y \wedge w \in B$ and

$$u + y \wedge w = (u + y) \wedge (u + w) = x \wedge (u + w) \le x,$$

it follows that $u + y \wedge w \le u$. Hence, $y \wedge w = 0$ holds, and so $y \in B^{\mathrm{d}}$. From $x = u + y$ we see that $E = B \oplus B^{\mathrm{d}}$, and therefore (3) holds with $A = B^{\mathrm{d}}$.

(3) $\Longrightarrow$ (1) This follows from Theorem 1.40. ■

Not every band is a projection band, and a Riesz space in which every band is a projection band is referred to as a Riesz space with the **projection**

property. From the preceding theorem it should be clear that in a Dedekind complete Riesz space every band is a projection band. This was proven by F. Riesz [**166**] is one of his early fundamental papers on Riesz spaces. Because it guarantees an abundance of order projections, we state it next as a separate theorem.

Theorem 1.42 (F. Riesz)**.** *If B is a band in a Dedekind complete Riesz space E, then $E = B \oplus B^{\mathrm{d}}$ holds.*

As usual, an operator $P\colon V \to V$ on a vector space is called a **projection** if $P^2 = P$. If a projection P is defined on a Riesz space and P is also a positive operator, then P will be referred to as a **positive projection**.

Now let B be a projection band in a Riesz space E. Thus, $E = B \oplus B^{\mathrm{d}}$ holds, and so every vector $x \in E$ has a unique decomposition $x = x_1 + x_2$, where $x_1 \in B$ and $x_2 \in B^{\mathrm{d}}$. Then it is easy to see that a projection $P_B\colon E \to E$ is defined via the formula

$$P_B(x) := x_1\,.$$

Clearly, P_B is a positive projection. Any projection of the form P_B is called an **order projection** (or a **band projection**). Thus, the order projections are associated with the projection bands in a one-to-one fashion.

Theorem 1.43. *If B is a projection band of a Riesz space E, then*

$$P_B(x) = \sup\bigl\{y \in B\colon\ 0 \le y \le x\bigr\} = \sup B \cap [0, x]$$

holds for all $x \in E^+$.

Proof. Let $x \in E^+$. Then (by Theorem 1.41) $u = \sup\bigl\{y \in B\colon\ 0 \le y \le x\bigr\}$ exists and belongs to B. We claim that $u = P_B(x)$.

To see this, write $x = x_1 + x_2$ with $0 \le x_1 \in B$ and $0 \le x_2 \in B^{\mathrm{d}}$, and note that $0 \le x_1 \le x$ implies $0 \le x_1 \le u$. Thus, $0 \le u - x_1 \le x - x_1 = x_2$, and hence $u - x_1 \in B^{\mathrm{d}}$, Since $u - x_1 \in B$ and $B \cap B^{\mathrm{d}} = \{0\}$, we see that $u = x_1$, as claimed. ■

Among projections the order projections are characterized as follows.

Theorem 1.44. *For an operator $T\colon E \to E$ on a Riesz space the following statements are equivalent.*

(1) *T is an order projection.*
(2) *T is a projection satisfying $0 \le T \le I$ (where, of course, I is the identity operator on E).*
(3) *T and $I - T$ have disjoint ranges, i.e., $Tx \perp y - Ty$ holds for all $x, y \in E$.*

Proof. (1) $\Longrightarrow$ (2) Obvious.

(2) $\Longrightarrow$ (3) Let $x, y \in E^+$. Put $z = Tx \wedge (I-T)y$. From the inequality $0 \le z \le (I-T)y$ it follows that $0 \le Tz \le T(I-T)y = (T-T^2)y = 0$, and so $Tz = 0$. Similarly, $(I-T)z = 0$, and hence $z = (I-T)z + Tz = 0$ holds. This shows that T and $I-T$ have disjoint ranges.

(3) $\Longrightarrow$ (1) Let A and B be the ideals generated by the ranges of T and $I-T$, respectively. By our hypothesis it follows that $A \perp B$, and from $x = Tx + (I-T)x$ we see that $E = A \oplus B$. But then, by Theorem 1.40 both A and B are projection bands of E. Now the identity

$$P_A x - Tx = P_A x - P_A Tx = P_A(x - Tx) = 0$$

shows that $T = P_A$ holds. Thus, T is an order projection, and the proof is finished. ■

A positive projection need not be an order projection. For instance, consider the operator $T\colon L_1[0,1] \to L_1[0,1]$ defined by

$$T(f) = \Big[\int_0^1 f(x)\,dx\Big] \cdot \mathbf{1}\,,$$

where $\mathbf{1}$ denotes the constant function one. Clearly, $0 \le T = T^2$ holds, and its is not difficult to see that T is not an order projection.

The basic properties of order projections are summarized in the next theorem.

Theorem 1.45. *If A and B are projection bands in a Riesz space E, then A^{d}, $A \cap B$, and $A + B$ are likewise projection bands. Moreover, we have:*

(1) $P_{A^{\mathrm{d}}} = I - P_A$.
(2) $P_{A\cap B} = P_A P_B = P_B P_A$.
(3) $P_{A+B} = P_A + P_B - P_A P_B$.

Proof. (1) From $E = A \oplus A^{\mathrm{d}}$ it follows that $A^{\mathrm{dd}} = A$ holds (see Theorem 1.40), and so A^{d} is a projection band. The identity $P_{A^{\mathrm{d}}} = I - P_A$ should be obvious.

(2) To see that $A \cap B$ is a projection band note first that the identity $B \cap [0,x] = [0, P_B x]$ implies $A \cap B \cap [0,x] = A \cap [0, P_B x]$ for each $x \in E^+$. Consequently,

$$P_A P_B x = \sup A \cap [0, P_B x] = \sup A \cap B \cap [0,x]$$

holds for each $x \in E^+$, which (by Theorem 1.41) shows that $A \cap B$ is a projection band and that $P_A P_B = P_{A\cap B}$ holds. Similarly, $P_B P_A = P_{A\cap B}$.

(3) Assume at the beginning that the two projection bands A and B satisfy $A \perp B$. Let $x \in E^+$. If $0 \le a + b \in A + B$ satisfies $a + b \le x$, then clearly $a \in A \cap [0, x]$ and $b \in B \cap [0, x]$, and so $a + b \le P_A x + P_B x \in A + B$ holds. This shows that

$$\sup(A + B) \cap [0, x] = P_A x + P_B x \in A + B\,,$$

and hence by Theorem 1.41 the ideal $A + B$ is a projection band. Also, $P_{A+B} = P_A + P_B$ holds.

For the general case observe that $A + B = (A \cap B^{\mathrm{d}}) \oplus B$. Now using the preceding case, we get

$$\begin{aligned} P_{A+B} &= P_{(A\cap B^{\mathrm{d}})\oplus B} = P_{A\cap B^{\mathrm{d}}} + P_B = P_A P_{B^{\mathrm{d}}} + P_B \\ &= P_A(I - P_B) + P_B = P_A - P_A P_B + P_B \\ &= P_A + P_B - P_{A\cap B}\,, \end{aligned}$$

and the proof is finished. ■

An immediate consequence of statement (2) of the preceding theorem is that two arbitrary order projections mutually commute.

A useful comparison property of order projections is described next.

Theorem 1.46. *If A and B are projection bands in a Riesz space, then the following statements are equivalent.*

(1) $A \subseteq B$.

(2) $P_A P_B = P_B P_A = P_A$.

(3) $P_A \le P_B$.

Proof. (1) $\Longrightarrow$ (2) Let $A \subseteq B$. Then from Theorem 1.45 it follows that

$$P_A P_B = P_B P_A = P_{A\cap B} = P_A\,.$$

(2) $\Longrightarrow$ (3) For each $0 \le x$ we have $P_A x = P_B P_A x \le P_B x$, and so $P_A \le P_B$ holds.

(3) $\Longrightarrow$ (1) If $0 \le x \in A$, then it follows from $0 \le x = P_A x \le P_B x \in B$ that $x \in B$. Therefore, $A \subseteq B$ holds, as required. ■

A vector x in a Riesz space E is said to be a **projection vector** whenever the principal band B_x generated by x (i.e., $B_x = \big\{y \in E\colon\ |y| \wedge n|x| \uparrow |y|\big\}$) is a projection band. If every vector in a Riesz space is a projection vector, then the Riesz space is said to have the **principal projection property**. For a projection vector x we shall write P_x for the order projection onto the band B_x.

Theorem 1.47. *A vector x in a Riesz space is a projection vector if and only if* $\sup\{y \wedge n|x|:\ n \in \mathbb{N}\}$ *exists for each $y \geq 0$. In this case*

$$P_x(y) = \sup\{y \wedge n|x|:\ n \in \mathbb{N}\}$$

holds for all $y \geq 0$.

Proof. Let $y \geq 0$. We claim that the sets $B_x \cap [0, y]$ and $\{y \wedge n|x|:\ n \in \mathbb{N}\}$ have the same upper bounds. To see this, note first that

$$\{y \wedge n|x|:\ n \in \mathbb{N}\} \subseteq B_x \cap [0, y]$$

holds. Now let $y \wedge n|x| \leq u$ for all n. If $z \in B_x \cap [0, y]$, then by Theorem 1.38 we have $z \wedge n|x| \uparrow z$. In view of $z \wedge n|x| \leq y \wedge n|x| \leq u$, we see that $z \leq u$, and so the two sets have the same upper bounds. Now to finish the proof invoke Theorems 1.41 and 1.43. ■

From the preceding theorem it follows immediately that in a Dedekind σ-complete Riesz space every principal band is a projection band. If $x, y \geq 0$ are projection vectors in a Riesz space, then note that the formulas of Theorem 1.45 take the form

$$P_{x \wedge y} = P_x P_y = P_y P_x \quad \text{and} \quad P_{x+y} = P_x + P_y - P_{x \wedge y}\,.$$

A vector $e > 0$ in a Riesz space E is said to be a **weak order unit** whenever the band generated by e satisfies $B_e = E$ (or, equivalently, whenever for each $x \in E^+$ we have $x \wedge ne \uparrow x$). Clearly, every vector $0 < x \in E$ is a weak order unit in the band it generates. Also, note that a vector $e > 0$ in an Archimedean Riesz space is a weak order unit if and only if $x \perp e$ implies $x = 0$.

Projection vectors satisfy the following useful properties.

Theorem 1.48. *In a Riesz space E the following statements hold:*

(1) *If u, v, and w are projection vectors satisfying $0 \leq w \leq v \leq u$, then for each $x \in E$ we have $(P_u - P_v)x \perp (P_v - P_w)x$.*

(2) *If $0 \leq u_\alpha \uparrow u$ holds in E with u and all the u_α projection vectors, then $P_{u_\alpha}(x) \uparrow P_u(x)$ holds for each $x \in E^+$.*

Proof. (1) By Theorem 1.46 we have $P_w \leq P_v \leq P_u$ and so if $x \in E$, then

$$\begin{aligned} 0 &\leq \big|(P_u - P_v)x\big| \wedge \big|(P_v - P_w)x\big| \\ &\leq (P_u - P_v)|x| \wedge (P_v - P_w)|x| \\ &\leq \big[P_u|x| - P_v(P_u|x|)\big] \wedge P_v(P_u|x|) = 0\,. \end{aligned}$$

(2) Let $x \in E^+$. Clearly, $P_{u_\alpha}(x) \uparrow\leq P_u(x)$. Thus, $P_u(x)$ is an upper bound for the net $\{P_{u_\alpha}(x)\}$, and we claim it is the least upper bound.

To see this, assume $P_{u_\alpha}(x) \le y$ for all α. Hence, $x \wedge nu_\alpha \le y$ holds for all α and n. Consequently, $u_\alpha \uparrow u$ implies $x \wedge nu \le y$ for all n, and therefore $P_u(x) = \sup\{x \wedge nu\colon\ n \in \mathbb{N}\} \le y$. Hence, $P_u(x)$ is the least upper bound of $\{P_{u_\alpha}(x)\}$, and thus $P_{u_\alpha}(x) \uparrow P_u(x)$. ■

Let e be a positive vector of a Riesz space E. A vector $x \in E^+$ is said to be a **component** of e whenever $x \wedge (e - x) = 0$. The collection of all components of e will be denoted by $\mathcal{C}_e$, i.e.,

$$\mathcal{C}_e := \{x \in E^+\colon\ x \wedge (e - x) = 0\}\,.$$

Clearly, $x \in \mathcal{C}_e$ implies $e - x \in \mathcal{C}_e$. Also, $P_B e \in \mathcal{C}_e$ for each projection band B.

Under the partial ordering induced by E, the set of components $\mathcal{C}_e$ is a Boolean algebra,[2] consisting precisely of the extreme points of the order interval $[0, e]$. The details follow.

Theorem 1.49. *For a positive vector e in a Riesz space E we have:*

(1) *If $x, y \in \mathcal{C}_e$ and $x \le y$ holds, then $y - x \in \mathcal{C}_e$.*

(2) *If $x_1, x_2, y_1, y_2 \in \mathcal{C}_e$ satisfy the inequalities $x_1 \le x_2 \le y_1 \le y_2$, then $x_2 - x_1 \perp y_2 - y_1$.*

(3) *If $x, y \in C_e$, then $x \vee y$ and $x \wedge y$ both belong to $\mathcal{C}_e$ (and so $\mathcal{C}_e$ is a Boolean algebra with smallest element 0 and largest element e).*

(4) *If E is Dedekind complete, then for every non-empty subset C of $\mathcal{C}_e$ the elements $\sup C$ and $\inf C$ both belong to $\mathcal{C}_e$ (and so in this case $\mathcal{C}_e$ is a Dedekind complete Boolean algebra).*

(5) *The set of components $\mathcal{C}_e$ of e is precisely the set of all extreme points of the convex set $[0, e]$.*[3]

Proof. (1) It follows immediately from the inequalities

$$\begin{aligned} 0 &\le (y - x) \wedge \big[e - (y - x)\big] = (y - x) \wedge \big[(e - y) + x\big] \\ &\le (y - x) \wedge (e - y) + (y - x) \wedge x \\ &\le y \wedge (e - y) + (e - x) \wedge x = 0 + 0 = 0\,. \end{aligned}$$

(2) Note that $0 \le (x_2 - x_1) \wedge (y_2 - y_1) \le y_1 \wedge (e - y_1) = 0$.

---

[2] Recall that a **Boolean algebra** $\mathcal{B}$ is a distributive lattice with smallest and largest elements that is complemented. That is, $\mathcal{B}$ is a partially ordered set that is a distributive lattice with a smallest element 0 and a largest element e such that for each $a \in \mathcal{B}$ there exists a (necessarily unique) element $a' \in \mathcal{B}$ (called the *complement* of a) satisfying $a \wedge a' = 0$ and $a \vee a' = e$. A Boolean algebra $\mathcal{B}$ is **Dedekind complete** if every nonempty subset of $\mathcal{B}$ has a supremum.

[3] Recall that a vector u in a convex set C is said to be an **extreme point** of C if it follows from $u = \lambda v + (1 - \lambda)w$ with $v, w \in C$ and $0 < \lambda < 1$ that $v = w = u$.

(3) Let $x, y \in \mathcal{C}_e$. Then, using the distributive laws, we see that

$$\begin{aligned}(x \vee y) \wedge (e - x \vee y) &= (x \vee y) \wedge \big[(e-x) \wedge (e-y)\big] \\ &= \big[x \wedge (e-x) \wedge (e-y)\big] \vee \big[y \wedge (e-x) \wedge (e-y)\big] \\ &= 0 \vee 0 = 0\,,\end{aligned}$$

and

$$\begin{aligned}(x \wedge y) \wedge (e - x \wedge y) &= (x \wedge y) \wedge \big[(e-x) \vee (e-y)\big] \\ &= \big[x \wedge y \wedge (e-x)\big] \vee \big[x \wedge y \wedge (e-y)\big] \\ &= 0 \vee 0 = 0\,.\end{aligned}$$

(4) Now assume that E is Dedekind complete and let C be a nonempty set of components of e. Put $u = \sup C$ and $v = \inf C$. Then, using the infinite distributive laws, we get

$$\begin{aligned}0 \le u \wedge (e-u) &= [\sup C] \wedge (e-u) = \sup\big\{c \wedge (e-u)\colon\; c \in C\big\} \\ &\le \sup\big\{c \wedge (e-c)\colon\; c \in C\big\} = 0\,.\end{aligned}$$

Similarly, we have

$$\begin{aligned}0 \le v \wedge (e-v) &= = v \wedge \big(e - \inf C\big) = v \wedge \sup\big\{e - c\colon\; c \in C\big\} \\ &= \sup\big\{v \wedge (e-c)\colon\; c \in C\big\} \\ &\le \sup\big\{c \wedge (e-c)\colon\; c \in C\big\} = 0\,.\end{aligned}$$

(5) Assume first that an element $x \in [0, e]$ is an extreme point of $[0, e]$. Let $y = x \wedge (e - x) \ge 0$. We must show that $y = 0$. Clearly, $0 \le x - y \le e$ and $0 \le x + y \le e$, and from the convex combination $x = \frac{1}{2}(x-y) + \frac{1}{2}(x+y)$ we get $x - y = x$. So $y = 0$, as desired.

For the converse, assume that $v \in \mathcal{C}_e$ and let $v = \lambda x + (1-\lambda)y$, where $x, y \in [0, e]$ and $0 < \lambda < 1$. From $v \wedge (e-v) = 0$, it follows that $x \wedge (e-v) = 0$, and so from part (1) of Lemma 1.4 we get

$$x = x \wedge e = x \wedge [(v + (e-v)] \le x \wedge v + x \wedge (e-v) = x \wedge v \le v\,.$$

Similarly, $y \le v$. Now if either $x < v$ or $y < v$ were true, then

$$v = \lambda x + (1-\lambda)y < \lambda v + (1-\lambda)v = v$$

also would be true, which is impossible. Hence $x = y = v$ holds, and so v is an extreme point of $[0, e]$. This completes the proof of the theorem. ■

When E has the principal projection property, Y. A. Abramovich [**1**] has described the lattice operations of $\mathcal{L}_b(E, F)$ in terms of components as follows.

Theorem 1.50 (Abramovich). *If a Riesz space E has the principal projection property and F is a Dedekind complete Riesz space, then for each pair $S,T \in \mathcal{L}_b(E,F)$ and each $x \in E^+$ we have:*

$$\begin{aligned} [S \vee T](x) &= \sup\{S(y) + T(z)\colon\ y \wedge z = 0 \ \text{ and } \ y + z = x\}\,. \\ [S \wedge T](x) &= \inf\{S(y) + T(z)\colon\ y \wedge z = 0 \ \text{ and } \ y + z = x\}\,. \end{aligned}$$

Proof. Notice that the first formula follows from the second by using the identity $S \vee T = -[(-S) \wedge (-T)]$. Also, if the second formula is true for the special case $S \wedge T = 0$, then it is true in general. This claim follows easily from the identity $(S - S \wedge T) \wedge (T - S \wedge T) = 0$. To complete the proof, assume that $S \wedge T = 0$ in $\mathcal{L}_b(E,F)$. Fix $x \in L^+$ and put

$$u = \inf\{S(y) + T(x-y)\colon\ y \wedge (x-y) = 0\}\,.$$

We must show that $u = 0$.

To this end, fix any $0 \le y \in E^+$ satisfy $0 \le y \le x$. Let P denote the order projection of E onto the band generated by $(2y-x)^+$ and put $z = P(x)$. From $x \le 2y + (x-2y)^+$ and $(x-2y)^+ \wedge (2y-x)^+ = 0$, it follows that $P(x) \le 2P(y) + P\big((x-2y)^+\big) = 2P(y) \le 2y$. Therefore,

$$z \le 2y\,. \tag{$\star$}$$

Also, from $(2y-x)^+ \le (2x-x)^+ = x$ we see that

$$2y - x \le (2y-x)^+ = P\big((2y-x)^+\big) \le P(x) = z\,,$$

and consequently

$$x - z \le 2(x-y)\,. \tag{$\star\star$}$$

Now combining ($\star$) and ($\star\star$), we get

$$0 \le u \le S(z) + T(x-z) \le 2[S(y) + T(x-y)]\,, \tag{$\star\star\star$}$$

for all elements $y \in E^+$ with $0 \le y \le x$. Taking into consideration that (according to Theorem 1.18) we have $\inf\{S(y)+T(x-y)\colon\ 0 \le y \le x\} = 0$, it follows from ($\star\star\star$) that $u = 0$, and the proof is finished. ■

It should be noted that Theorem 1.50 is false without assuming that E has the principal projection property. For instance, let $E = C[0,1]$, $F = \mathbb{R}$, and let $S,T\colon E \to F$ be defined by $S(f) = f(0)$ and $T(f) = f(1)$. Then $S \wedge T = 0$ holds, while

$$\begin{aligned} &\inf\{S(f) + T(g)\colon\ f \wedge g = 0 \ \text{and} \ f + g = \mathbf{1}\} \\ &\quad = \inf\{S(f) + T(\mathbf{1} - f)\colon\ f = 0 \ \text{or} \ f = \mathbf{1}\} = 1\,. \end{aligned}$$

When E has the principal projection property, the lattice operations of $\mathcal{L}_b(E,F)$ also can be expressed in terms of directed sets involving components as follows.

Theorem 1.51. *Assume that E has the principal projection property and that F is Dedekind complete. Then for all $S, T \in \mathcal{L}_{\mathrm{b}}(E,F)$ and $x \in E^+$ we have:*

(1) $\Big\{\sum_{i=1}^{n} S(x_i) \vee T(x_i)\colon\ x_i \wedge x_j = 0 \textit{ for } i \neq j \textit{ and } \sum_{i=1}^{n} x_i = x\Big\} \uparrow [S \vee T](x)$.

(2) $\Big\{\sum_{i=1}^{n} S(x_i) \wedge T(x_i)\colon\ x_i \wedge x_j = 0 \textit{ for } i \neq j \textit{ and } \sum_{i=1}^{n} x_i = x\Big\} \downarrow [S \wedge T](x)$.

(3) $\Big\{\sum_{i=1}^{n} |T(x_i)|\colon\ x_i \wedge x_j = 0 \textit{ for } i \neq j \textit{ and } \sum_{i=1}^{n} x_i = x\Big\} \uparrow |T|(x)$.

Proof. Since (2) and (3) follow from (1) by using the usual lattice identities $S \wedge T = -[(-S) \vee (-T)]$ and $|T| = T \vee (-T)$, we prove only the first formula. Put

$$D = \Big\{\sum_{i=1}^{n} S(x_i) \vee T(x_i)\colon\ x_i \wedge x_j = 0 \text{ for } i \neq j \text{ and } \sum_{i=1}^{n} x_i = x\Big\},$$

where $x \in E^+$ is fixed, and note that $\sup D \leq [S \vee T](x)$ holds in F. On the other hand, if $y, z \in E^+$ satisfy $y \wedge z = 0$ and $y + z = x$, then the relation

$$S(y) + T(z) \leq S(y) \vee T(y) + S(z) \vee T(z) \in D\,,$$

in conjunction with Theorem 1.50, shows that $\sup D = [S \vee T](x)$ holds. Therefore, what remains to be shown is that D is directed upward.

To this end, let $\{x_1, \ldots, x_n\}$ and $\{y_1, \ldots, y_m\}$ be two subsets of E^+ each of which is pairwise disjoint such that $\sum_{i=1}^{n} x_i = \sum_{j=1}^{m} y_j = x$. Then note that the finite set $\{x_i \wedge y_j\colon\ i = 1, \ldots, n;\ j = 1, \ldots, m\}$ is pairwise disjoint and

$$\sum_{i=1}^{n}\sum_{j=1}^{m} x_i \wedge y_j = \sum_{i=1}^{n} x_i \wedge \Big[\sum_{j=1}^{m} y_j\Big] = \sum_{i=1}^{n} x_i \wedge x = \sum_{i=1}^{n} x_i = x\,.$$

In addition, we have

$$\begin{aligned}
\sum_{i=1}^{n} S(x_i) \vee T(x_i) &= \sum_{i=1}^{n} S\Big(x_i \wedge \sum_{j=1}^{n} y_j\Big) \vee T\Big(x_i \wedge \sum_{j=1}^{m} y_j\Big) \\
&= \sum_{i=1}^{n} \Big[\sum_{j=1}^{n} S(x_i \wedge y_j)\Big] \vee \Big[\sum_{j=1}^{m} T(x_i \wedge y_j)\Big] \\
&\leq \sum_{i=1}^{n}\sum_{j=1}^{m} S(x_i \wedge y_j) \vee T(x_i \wedge y_j)\,,
\end{aligned}$$

and, similarly,

$$\sum_{j=1}^{m} S(y_j) \vee T(y_j) \le \sum_{i=1}^{n}\sum_{j=1}^{m} S(x_i \wedge y_j) \vee T(x_i \wedge y_j)\,.$$

Therefore, $D \uparrow [S \vee T](x)$ holds. ■

The final result of this section deals with retracts of Riesz spaces. Let us say that a Riesz subspace G of a Riesz space E is a **retract** (or that E is **retractable** on G) whenever there exists a positive projection $P\colon E \to E$ whose range is G.

Theorem 1.52. *For a Riesz subspace G of a Riesz space E we have the following:*

(1) *If G is a retract of E and E is Dedekind complete, then G is a Dedekind complete Riesz space in its own right.*

(2) *If G is Dedekind complete in its own right and G majorizes E, then G is a retract of E.*

Proof. (1) Let $P\colon E \to E$ be a positive projection whose range is the Riesz subspace G, and let $0 \le x_\alpha \uparrow \le x$ in G. Then there exists some $y \in E$ with $0 \le x_\alpha \uparrow y \le x$ in E, and so $0 \le x_\alpha = Px_\alpha \le Py$ holds in G for each α. On the other hand, if for some $z \in G$ we have $0 \le x_\alpha \le z$ for all α, then $y \le z$, and hence $Py \le Pz = z$. In other words, $0 \le x_\alpha \uparrow Py$ holds in G, which proves that G is a Dedekind complete Riesz space.

(2) Apply Theorem 1.32 to the identity operator $I\colon G \to G$. ■

Exercises

1. For two nets $\{x_\alpha\}$ and $\{y_\beta\}$ in a Riesz space satisfying $x_\alpha \xrightarrow{o} x$ and $y_\beta \xrightarrow{o} y$ establish the following properties.

(a) If $x_\alpha \xrightarrow{o} u$, then $u = x$ (and so the order limits whenever they exist are uniquely determined).

(b) $\lambda x_\alpha + \mu y_\beta \xrightarrow{o} \lambda x + \mu y$ for all $\lambda, \mu \in \mathbb{R}$.

(c) $|x_\alpha| \xrightarrow{o} |x|$.

(d) $x_\alpha \vee y_\beta \xrightarrow{o} x \vee y$ and $x_\alpha \wedge y_\beta \xrightarrow{o} x \wedge y$.

(e) $(x_\alpha - y_\beta)^+ \xrightarrow{o} (x - y)^+$.

(f) If $x_\alpha \le z$ holds for all $\alpha \succeq \alpha_0$, then $x \le z$.

2. Show that the intersection of two order dense ideals is also an order dense ideal.

3. Let $0 \le y \le x \le e$ hold in a Riesz space. If y is a component of x and x is a component of e, then show that y is a component of e.

4. If $\mathbf{1}$ denotes the constant function one on $[0,1]$, then compute $\mathcal{C}_{\mathbf{1}}$ in:

$$\text{(a) } C[0,1]\,; \quad \text{(b) } L_1[0,1]\,; \quad \text{(c) } \ell_\infty[0,1]\,.$$

5. Show that in an Archimedean Riesz space a vector $e > 0$ is a weak order unit if and only if $x \perp e$ implies $x = 0$.

6. If E has the principal projection property, then show that $P_{x^+}(x) = x^+$ holds for all $x \in E$.

7. Let E be a Riesz space satisfying the principal projection property, let $0 \le y \le x$, and let $\epsilon \in \mathbb{R}$. If P denotes the order projection onto the band generated by $(y - \epsilon x)^+$, then show that $\epsilon P(x) \le y$ holds.

8. If A and B are two projection bands in a Riesz space E, then show that:
(a) $P_{A \cap B}(x) = P_A(x) \wedge P_B(x)$ holds for all $x \in E^+$.
(b) $P_{A+B}(x) = P_A(x) \vee P_B(x)$ holds for all $x \in E^+$.
(c) $P_{A+B} = P_A + P_B$ holds if and only if $A \perp B$.

9. If P and Q are order projections on a Riesz space E, then show that

$$P(x) \wedge Q(y) = PQ(x \wedge y)$$

for all $x, y \in E^+$.

10. For an order projection P on a Riesz space E establish the following:
(a) $|Px| = P(|x|)$ holds for all $x \in E$.
(b) If D is a nonempty subset of E for which $\sup D$ exists in E, then $\sup P(D)$ exists in E and $\sup P(D) = P(\sup D)$.

11. Let E and F be two Riesz spaces with F Dedekind complete. Show that:
(a) If P is an order projection on E and Q is an order projection on F, then the operator (transformer) $T \mapsto QTP$ is an order projection on $\mathcal{L}_b(E,F)$.
(b) If P_1, P_2 are order projections on E and Q_1, Q_2 are order projections on F, then

$$(Q_1TP_1) \wedge (Q_2SP_2) = Q_1Q_2(T \wedge S)P_1P_2$$

holds in $\mathcal{L}_b(E,F)$ for all $S, T \in \mathcal{L}_b^+(E,F)$.

12. Let E and F be two Riesz spaces with F Dedekind complete. Show that:
(a) If P is an order projection on E, then $|TP| = |T|P$ holds for all $T \in \mathcal{L}_b(E,F)$.
(b) If Q is an order projection on F, then $|QT| = Q|T|$ holds for all $T \in \mathcal{L}_b(E,F)$.

1.4. Order Continuous Operators

In this section the basic properties of order continuous operators will be studied. Our discussion starts with their definition introduced by T. Ogasawara around 1940; see the work of M. Nakamura [**146**]. Recall that a net $\{x_\alpha\}$ in a Riesz space is order convergent to some vector x, denoted

$x_\alpha \xrightarrow{o} x$, whenever there exists another net $\{y_\alpha\}$ with the same index set satisfying $|x_\alpha - x| \le y_\alpha \downarrow 0$.

Definition 1.53. *An operator $T\colon E \to F$ between two Riesz spaces is said to be:*

(a) ***Order continuous***, *if $x_\alpha \xrightarrow{o} 0$ in E implies $Tx_\alpha \xrightarrow{o} 0$ in F.*

(b) ***σ-order continuous***, *if $x_n \xrightarrow{o} 0$ in E implies $Tx_n \xrightarrow{o} 0$ in F.*

It is useful to note that a positive operator $T\colon E \to F$ between two Riesz spaces is order continuous if and only if $x_\alpha \downarrow 0$ in E implies $Tx_\alpha \downarrow 0$ in F (and also if and only if $0 \le x_\alpha \uparrow x$ in E implies $Tx_\alpha \uparrow Tx$ in F.) In the terminology of directed sets a positive operator $T\colon E \to F$ is, of course, order continuous if and only if $D \downarrow 0$ in E implies $T(D) \downarrow 0$ in F. Similar observations hold true for positive σ-order continuous operators.

Lemma 1.54. *Every order continuous operator is order bounded.*

Proof. Let $T\colon E \to F$ be an order continuous operator and let $x \in E^+$. If we consider the order interval $[0, x]$ as a net $\{x_\alpha\}$, where $x_\alpha = \alpha$ for each $\alpha \in [0, x]$, then $x_\alpha \downarrow 0$. So, by the order continuity of T, there exists a net $\{y_\alpha\}$ of F with the same index $[0, x]$ such that $|Tx_\alpha| \le y_\alpha \downarrow 0$. Consequently, if $\alpha \in [0, x]$, then we have $|T\alpha| = |Tx_\alpha| \le y_\alpha \le y_x$, and this shows that $T[0, x]$ is an order bounded subset of F. ■

A σ-order continuous operator need not be order continuous, as the next example shows.

Example 1.55. Let E be the vector space of all Lebesgue integrable real-valued functions defined on $[0, 1]$. Note that two functions that differ at one point are considered to be different. Under the pointwise ordering (i.e., $f \ge g$ means $f(x) \ge g(x)$ for all $x \in [0, 1]$), E is a Riesz space—in fact, it is a function space. Also, note that $f_\alpha \uparrow f$ holds in E if and only if $f_\alpha(x) \uparrow f(x)$ holds in $\mathbb{R}$ for all $x \in [0, 1]$.

Now define the operator $T\colon E \to \mathbb{R}$ by

$$T(f) = \int_0^1 f(x)\, dx\,.$$

Clearly, T is a positive operator, and from the Lebesgue dominated convergence theorem it easily follows that T is σ-order continuous. However, T is not order continuous.

To see this, note first that if $\mathcal{F}$ denotes the collection of all finite subsets of $[0, 1]$, then the net $\{\chi_\alpha\colon\ \alpha \in \mathcal{F}\} \subseteq E$ (where χ_α is the characteristic function of α) satisfies $\chi_\alpha \uparrow \mathbf{1}$ (= the constant function one). On the other hand, observe that $T(\chi_\alpha) = 0 \not\to T(\mathbf{1}) = 1$. ■

The order continuous operators have a number of nice characterizations.

Theorem 1.56. *For an order bounded operator $T\colon E \to F$ between two Riesz spaces with F Dedekind complete, the following statements are equivalent.*

(1) *T is order continuous.*

(2) *If $x_\alpha \downarrow 0$ holds in E, then $Tx_\alpha \xrightarrow{o} 0$ holds in F.*

(3) *If $x_\alpha \downarrow 0$ holds in E, then $\inf\{|Tx_\alpha|\} = 0$ in F.*

(4) *T^+ and T^- are both order continuous.*

(5) *$|T|$ is order continuous.*

Proof. (1) $\Longrightarrow$ (2) and (2) $\Longrightarrow$ (3) are obvious.

(3) $\Longrightarrow$ (4) It is enough to show that T^+ is order continuous. To this end, let $x_\alpha \downarrow 0$ in E. Let $T^+x_\alpha \downarrow z \geq 0$ in F. We have to show that $z = 0$. Fix some index β and put $x = x_\beta$.

Now for each $0 \leq y \leq x$ and each $\alpha \succeq \beta$ we have

$$0 \leq y - y \wedge x_\alpha = y \wedge x - y \wedge x_\alpha \leq x - x_\alpha\,,$$

and consequently

$$T(y) - T(y \wedge x_\alpha) = T(y - y \wedge x_\alpha) \leq T^+(x - x_\alpha) = T^+x - T^+x_\alpha\,,$$

from which it follows that

$$0 \leq z \leq T^+x_\alpha \leq T^+x + \big|T(y \wedge x_\alpha)\big| - Ty \tag{$\star$}$$

holds for all $\alpha \succeq \beta$ and all $0 \leq y \leq x$. Now since for each fixed vector $0 \leq y \leq x$ we have $y \wedge x_\alpha \downarrow_{\alpha \succeq \beta} 0$, it then follows from our hypothesis that $\inf_{\alpha \succeq \beta}\{|T(y \wedge x_\alpha)|\} = 0$, and hence from ($\star$) we see that $0 \leq z \leq T^+x - Ty$ holds for all $0 \leq y \leq x$. In view of $T^+x = \sup\{Ty\colon\ 0 \leq y \leq x\}$, the latter inequality yields $z = 0$, as desired.

(4) $\Longrightarrow$ (5) The implication follows from the identity $|T| = T^+ + T^-$.

(5) $\Longrightarrow$ (1) The implication follows easily from the lattice inequality $|Tx| \leq |T|(|x|)$. ■

The reader can formulate by himself the analogue of Theorem 1.56 for σ-order continuous operators.

The collection of all order continuous operators of $\mathcal{L}_{\mathrm{b}}(E, F)$ will be denoted by $\mathcal{L}_{\mathrm{n}}(E, F)$; the subscript n is justified by the fact that the order continuous operators are also known as normal operators. That is,

$$\mathcal{L}_{\mathrm{n}}(E, F) := \big\{T \in \mathcal{L}_{\mathrm{b}}(E, F)\colon\ T \text{ is order continuous}\big\}\,.$$

Similarly, $\mathcal{L}_c(E,F)$ will denote the collection of all order bounded operators from E to F that are σ-order continuous. That is,

$$\mathcal{L}_c(E,F) := \{T \in \mathcal{L}_b(E,F):\ T \text{ is } \sigma\text{-order continuous}\}.$$

Clearly, $\mathcal{L}_n(E,F)$ and $\mathcal{L}_c(E,F)$ are both vector subspaces of $\mathcal{L}_b(E,F)$, and moreover $\mathcal{L}_n(E,F) \subseteq \mathcal{L}_c(E,F)$ holds. When F is Dedekind complete T. Ogasawara [**156**] has shown that both $\mathcal{L}_n(E,F)$ and $\mathcal{L}_c(E,F)$ are bands of $\mathcal{L}_b(E,F)$. The details follow.

Theorem 1.57 (Ogasawara). *If E and F are Riesz spaces with F Dedekind complete, then $\mathcal{L}_n(E,F)$ and $\mathcal{L}_c(E,F)$ are both bands of $\mathcal{L}_b(E,F)$.*

Proof. We shall establish that $\mathcal{L}_n(E,F)$ is a band of $\mathcal{L}_b(E,F)$. That $\mathcal{L}_c(E,F)$ is a band can be proven in a similar manner.

Note first that if $|S| \le |T|$ holds in $\mathcal{L}_b(E,F)$ with $T \in \mathcal{L}_n(E,F)$, then from Theorem 1.56 it follows that $S \in \mathcal{L}_n(E,F)$. That is, $\mathcal{L}_n(E,F)$ is an ideal of $\mathcal{L}_b(E,F)$.

To see that the ideal $\mathcal{L}_n(E,F)$ is a band, let $0 \le T_\lambda \uparrow T$ in $\mathcal{L}_b(E,F)$ with $\{T_\lambda\} \subseteq \mathcal{L}_n(E,F)$, and let $0 \le x_\alpha \uparrow x$ in E. Then for each fixed index λ we have

$$0 \le T(x - x_\alpha) \le (T - T_\lambda)(x) + T_\lambda(x - x_\alpha),$$

and $x - x_\alpha \downarrow 0$, in conjunction with $T_\lambda \in \mathcal{L}_n(E,F)$, implies

$$0 \le \inf_\alpha \{T(x - x_\alpha)\} \le (T - T_\lambda)(x)$$

for all λ. From $T - T_\lambda \downarrow 0$ we see that $\inf_\alpha\{T(x - x_\alpha)\} = 0$, and hence $T(x_\alpha) \uparrow T(x)$. Thus, $T \in \mathcal{L}_n(E,F)$, and the proof is finished. ∎

Now consider two Riesz spaces E and F with F Dedekind complete. The band of all operators in $\mathcal{L}_b(E,F)$ that are disjoint from $\mathcal{L}_c(E,F)$ will be denoted by $\mathcal{L}_s(E,F)$, i.e., $\mathcal{L}_s(E,F) := \mathcal{L}_c^d(E,F)$, and its nonzero members will be referred to as **singular operators**. Since $\mathcal{L}_b(E,F)$ is a Dedekind complete Riesz space (see Theorem 1.18), it follows from Theorem 1.42 that $\mathcal{L}_c(E,F)$ is a projection band, and so

$$\mathcal{L}_b(E,F) = \mathcal{L}_c(E,F) \oplus \mathcal{L}_s(E,F)$$

holds. In particular, each operator $T \in \mathcal{L}_b(E,F)$ has a unique decomposition $T = T_c + T_s$, where $T_c \in \mathcal{L}_c(E,F)$ and $T_s \in \mathcal{L}_s(E,F)$. The operator T_c is called the **σ-order continuous component** of T, and T_s is called the **singular component** of T. Similarly,

$$\mathcal{L}_b(E,F) = \mathcal{L}_n(E,F) \oplus \mathcal{L}_\sigma(E,F),$$

where $\mathcal{L}_\sigma(E,F) := \mathcal{L}_{\rm n}^{\rm d}(E,F)$. Thus, every operator $T \in \mathcal{L}_{\rm b}(E,F)$ also has a unique decomposition $T = T_{\rm n} + T_\sigma$, where $T_{\rm n} \in \mathcal{L}_{\rm n}(E,F)$ and $T_\sigma \in \mathcal{L}_\sigma(E,F)$. The operator $T_{\rm n}$ is called the **order continuous component** of T.

The next examples shows that $\mathcal{L}_{\rm c}(E,F) = \{0\}$ is possible.

Example 1.58. For each $1 < p < \infty$ we have

$$\mathcal{L}_{\rm c}(C[0,1], L_p[0,1]) = \{0\}\,.$$

That is, the zero operator is the only σ-order continuous positive operator from $C[0,1]$ to $L_p[0,1]$.

To establish this, we need to show first that the only positive σ-order continuous operator from $C[0,1]$ to $\mathbb{R}$ is the zero operator. To this end, let $\phi\colon C[0,1] \to \mathbb{R}$ be a positive σ-order continuous operator.

Let $\{r_1, r_2, \ldots\}$ be an enumeration of all rational numbers of $[0,1]$. For each pair $m, n \in \mathbb{N}$ choose some $x_{m,n} \in C[0,1]$ such that:

(a) $0 \le x_{m,n}(t) \le 1$ for all $t \in [0,1]$.

(b) $x_{m,n}(r_n) = 1$.

(c) $x_{m,n}(t) = 0$ for all $t \in [0,1]$ with $|t - r_n| > \frac{1}{2^{n+m}}$.

Put $y_{m,n} = \bigvee_{i=1}^n x_{m,i}$, and note that for each fixed m we have $y_{m,n} \uparrow_n$ in $C[0,1]$. In view of $y_{m,n}(r_n) = 1$, it follows that $y_{m,n} \uparrow_n \mathbf{1}$ (= the constant function one). Since ϕ is a positive σ-order continuous operator, we see that $\phi(y_{m,n}) \uparrow_n \phi(\mathbf{1})$ holds in $\mathbb{R}$ for each fixed m.

Put $\epsilon > 0$. For each m choose some $n_m \in \mathbb{N}$ with $\phi(\mathbf{1}) - \phi(y_{m,n_m}) < \frac{1}{2^m}\epsilon$, and then put $z_n = \bigwedge_{m=1}^n y_{m,n_m}$. Clearly, $z_n \downarrow$ holds in $C[0,1]$, and since each set $\{t \in [0,1]\colon\ y_{m,n}(t) > 0\}$ has Lebesgue measure less that $\frac{1}{2^m}$, it follows that $z_n \downarrow 0$. Now the inequalities

$$\begin{aligned} 0 \ &\le\ \phi(\mathbf{1}) - \phi(z_n) = \phi(\mathbf{1} - z_n) = \phi\Big(\bigvee_{m=1}^n \big(\mathbf{1} - y_{m,n_m}\big)\Big) \\ &\le\ \phi\Big(\sum_{m=1}^n \big(\mathbf{1} - y_{m,n_m}\big)\Big) = \sum_{m=1}^n \phi\big(\mathbf{1} - y_{m,n_m}\big) < \epsilon\,, \end{aligned}$$

in conjunction with $\phi(z_n) \downarrow 0$, imply $0 \le \phi(\mathbf{1}) \le \epsilon$ for all $\epsilon > 0$. Therefore, $\phi(\mathbf{1}) = 0$, and from this we see that $\phi = 0$.

Now let $T\colon C[0,1] \to L_p[0,1]$ be a positive σ-order continuous operator. Then for each fixed $0 \le g \in L_q[0,1]$, where $\frac{1}{p} + \frac{1}{q} = 1$, the positive operator $\psi\colon C[0,1] \to \mathbb{R}$ defined by

$$\psi(f) = \int_0^1 g(t)\big[Tf(t)\big]\,dt$$

is σ-order continuous. Hence, by the previous case $\int_0^1 g(t)[Tf(t)]\,dt = 0$ for all $g \in L_q[0,1]$ and all $f \in C[0,1]$. The latter easily implies $T = 0$, as claimed. ■

If x and y are vectors in a Riesz space and ϵ is any real number, then from the identity $x - y = (1-\epsilon)x + (\epsilon x - y)$ we see that

$$x - y \le (1-\epsilon)x + (\epsilon x - y)^+ .$$

This simple inequality is useful in many contexts and was introduced by T. Andô [**125**, Note XIV]. In the sequel it will be referred to as **Andô's inequality**.

The σ-order continuous and order continuous components of a positive operator are described by formulas as follows.

Theorem 1.59. *Let E and F be two Riesz spaces with F Dedekind complete. If $T\colon E \to F$ is a positive operator, then*

(1) $T_{\mathrm{c}}(x) = \inf\bigl\{\sup T(x_n)\colon\ 0 \le x_n \uparrow x\bigr\}$, *and*

(2) $T_{\mathrm{n}}(x) = \inf\bigl\{\sup T(x_\alpha)\colon\ 0 \le x_\alpha \uparrow x\bigr\}$

hold for each $x \in E^+$.[4]

Proof. We prove the formula for T_{n} and leave the identical arguments for T_{c} to the reader.

For each positive operator $S\colon E \to F$ define $S^\star\colon E^+ \to F^+$ by

$$S^\star(x) = \inf\bigl\{\sup S(x_\alpha)\colon\ 0 \le x_\alpha \uparrow x\bigr\},\ \ x \in E^+ .$$

Clearly, $0 \le S^\star(x) \le S(x)$ holds for all $x \in E^+$, and $S^\star(x) = S(x)$ whenever $S \in \mathcal{L}_{\mathrm{n}}(E,F)$. Moreover, it is not difficult to see that $S^\star$ is additive on E^+, and hence (by Theorem 1.10), $S^\star$ extends to a positive operator from E to F. On the other hand, it is easy to see that $S \mapsto S^\star$, from $\mathcal{L}_{\mathrm{b}}^+(E,F)$ to $\mathcal{L}_{\mathrm{b}}^+(E,F)$, is likewise additive, i.e., $(S_1+S_2)^\star = S_1^\star + S_2^\star$ holds, and hence $S \mapsto S^\star$ defines a positive operator from $\mathcal{L}_{\mathrm{b}}(E,F)$ to $\mathcal{L}_{\mathrm{b}}(E,F)$. From the inequality $0 \le S^\star \le S$ we also see that $S \mapsto S^\star$ is order continuous, i.e., $S_\alpha \downarrow 0$ in $\mathcal{L}_{\mathrm{b}}(E,F)$ implies $S_\alpha^\star \downarrow 0$.

[4] These formulas have an interesting history. When $F = \mathbb{R}$, the formula for T_{c} is due to W. A. J. Luxemburg and A. C. Zaanen [**130**, Note VI, Theorem 20.4, p. 663], and for the same case, the formula for T_{n} is due to W. A. J. Luxemburg [**125**]. When $\mathcal{L}_{\mathrm{n}}(F,\mathbb{R})$ separates the points of F, the formulas were established by C. D. Aliprantis [**6**]. In 1975 A. R. Schep announced the validity of the formulas in the general setting and later published his proof in [**176**]. An elementary proof for the T_{c} formula also was obtained by P. van Eldik in [**59**]. The proof presented here is due to the authors [**12**].

Now let $T\colon E \to F$ be a fixed positive operator. It is enough to show that $T^\star$ is order continuous. If this is done, then the inequality $T^\star \leq T$ implies $T^\star = (T^\star)_{\rm n} \leq T_{\rm n}$, and since $T_{\rm n} \leq T^\star$ is trivially true, we see that $T_{\rm n} = T^\star$. To this end, let $0 \leq y_\lambda \uparrow y$ in E. We must show that $T^\star(y - y_\lambda) \downarrow 0$.

Fix $0 < \epsilon < 1$, and let T_λ denote the operator defined in Theorem 1.28 that agrees with T on the ideal generated by $(\epsilon y - y_\lambda)^+$ and vanishes on $(\epsilon y - y_\lambda)^-$. Clearly, $T \geq T_\lambda \downarrow \geq 0$, and $T_\lambda(y_\lambda - \epsilon y)^+ = 0$ holds for all λ. Let $T_\lambda \downarrow R$ in $\mathcal{L}_{\rm b}(E,F)$. From $0 \leq (y_\lambda - \epsilon y)^+ \uparrow (1-\epsilon)y$ and $R(y_\lambda - \epsilon y)^+ = 0$ for each λ, we see that $R^\star(y) = 0$. From Andô's inequality

$$0 \leq y - y_\lambda \leq (1-\epsilon) + (\epsilon - y_\lambda)^+ \,,$$

it follows that

$$0 \leq T^\star(y - y_\lambda) \leq (1-\epsilon)T^\star(y) + T^\star(\epsilon y - y_\lambda)^+ \,. \qquad (\dagger)$$

Now since $0 \leq x \leq (\epsilon y - y_\lambda)^+$ implies $T(x) = T_\lambda(x)$, we see that

$$\begin{aligned} T^\star(\epsilon y - y_\lambda)^+ &= \inf\bigl\{\sup T(x_\alpha)\colon\ 0 \leq x_\alpha \uparrow (\epsilon y - y_\lambda)^+\bigr\} \\ &= \inf\bigl\{\sup T_\lambda(x_\alpha)\colon\ 0 \leq x_\alpha \uparrow (\epsilon y - y_\lambda)^+\bigr\} \\ &= T_\lambda^\star(\epsilon y - y_\lambda)^+ \leq T_\lambda^\star(y)\,, \end{aligned}$$

and so, substituting into (†), we obtain

$$0 \leq T^\star(y - y_\lambda) \leq (1-\epsilon)T^\star(y) + T_\lambda^\star(y)\,. \qquad (\dagger\dagger)$$

From $T_\lambda \downarrow R$ and the order continuity of $S \mapsto S^\star$, it follows that $T_\lambda^\star \downarrow R^\star$. In particular, $T_\lambda^\star(y) \downarrow R^\star(y) = 0$, and so from (††) we see that

$$0 \leq \inf_\lambda \bigl\{T^\star(y - y_\lambda)\bigr\} \leq (1-\epsilon)T^\star(y)$$

holds for all $0 < \epsilon < 1$. Hence, $T^\star(y - y_\lambda) \downarrow 0$, as desired. ■

Consider an order bounded operator $T\colon E \to F$ between two Riesz spaces with F Dedekind complete. Then the **null ideal** N_T of T is defined by

$$N_T := \bigl\{x \in E\colon\ |T|(|x|) = 0\bigr\}\,.$$

Note that N_T is indeed an ideal of E. The disjoint complement of N_T is referred to as the **carrier** of T and is denoted by C_T. That is,

$$C_T := N_T^{\rm d} = \bigl\{x \in E\colon\ x \perp N_T\bigr\}\,.$$

Clearly, $|T|$ is strictly positive on C_T, i.e., $0 < x \in C_T$ implies $0 < |T|(x)$.

When an order bounded operator is, in addition, order continuous, then it is easy to see that its null ideal is a band. However, the converse is false.

Example 1.60. Consider an infinite set X, and let $X_\infty = X \cup \{\infty\}$ be the one-point compactification of X considered equipped with the discrete topology. Thus, a function $f\colon X \to \mathbb{R}$ belongs to $C(X_\infty)$ if and only if there exists some constant c (depending upon f) such that for each $\epsilon > 0$ we have $\big|f(x) - c\big| < \epsilon$ for all but a finite number of x, in which case $c = f(\infty)$.

Now fix a countable subset $\{x_1, x_2, \ldots\}$ of X, and then define the operator $T\colon C(X_\infty) \to \mathbb{R}$ by

$$T(f) = f(\infty) + \sum_{n=1}^{\infty} 2^{-n} f(x_n)\,.$$

Clearly, T is a positive operator, and

$$N_T = \big\{f \in C(X_\infty)\colon\ f(x_n) = 0 \ \text{ for } \ n = 1, 2, \ldots\big\}\,.$$

Since $f_\alpha \uparrow f$ holds in $C(X_\infty)$ if and only if $f_\alpha(x) \uparrow f(x)$ holds in $\mathbb{R}$ for all $x \in X$ (why?), it follows that N_T is a band of $C(X_\infty)$. On the other hand, we claim that T is not order continuous.

To see this, consider the net $\{\chi_\alpha\} \subseteq C(X_\infty)$, where α runs over the collection of all finite subsets of X. Then $0 \le \chi_\alpha \uparrow \mathbf{1}$ holds in $C(X_\infty)$, while $T(\chi_\alpha) \not\to T(\mathbf{1})$. Also, it is interesting to observe that if X is countable, then T is necessarily σ-order continuous! ■

In terms of null ideals the order and σ-order continuous operators are characterized as follows. (Recall that an ideal A of a Riesz space is said to be a **σ-ideal** whenever $\{x_n\} \subseteq A$ and $0 \le x_n \uparrow x$ imply $x \in A$.)

Theorem 1.61. *For an order bounded operator $T\colon E \to F$ between two Riesz spaces with F Dedekind complete we have the following.*

1. *T is order continuous if and only if the null ideal N_S is a band for every operator S in the ideal $\mathcal{A}_T$ generated by T is $\mathcal{L}_\mathrm{b}(E, F)$.*
2. *T is σ-order continuous if and only if the null ideal N_S is a σ-ideal for each $S \in \mathcal{A}_T$.*

Proof. We shall only prove (1) since the proof of (2) is similar. The "only if" part follows immediately from Theorem 1.56. For the "if" part (in view of Theorem 1.56) we can assume that $T \ge 0$. Let $0 \le x_\alpha \uparrow x$ in E, and let $0 \le Tx_\alpha \uparrow y \le Tx$ in F. We must show that $y = Tx$ holds.

To this end, let $0 < \epsilon < 1$. For each α, let T_α be the operator given by Theorem 1.28 that agrees with T on the ideal generated by $(\epsilon x - x_\alpha)^+$ and vanishes on $(\epsilon x - x_\alpha)^-$. Clearly, $T \ge T_\alpha \downarrow \ge 0$, and $T_\alpha(\epsilon x - x_\alpha)^- = 0$ for each α. Let $T_\alpha \downarrow S \ge 0$ in $\mathcal{L}_\mathrm{b}(E, F)$, and note that $S \in \mathcal{A}_T$. Also, $S(\epsilon x - x_\alpha)^- = 0$ holds for each α, and so $\big\{(\epsilon x - x_\alpha)^-\big\} \subseteq N_S$. On the other hand, $0 \le (\epsilon x - x_\alpha)^- \uparrow (1-\epsilon)x$ holds in E, and hence, since by our hypothesis N_S is a band, $x \in N_S$. Therefore, $Sx = 0$. Now the relation

$$0 \le T(\epsilon x - x_\alpha)^+ = T_\alpha(\epsilon x - x_\alpha)^+ \le T_\alpha(x)\,,$$

in conjunction with Andô's inequality $0 \le x - x_\alpha \le (1-\epsilon)x + (\epsilon x - x_\alpha)^+$, yields

$$0 \le Tx - y \le T(x - x_\alpha) \le (1-\epsilon)Tx + T(\epsilon x - x_\alpha)^+ \le (1-\epsilon)Tx + T_\alpha(x)\,.$$

Taking into consideration that $T_\alpha(x) \downarrow S(x) = 0$, the preceding inequality yields $0 \le Tx - y \le (1-\epsilon)Tx$ for all $0 < \epsilon < 1$. Hence, $y = Tx$ holds, as required. ■

To illustrate the previous theorem, consider the operator $T\colon C(X_\infty) \to R$ of Example 1.60 defined by

$$T(f) = f(\infty) + \sum_{n=1}^{\infty} 2^{-n} f(x_n)\,.$$

As we have seen before, $N_T = \big\{f \in C(X_\infty)\colon\ f(x_n) = 0 \text{ for } n = 1,2,\dots\big\}$, and this shows that N_T is a band of $C(X_\infty)$. On the other hand, if $S\colon C(X_\infty) \to \mathbb{R}$ is defined by

$$S(f) = f(\infty)\,,$$

then S is a positive operator satisfying $0 \le S \le T$. Clearly, the null ideal of S is given by $N_S = \big\{f \in C(X_\infty)\colon\ f(\infty) = 0\big\}$. Now note that the net $\{\chi_\alpha\}$ of all characteristic functions of the finite subsets of X satisfies $\{\chi_\alpha\} \subseteq N_S$ and $\chi_\alpha \uparrow \mathbf{1}$. Since $\mathbf{1} \notin N_S$, we see that N_S is not a band of $C(X_\infty)$, in accordance with part (1) of Theorem 1.61.

Consider two Riesz spaces E and F with F Dedekind complete. An operator $T \in \mathcal{L}_{\rm b}(E,F)$is said to have **zero carrier** whenever $C_T = \{0\}$ (or, equivalently, whenever N_T is order dense in E). It is easy to check that the zero operator is the only order continuous operator with zero carrier. On the other hand, If $T \in \mathcal{L}_{\rm b}(E,F)$ has a zero carrier, then $T \perp \mathcal{L}_{\rm n}(E,F)$, that is, $T \in \mathcal{L}_\sigma(E,F)$. (To see this, write $T = T_{\rm n} + T_\sigma$, and note that $|T| = |T_{\rm n}| + |T_\sigma|$; see Exercise 2 of Section 1.1. Therefore, $N_T \subseteq N_{T_{\rm n}}$ holds, and so by the order denseness of N_T we see that $N_{T_{\rm n}} = E$. That is, $T_{\rm n} = 0$ and so $T = T_\sigma \in \mathcal{L}_\sigma(E,F)$.) From $|T+S| \le |T| + |S|$, it follows that $N_T \cap N_S \subseteq N_{T+S}$, and using the fact that the intersection of two order dense ideals is an order dense ideal (why?), we see that the operators of $\mathcal{L}_{\rm b}(E,F)$ with zero carriers form an ideal. The next theorem tells us that this ideal is always order dense in $\mathcal{L}_\sigma(E,F)$.

Theorem 1.62. *Let E and F be two Riesz spaces with F Dedekind complete. Then the ideal*

$$\big\{T \in \mathcal{L}_{\rm b}(E,F)\colon\ C_T = \{0\}\big\}$$

is order dense in $\mathcal{L}_\sigma(E,F)$.

Proof. We have mentioned before that the set $\{T \in \mathcal{L}_b(E,F)\colon\ C_T = \{0\}\}$ is an ideal that is included in $\mathcal{L}_\sigma(E,F)$. For the order denseness assume that $0 < T \in \mathcal{L}_\sigma(E,F)$.

Since T is not order continuous, there exists (by Theorem 1.61) an operator $0 < S \le T$ such that N_S is not a band. Denote by B the band generated by N_S. Let R be the operator determined by Theorem 1.28 such that $R = S$ on B and $R = 0$ on B^{d}. Clearly, $N_S \subseteq N_R$ and $0 < R \le S$. On the other hand, since $R = 0$ holds on $B^{\mathrm{d}} = N_S^{\mathrm{d}} = C_S$, we see that $N_S \oplus C_S \subseteq N_R$, and this (in view of Theorem 1.36) shows that N_R is order dense in E. Thus, R has zero carrier. Now to complete the proof note that $0 < R \le T$ holds. ■

The preceding theorem shows that $\mathcal{L}_\sigma(E,F) = \{0\}$ holds (or, equivalently, $\mathcal{L}_b(E,F) = \mathcal{L}_n(E,F)$) if and only if every nonzero operator from E to F has a nonzero carrier. Thus, in view of Theorem 1.61 we see that the following theorem of the authors [**12**] holds.

Theorem 1.63 (Aliprantis–Burkinshaw)**.** *For a pair of Riesz spaces E and F with F Dedekind complete, the following statements are equivalent.*

(1) *Every order bounded operator from E to F is order continuous, i.e., $\mathcal{L}_b(E,F) = \mathcal{L}_n(E,F)$.*

(2) *Every nonzero order bounded operator from E to F has a nonzero carrier.*

(3) *The null ideal of every order bounded operator from E to F is a band.*

The next result tells us when a positive operator is order continuous on a given ideal.

Theorem 1.64. *Let $T\colon E \to F$ be a positive operator between two Riesz spaces with F Dedekind complete, and let A be an ideal of E. Then the operator T is order* (*resp. σ-order*) *continuous on A if and only if T_A is an order* (*resp. σ-order*) *continuous operator.*

Proof. We establish the result for the "order continuous" case; the "σ-order continuous" case can be proven in a similar fashion. Recall that for each $x \in E^+$ the operator T_A is given (according to Theorem 1.28) by

$$T_A(x) = \sup\{T(y)\colon\ y \in A \text{ and } 0 \le y \le x\}\,.$$

Since $T_A = T$ holds on A, it should be obvious that if T_A is an order continuous operator, then T must be order continuous on A. For the converse, assume that T is order continuous on A, and let $0 \le x_\alpha \uparrow x$ in E. Let

$T_A(x_\alpha) \uparrow z \le T_A(x)$. Now fix $y \in A \cap [0, x]$. Then $0 \le y \wedge x_\alpha \uparrow y$ holds in A, and so $T(y \wedge x_\alpha) \uparrow T(y)$ holds in F. From

$$T(y \wedge x_\alpha) = T_A(y \wedge x_\alpha) \le z \le T_A(x)\,,$$

it follows that $T(y) \le z \le T_A(x)$ holds for all $y \in A \cap [0, x]$. Hence,

$$T_A(x) = \sup T(A \cap [0, x]) \le z \le T_A(x)\,,$$

and so $z = T_A(x)$, proving that T_A is an order continuous operator. ■

The final result of this section is an extension theorem for positive order continuous operators and is due to A. I. Veksler [**188**].

Theorem 1.65 (Veksler)**.** *Let G be an order dense majorizing Riesz subspace of a Riesz space E, and let F be Dedekind complete. If $T\colon G \to F$ is a positive order continuous operator, then the formula*

$$T(x) = \sup\bigl\{T(y)\colon\ y \in G \ \textit{and}\ 0 \le y \le x\bigr\}\,, \quad x \in E^+,$$

defines a unique order continuous linear extension of T to all of E.

Proof. Since G majorizes E, it is easy to see that

$$S(x) = \sup\bigl\{T(y)\colon\ y \in G \ \text{and}\ 0 \le y \le x\bigr\}$$

exists in F for each $x \in E^+$. Also, note that if $\{x_\alpha\} \subseteq G$ satisfies $0 \le x_\alpha \uparrow x$, then $T(x_\alpha) \uparrow S(x)$ holds. Indeed, if $0 \le y \in G$ satisfies $0 \le y \le x$, then $0 \le x_\alpha \wedge y \uparrow y$ holds in G, and so by the order continuity of $T\colon G \to F$ we see that

$$T(y) = \sup\bigl\{T(x_\alpha \wedge y)\bigr\} \le \sup\bigl\{T(x_\alpha)\bigr\} \le S(x)\,.$$

This easily implies that $T(x_\alpha) \uparrow S(x)$.

Now let $x, y \in E^+$. Pick two nets $\{x_\alpha\}$ and $\{y_\beta\}$ of G^+ with $0 \le x_\alpha \uparrow x$ and $0 \le y_\beta \uparrow y$ (see Theorem 1.34). Then $0 \le x_\alpha + y_\beta \uparrow x + y$ holds, and so by the above discussion

$$T(x_\alpha) + T(y_\beta) = T(x_\alpha + y_\beta) \uparrow S(x + y)\,.$$

From $T(x_\alpha) \uparrow S(x)$ and $T(y_\beta) \uparrow S(y)$, we get $S(x+y) = S(x) + S(y)$. That is, $S\colon E^+ \to F^+$ is additive, and thus by Theorem 1.10 it extends uniquely to a positive operator from E to F. Clearly, S is an extension of T.

Finally, it remains to be shown that S is order continuous. To this end, let $0 \le x_\alpha \uparrow x$ in E. Put

$$D = \bigl\{y \in G^+\colon\ \text{there exists some}\ \alpha\ \text{with}\ y \le x_\alpha\bigr\}\,,$$

and note that $\sup T(D) \le \sup\bigl\{S(x_\alpha)\bigr\} \le S(x)$ holds in F. Since G is order dense in E, it is easy to see that $D \uparrow x$ holds. Thus, by the above discussion $\sup T(D) = S(x)$, and so $S(x_\alpha) \uparrow S(x)$, proving that S is order continuous. The proof of the theorem is now complete. ■

Exercises

1. A Riesz space is said to have the **countable sup property**, if whenever an arbitrary subset D has a supremum, then there exists an at most countable subset C of D with $\sup C = \sup D$.

(a) Show that if F is an Archimedean Riesz space with the countable sup property and $T\colon E \to F$ is a **strictly positive operator** (i.e., $x > 0$ implies $Tx > 0$), then E likewise has the countable sup property.

(b) Let $T\colon E \to F$ be a positive operator between two Riesz spaces with E having the countable sup property. Then show that T is order continuous if and only if T is σ-order continuous.

2. Let E be Dedekind σ-complete, and let F be **super Dedekind complete** (i.e., let F be Dedekind complete with the countable sup property), and let $T\colon E \to F$ be a positive σ-order continuous operator. Show that:

(a) C_T is a super Dedekind complete Riesz space and that T restricted to C_T is strictly positive and order continuous.

(b) C_T is a projection band.

(c) T is order continuous if and only if N_T is a band.

3. Let E and F be two Riesz spaces with F Dedekind complete. Consider the band $\mathcal{L}_{c\sigma}(E,F) := \mathcal{L}_c(E,F) \cap \mathcal{L}_\sigma(E,F)$, and note that

$$\mathcal{L}_b(E,F) = \mathcal{L}_n(E,F) \oplus \mathcal{L}_{c\sigma}(E,F) \oplus \mathcal{L}_s(E,F)\,.$$

Thus, every operator $T \in \mathcal{L}_b(E,F)$ has a unique decomposition of the form $T = T_n + T_{c\sigma} + T_s$, where $T_n \in \mathcal{L}_n(E,F)$, $T_{c\sigma} \in \mathcal{L}_{c\sigma}(E,F)$, and $T_s \in \mathcal{L}_s(E,F)$. Clearly, $T_c = T_n + T_{c\sigma}$ and $T_\sigma = T_{c\sigma} + T_s$ hold.

If F is super Dedekind complete and $T \in \mathcal{L}_c(E,F)$, then prove the following statements.

(a) $T \in \mathcal{L}_{c\sigma}(E,F)$ if and only if $C_T = \{0\}$ (or, equivalently, if and only if N_T is order dense in E).

(b) $N_T \oplus C_T \subseteq N_{T_{c\sigma}}$.

(c) The largest ideal of E on which T is order continuous is the order dense ideal $N_{T_{c\sigma}}$.

4. Let $T\colon E \to F$ be a positive operator between two Riesz spaces with F Dedekind complete. Then show that:

(a) In the formula

$$T_n(x) = \inf\bigl\{\sup T(x_\alpha)\colon\ 0 \le x_\alpha \uparrow x\bigr\}\,,$$

the greatest lower bound is attained for each $x \in E^+$ if and only if N_{T_σ} is order dense in E.

(b) In the formula

$$T_c(x) = \inf\bigl\{\sup T(x_n)\colon\ 0 \le x_n \uparrow x\bigr\}\,,$$

the greatest lower bound is attained for each $x \in E^+$ if and only if N_{T_s} is super order dense in E. (Recall that an ideal A in a Riesz space E is said to be **super order dense** whenever for each $x \in E^+$ there exists a sequence $\{x_n\} \subseteq A$ such that $0 \le x_n \uparrow x$.)

5. Let E and F be two Riesz spaces with F Dedekind complete. Show that for each $T \in \mathcal{L}_b(E,F)$ the ideal N_{T_σ} (resp. N_{T_s}) is the largest ideal of E on which T is order (resp. σ-order) continuous.

6. Consider the operator T of Example 1.60. Show that T is σ-order continuous if and only if X is an uncountable set.

7. For a pair of Riesz spaces E and F with F Dedekind complete show that the following statements are equivalent.
(a) Every order bounded operator from E to F is σ-order continuous.
(b) The null ideal of every order bounded operator from E to F is a σ-ideal.

8. Let $T\colon E \to E$ be an order continuous positive operator on a Riesz space, and let $\{T_\alpha\}$ be a net of positive order continuous operators from E to E satisfying $T_\alpha(x) \uparrow T(x)$ in E for each $x \in E^+$. Show that:
(a) If $0 \le x_\lambda \uparrow x$ in E, then $T_\alpha(x_\lambda) \uparrow_{\alpha,\lambda} T(x)$ holds in E.
(b) If $x \in E^+$, then $T_\alpha^k(x) \uparrow T^k(x)$ holds in E for each k.
Also, establish the sequential analogues of the above statements.

9. Let $T\colon E \to F$ be a positive operator between two Riesz spaces with F Dedekind complete. Then show that the components T_σ and T_s of T for each $x \in E^+$ are given by the formulas

$$T_\sigma(x) = \sup\bigl\{\inf T(x_\alpha)\colon\ x \ge x_\alpha \downarrow 0\bigr\}$$

and

$$T_s(x) = \sup\bigl\{\inf T(x_n)\colon\ x \ge x_n \downarrow 0\bigr\}\,.$$

10. Show that an order bounded operator $T\colon E \to F$ between two Riesz spaces with F Dedekind complete is order continuous if and only if $T \perp S$ holds for each operator $S \in \mathcal{L}_b(E,F)$ with $C_S = \{0\}$.

11. As usual, if $\{x_\alpha\}$ is an order bounded net in a Dedekind complete Riesz space, then we define

$$\limsup x_\alpha := \bigwedge_\alpha \bigvee_{\beta \succeq \alpha} x_\beta \qquad \text{and} \qquad \liminf x_\alpha := \bigvee_\alpha \bigwedge_{\beta \succeq \alpha} x_\beta\,.$$

(a) Show that in a Dedekind complete Riesz space an order bounded net $\{x_\alpha\}$ satisfies $x_\alpha \xrightarrow{o} x$ if and only if $x = \limsup x_\alpha = \liminf x_\alpha$.
(b) If $T\colon E \to F$ is a positive operator between two Riesz spaces with F Dedekind complete, then show that

$$T_c(x) = \inf\bigl\{\liminf T(x_n)\colon\ 0 \le x_n \le x \ \text{and}\ x_n \xrightarrow{o} x\bigr\}$$

and

$$T_n(x) = \inf\bigl\{\liminf T(x_\alpha)\colon\ 0 \le x_\alpha \le x \ \text{and}\ x_\alpha \xrightarrow{o} x\bigr\}$$

hold for each $x \in E^+$.

12. For two Riesz spaces E and F with F Dedekind complete establish the following:
(a) If A is an ideal of E, then its **annihilator**

$$A^{\mathrm{o}} := \bigl\{T \in \mathcal{L}_b(E,F)\colon\ T = 0 \ \text{on}\ A\bigr\}$$

is a band of $\mathcal{L}_b(E,F)$.

(b) If $\mathcal{A}$ is an ideal of $\mathcal{L}_b(E,F)$, then its **inverse annihilator**

$$^{\circ}\mathcal{A} := \{x \in E\colon\ T(x) = 0 \text{ for each } T \in \mathcal{A}\}$$

is an ideal of E.

(c) Every order bounded operator from E to F is order continuous (i.e., $\mathcal{L}_b(E,F) = \mathcal{L}_n(E,F)$ holds) if and only if for every order dense ideal A of E we have $A^{\circ} = \{0\}$.

13. Consider two Riesz spaces E and F with F Dedekind complete. As usual, we say that $\mathcal{L}_b(E,F)$ **separates the points** of E whenever for each $x \neq 0$ in E there exists some $T \in \mathcal{L}_b(E,F)$ with $T(x) \neq 0$.

Show that if $\mathcal{L}_b(E,F)$ separates the points of E and $(^{\circ}\mathcal{B})^{\circ} = \mathcal{B}$ holds for each band $\mathcal{B}$ of $\mathcal{L}_b(E,F)$ (for notation see the preceding exercise), then every order bounded operator from E to F is order continuous.

1.5. Positive Linear Functionals

Let E be a Riesz space. A linear functional $f\colon E \to \mathbb{R}$ is said to be **positive** whenever $f(x) \geq 0$ holds for each $x \in E^+$. Also, a linear functional f is called **order bounded** if f maps order bounded subsets of E to bounded subsets of $\mathbb{R}$. The vector space $E^\sim$ of all order bounded linear functionals on E is called the **order dual** of E, i.e., $E^\sim = \mathcal{L}_b(E,\mathbb{R})$. Since $\mathbb{R}$ is a Dedekind complete Riesz space, it follows at once from Theorem 1.18 that $E^\sim$ is precisely the vector space generated by the positive linear functionals. Moreover, $E^\sim$ is a Dedekind complete Riesz space. Recall that $f \geq g$ in $E^\sim$ means $f(x) \geq g(x)$ for all $x \in E^+$. Also, note that if $f, g \in E^\sim$ and $x \in E^+$, then according to Theorem 1.18 we have:

(1) $f^+(x) = \sup\{f(y)\colon\ 0 \leq y \leq x\}$.

(2) $f^-(x) = \sup\{-f(y)\colon\ 0 \leq y \leq x\}$.

(3) $|f|(x) = \sup\{|f(y)|\colon\ |y| \leq x\}$.

(4) $[f \vee g](x) = \sup\{f(y) + g(z)\colon\ y, z \in E^+ \text{ and } y + z = x\}$.

(5) $[f \wedge g](x) = \inf\{f(y) + g(z)\colon\ y, z \in E^+ \text{ and } y + z = x\}$.

Observe that from formula (5) the following important characterization of disjointness in $E^\sim$ holds: For $f, g \in E^\sim$ we have $f \perp g$ if and only if for each $\epsilon > 0$ and each $x \in E^+$ there exist $y, z \in E^+$ with $y + z = x$ and $|f|(y) < \epsilon$ and $|g|(z) < \epsilon$.

The order dual $E^\sim$ may happen to be trivial. For instance, if $0 < p < 1$, then it has been shown by M. M. Day that the Riesz space $E = L_p[0,1]$ satisfies $E^\sim = \{0\}$; see our book [**7**, Theorem 5.24, p. 128]. In this book, Riesz spaces with trivial order dual will be of little interest. As a matter of fact, we are interested in Riesz spaces whose order duals separate the points of the spaces. Recall that the expression $E^\sim$ **separates the points** of E means that for each $x \neq 0$ there exists some $f \in E^\sim$ with $f(x) \neq 0$. Since E is a Riesz space, it is easy to see that $E^\sim$ separates the points of E if and only if for each $0 < x \in E$ there exists some $0 < f \in E^\sim$ with $f(x) \neq 0$.

Theorem 1.66. *If $E^\sim$ separates the points of the Riesz space E, then a vector $x \in E$ satisfies $x \geq 0$ if and only if $f(x) \geq 0$ holds for all $0 \leq f \in E^\sim$.*

Proof. Clearly, if $x \geq 0$ holds, then $f(x) \geq 0$ likewise holds for every $0 \leq f \in E^\sim$.

For the converse, assume that some vector $x \in E$ satisfies $f(x) \geq 0$ for all $0 \leq f \in E^\sim$. If $0 \leq f \in E^\sim$ is fixed, then by Theorem 1.23 there exists some $0 \leq g \leq f$ with $f(x^-) = -g(x)$. Since by our hypothesis $g(x) \geq 0$ holds, it follows that $0 \leq f(x^-) = -g(x) \leq 0$, and so $f(x^-) = 0$ holds for all $0 \leq f \in E^\sim$. Since $E^\sim$ separates the points of E, we see that $x^- = 0$. Consequently, $x = x^+ - x^- = x^+ \geq 0$ holds, and the proof is finished. ■

Besides the order dual of a Riesz space, we shall need to consider the bands of order continuous and σ-order continuous linear functionals.

Let E be a Riesz space. The vector space $\mathcal{L}_{\rm n}(E, \mathbb{R})$ of all order continuous linear functionals on E will be denoted by $E_{\rm n}^\sim$. Similarly, the vector space $\mathcal{L}_{\rm c}(E, \mathbb{R})$ of all σ-order continuous linear functionals on E will be denoted by $E_{\rm c}^\sim$. That is,

$$E_{\rm n}^\sim := \mathcal{L}_{\rm n}(E, \mathbb{R}) \quad \text{and} \quad E_{\rm c}^\sim := \mathcal{L}_{\rm c}(E, \mathbb{R})\,.$$

Note that a positive linear functional f on E is order continuous if and only if $x_\alpha \downarrow 0$ in E implies $f(x_\alpha) \downarrow 0$ in $\mathbb{R}$. Likewise, f is σ-order continuous if and only if for every sequence $\{x_n\}$ with $x_n \downarrow 0$ we have $f(x_n) \downarrow 0$ in $\mathbb{R}$. Clearly, we have

$$E_{\rm n}^\sim \subseteq E_{\rm c}^\sim \subseteq E^\sim\,.$$

By Theorem 1.57 both $E_{\rm c}^\sim$ and $E_{\rm n}^\sim$ are bands of $E^\sim$. The band $E_{\rm n}^\sim$ will be referred to as the **order continuous dual** of E, and the band $E_{\rm c}^\sim$ as the **σ-order continuous dual** of E.

Here are two examples of Riesz spaces and their duals. (For a justification of their duals see Section 4.1.)

(1) Let $1 \leq p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$.
   (a) If $E = \ell_p$, then $E^\sim = E_{\rm c}^\sim = E_{\rm n}^\sim = \ell_q$; and
   (b) if $E = L_p[0,1]$, then $E^\sim = E_{\rm c}^\sim = E_{\rm n}^\sim = L_q[0,1]$.

(2) Consider $E = C[0,1]$. Then $E_{\rm c}^\sim = E_{\rm n}^\sim = \{0\}$, and $E^\sim$ is the Riesz space of all regular Borel measures on $[0,1]$.

Recall that the null ideal of an arbitrary linear functional $f \in E^\sim$ is the ideal $N_f := \big\{x \in E\colon\ |f|(|x|) = 0\big\}$, and its carrier is the band $C_f := N_f^{\rm d}$.

H. Nakano [**150**, Theorem 20.1, p. 74] has shown that two linear functionals in $E_{\mathrm{n}}^{\sim}$ are disjoint if and only if their carriers are disjoint sets. This remarkable result is stated next.

Theorem 1.67 (Nakano). *If E is Archimedean, then for a pair $f, g \in E_{\mathrm{n}}^{\sim}$ the following statements are equivalent.*

(1) $f \perp g$.

(2) $C_f \subseteq N_g$.

(3) $C_g \subseteq N_f$.

(4) $C_f \perp C_g$.

Proof. Without loss of generality we can assume that $0 \leq f, g \in E_{\mathrm{n}}^{\sim}$.

(1) $\Longrightarrow$ (2) Let $0 \leq x \in C_f = N_f^{\mathrm{d}}$, and let $\epsilon > 0$. In view of $f \wedge g = 0$, there exists a sequence $\{x_n\} \subseteq E^+$ satisfying

$$0 \leq x_n \leq x \quad \text{and} \quad f(x_n) + g(x - x_n) < 2^{-n}\epsilon \ \text{for all}\ n\,.$$

Put $y_n = \bigwedge_{i=1}^{n} x_i$, and note that $y_n \downarrow 0$ in E. Indeed, if $0 \leq y \leq y_n$ holds for all n, then $0 \leq f(y) \leq f(y_n) < 2^{-n}\epsilon$ also holds for all n, and consequently $f(y) = 0$. Thus, $y \in C_f \cap N_f = \{0\}$, and so $y = 0$.

Now since $0 \leq g \in E_{\mathrm{n}}^{\sim}$, we see that $g(x - y_n) \uparrow g(x)$. On the other hand, from

$$0 \leq g(x - y_n) = g\Big(\bigvee_{i=1}^{n}(x - x_i)\Big) \leq \sum_{i=1}^{n} g(x - x_i) < \epsilon\,,$$

it follows that $0 \leq g(x) \leq \epsilon$ holds for all $\epsilon > 0$. Thus, $g(x) = 0$, so that $C_f \subseteq N_g$ holds.

(2) $\Longrightarrow$ (3) Since N_f is a band, it follows from $C_f = N_f^{\mathrm{d}} \subseteq N_g$ and Theorem 1.39 that

$$C_g = N_g^{\mathrm{d}} \subseteq N_f^{\mathrm{dd}} = N_f\,.$$

(3) $\Longrightarrow$ (4) Since $C_g \subseteq N_f$ is true by our hypothesis and $N_f \perp C_f$, we see that $C_g \perp C_f$ holds.

(4) $\Longrightarrow$ (1) From $C_f \perp C_g$ it follows that $C_g \subseteq C_f^{\mathrm{d}} = N_f^{\mathrm{dd}} = N_f$. Now if $0 \leq x = y + z \in N_g \oplus C_g$, then

$$0 \leq [f \wedge g](x) = [f \wedge g](y) + [f \wedge g](z) \leq g(y) + f(z) = 0\,,$$

and thus $f \wedge g = 0$ holds on the order dense ideal $N_g \oplus C_g$ (see Theorem 1.36). Since $f \wedge g \in E_{\mathrm{n}}^{\sim}$, it follows that $[f \wedge g](x) = 0$ holds for all $x \in E$, and the proof is finished. ■

It should be noted that the above proof of the implication (4) $\Longrightarrow$ (1) shows that the following general result is true.

- *If two positive order continuous operators S and T satisfy $C_S \perp C_T$, then $S \perp T$.*

However, as the next example shows, the converse is not true.

Example 1.68. Let $A = \left[0, \frac{1}{2}\right]$ and $B = \left[\frac{1}{2}, 1\right]$ and consider the two positive operators $S, T \colon L_1[0,1] \to L_1[0,1]$ defined by

$$S(f) = \Big[\int_0^1 f(x)\,dx\Big]\chi_A \quad \text{and} \quad T(f) = \Big[\int_0^1 f(x)\,dx\Big]\chi_B \,.$$

The Lebesgue dominated convergence theorem shows that S and T are both order continuous operators. On the other hand, note that if $0 \le f \in L_1[0,1]$, then we have

$$0 \le [S \wedge T](f) \le S(f) \wedge T(f) = \Big[\int_0^1 f(x)\,dx\Big] \cdot \chi_A \wedge \chi_B = 0\,,$$

and so $S \wedge T = 0$ holds in $\mathcal{L}_{\rm b}(L_1[0,1])$.

Finally, note that $N_S = N_T = \{0\}$, and so $C_S = C_T = L_1[0,1]$, proving that C_S and C_T are not disjoint sets. ■

If E is a Riesz space, then its order dual $E^\sim$ is again a Riesz space. Thus, we can consider the Riesz space of all order bounded linear functionals on $E^\sim$. The **second order dual** $E^{\sim\sim}$ of E is the order dual of $E^\sim$, that is, $E^{\sim\sim} := (E^\sim)^\sim$. For each $x \in E$ an order bounded linear functional $\widehat{x}$ can be defined on $E^\sim$ via the formula

$$\widehat{x}(f) := f(x)\,, \quad f \in E^\sim\,.$$

Clearly, $x \ge 0$ implies $\widehat{x} \ge 0$. Also, since $f_\alpha \downarrow 0$ in $E^\sim$ holds if and only if $\widehat{x}(f_\alpha) = f_\alpha(x) \downarrow 0$ for all $x \in E^+$, it easily follows that each $x \in E$ defines an order continuous linear functional on $E^\sim$. Thus, a positive operator $x \mapsto \widehat{x}$ can be defined from E to $E^{\sim\sim}$. This operator is called the **canonical embedding** of E into $E^{\sim\sim}$. The canonical embedding always preserves finite suprema and infima, and when $E^\sim$ separates the points of E, it is also one-to-one. The details follow.

Theorem 1.69. *Let E be a Riesz space. Then the canonical embedding $x \mapsto \widehat{x}$ is a lattice preserving operator (from E to $E^{\sim\sim}$).*

In particular, if $E^\sim$ separates the points of E, then $x \mapsto \widehat{x}$ is also one-to-one (and hence, in this case E, identified with its canonical image in $E^{\sim\sim}$, can be considered as a Riesz subspace of $E^{\sim\sim}$).

Proof. Only the preservation of the lattice operations needs verification. To this end, let $x \in E$ and $0 \le f \in E^\sim$. Applying Theorems 1.18 and 1.23

consecutively, we see that

$$\begin{aligned}(\widehat{x})^+(f) &= \sup\bigl\{\widehat{x}(g):\ g \in E^\sim \text{ and } 0 \le g \le f\bigr\}\\ &= \sup\bigl\{g(x):\ g \in E^\sim \text{ and } 0 \le g \le f\bigr\}\\ &= f(x^+) = \widehat{(x^+)}(f)\,.\end{aligned}$$

That is, $(\widehat{x})^+ = \widehat{(x^+)}$ holds. Now by using the lattice identity

$$x \vee y = (x-y)^+ + y = -\bigl[(-x) \wedge (-y)\bigr]\,,$$

we see that the canonical embedding $x \mapsto \widehat{x}$ preserves finite suprema and infima. ■

It should be noted that the canonical embedding of E into $E^{\sim\sim}$ does not necessarily preserve infinite suprema and infima; see Exercise 10 at the end of this section. In the sequel the vectors of a Riesz space E will play a double role. Besides being the vectors of E, they also will be considered (by identifying x with $\widehat{x}$) as order bounded linear functionals on $E^\sim$.

Now let E be a Riesz space, and let A be an ideal of $E^\sim$. Then it is easy to see that for each $x \in E$, the restriction of $\widehat{x}$ to A defines an order continuous linear functional (and hence order bounded) on A. Therefore, there exists a natural embedding $x \mapsto \widehat{x}$ of E into $A_{\rm n}^\sim$ defined by

$$\widehat{x}(f) := f(x)\,,\quad f \in A\,.$$

As in Theorem 1.69 we can see that the natural embedding $x \mapsto \widehat{x}$, from E into $A_{\rm n}^\sim$, is lattice preserving and is one-to-one if and only if the ideal A separates the points of E.

When A consists of order continuous linear functionals, H. Nakano [**150**, Theorem 22.6, p. 83] has shown (among other things) that $x \mapsto \widehat{x}$ preserves arbitrary suprema and infima. The details are included in the next theorem.

Theorem 1.70 (Nakano)**.** *Let E be an Archimedean Riesz space, and let A be an ideal of $E_{\rm n}^\sim$. Then the embedding $x \mapsto \widehat{x}$ is an order continuous lattice preserving operator from E to $A_{\rm n}^\sim$ whose range is an order dense Riesz subspace of $A_{\rm n}^\sim$.*

Proof. To see that $x \mapsto \widehat{x}$ is order continuous, note that if $x_\alpha \downarrow 0$ holds in E, then $\widehat{x}_\alpha(f) = f(x_\alpha) \downarrow 0$ holds for each $0 \le f \in A$, and so $\widehat{x}_\alpha \downarrow 0$ holds in $A_{\rm n}^\sim$. That is, $x \mapsto \widehat{x}$ is an order continuous operator.

Now let us establish that the range of $x \mapsto \widehat{x}$ is an order dense Riesz subspace of $A_{\rm n}^\sim$. To this end, let $0 < \phi \in A_{\rm n}^\sim$. Pick some $0 < f \in C_\phi$, and then choose $0 < x \in C_f$. Clearly, $f(x) > 0$. If $\widehat{x} \wedge \phi = 0$ holds, then by Theorem 1.67 we have $\widehat{x}(C_\phi) = \{0\}$, and so $\widehat{x}(f) = f(x) = 0$, a contradiction. Thus, $\widehat{x} \wedge \phi > 0$ holds, and hence, by replacing ϕ with $\widehat{x} \wedge \phi$, we can assume that $0 < \phi \le \widehat{x}$ holds in $A_{\rm n}^\sim$ for some $x \in E$. Next fix some

$0 < \epsilon < 1$ with $\psi = (\phi - \epsilon x)^+ > 0$. Choose some $0 < g \in C_\psi$, and then select some $0 < y \in C_g$. We claim that the vector $z = y \wedge \epsilon x \in E$ satisfies $0 < \widehat{z} \leq \phi$ in $A_n^\sim$.

To see that $\widehat{z} > 0$ holds, note that if $\widehat{z} = \widehat{y} \wedge \epsilon\widehat{x} = 0$, then $\widehat{y} \wedge \widehat{x} = 0$, and so in view of $0 \leq \psi \leq \widehat{x}$, we see that $\widehat{y} \wedge \psi = 0$. By Theorem 1.67 we have $\widehat{y}(C_\psi) = \{0\}$, and hence $\widehat{y}(f) = f(y) = 0$, a contradiction. Thus, $\widehat{z} > 0$.

Finally, let us establish that $\widehat{z} \leq \phi$ holds. To this end, assume by way of contradiction that $\omega = (\widehat{z} - \phi)^+ > 0$. Choose $0 < h \in C_\omega$, and note that, in view of $0 < \omega \leq (\epsilon\widehat{x} - \phi)^+ = (\phi - \epsilon\widehat{x})^-$, we have $\omega \perp \psi$ and so by Theorem 1.67 we get $C_\omega \perp C_\psi$. In particular, $h \perp g$ holds, and by applying Theorem 1.67 once more, we get $h(C_g) = \{0\}$. Therefore,

$$0 < \omega(h) = (\widehat{z} - \phi)^+(h) \leq \widehat{z}(h) \leq \widehat{y}(h) = h(y) = 0$$

holds, which is impossible. Hence, $\widehat{z} \leq \phi$, and the proof is complete. ■

As an application of Theorem 1.70, we shall characterize the perfect Riesz spaces. A Riesz space E is said to be **perfect** whenever the natural embedding $x \mapsto \widehat{x}$ from E to $(E_n^\sim)_n^\sim$ is one-to-one and onto. Clearly, every perfect Riesz space must be Dedekind complete. H. Nakano [**150**, Section 24] has characterized the perfect Riesz spaces as follows.

Theorem 1.71 (Nakano)**.** *A Riesz space E is a perfect Riesz space if and only if the following two conditions hold:*

(1) *$E_n^\sim$ separates the points of E.*

(2) *Whenever a net $\{x_\alpha\} \subseteq E$ satisfies $0 \leq x_\alpha \uparrow$ and $\sup\{f(x_\alpha)\} < \infty$ for each $0 \leq f \in E_n^\sim$, then there exists some $x \in E$ satisfying $0 \leq x_\alpha \uparrow x$ in E.*

Proof. Assume that E is a perfect Riesz space, i.e., assume that $x \mapsto \widehat{x}$ from E to $(E_n^\sim)_n^\sim$ is one-to-one and onto. Then, clearly, $E_n^\sim$ separates the points of E. On the other hand, if a net $\{x_\alpha\} \subseteq E^+$ satisfies $0 \leq x_\alpha \uparrow$ and $\phi(f) = \sup\{f(x_\alpha)\} < \infty$ for each $0 \leq f \in E_n^\sim$, then it easily follows that the mapping $\phi\colon (E_n^\sim)^+ \to \mathbb{R}^+$ is additive, and hence ϕ defines a positive linear functional on $E_n^\sim$. In view of $\widehat{x}_\alpha \uparrow \phi$ in $(E_n^\sim)^\sim$, it follows (from Theorem 1.57) that $\phi \in (E_n^\sim)_n^\sim$. Pick some $x \in E$ with $\phi = \widehat{x}$, and note that $0 \leq x_\alpha \uparrow x$ holds in E.

For the converse assume that E satisfies the two conditions. Then, by Theorem 1.70, the operator $x \mapsto \widehat{x}$ from E to $(E_n^\sim)_n^\sim$ is order continuous, one-to-one, and lattice preserving whose range is order dense in $(E_n^\sim)_n^\sim$. Now let $0 \leq \phi \in (E_n^\sim)_n^\sim$. Pick a net $\{x_\alpha\} \subseteq E^+$ with $0 \leq \widehat{x}_\alpha \uparrow \phi$ in $(E_n^\sim)_n^\sim$. Then $\{x_\alpha\}$ satisfies condition (2), and so there exists some $x \in E$ with $0 \leq x_\alpha \uparrow x$

in E. It follows that $0 \le \widehat{x}_\alpha \uparrow \widehat{x}$ holds in $(E_n^\sim)_n^\sim$, and thus $\phi = \widehat{x}$, proving that $x \mapsto \widehat{x}$ is also onto. ■

The (order bounded) finite rank operators will be of great importance. If $f \in E^\sim$ and $u \in F$, then the symbol $f \otimes u$ will denote the order bounded operator of $\mathcal{L}_b(E,F)$ defined by

$$[f \otimes u](x) := f(x)u$$

for each $x \in E$. Every operator of the form $f \otimes u$ is referred to as a **rank one operator**. Note that if $f \in E_n^\sim$ (resp. $f \in E_c^\sim$), then $f \otimes u$ is an order (resp. σ-order) continuous operator. Every operator $T\colon E \to F$ of the form $T = \sum_{i=1}^n f_i \otimes u_i$, where $f_i \in E^\sim$ and $u_i \in F$ $(i = 1, \ldots, n)$, is called a **finite rank operator**. In general, if G is a vector subspace of $E^\sim$, then we define

$$G \otimes F := \Big\{ T \in \mathcal{L}(E,F)\colon\ \exists\, n,\ f_i \in G,\ u_i \in F\ (1 \le i \le n) \text{ with } T = \sum_{i=1}^n f_i \otimes u_i \Big\}.$$

Clearly, $G \otimes F$ is a vector subspace of $\mathcal{L}_b(E,F)$.

The next theorem describes some basic lattice properties of the rank one operators.

Theorem 1.72. *For a pair of Riesz spaces E and F we have the following:*

(1) *If $0 \le f \in E^\sim$ and $u, v \in F$, then $(f \otimes u) \vee (f \otimes v)$ and $(f \otimes u) \wedge (f \otimes v)$ both exist in $\mathcal{L}(E,F)$ and*
$$(f \otimes u) \vee (f \otimes v) = f \otimes (u \vee v)$$
and
$$(f \otimes u) \wedge (f \otimes v) = f \otimes (u \wedge v)\,.$$

(2) *If $0 \le u \in F$ and $f, g \in E^\sim$, then $(f \otimes u) \vee (g \otimes u)$ and $(f \otimes u) \wedge (g \otimes u)$ both exist in $\mathcal{L}(E,F)$ and*
$$(f \otimes u) \vee (g \otimes u) = (f \vee g) \otimes u$$
and
$$(f \otimes u) \wedge (g \otimes u) = (f \wedge g) \otimes u\,.$$

(3) *If $f \in E^\sim$ and $u \in F$, then the modulus of $f \otimes u$ exists in $\mathcal{L}(E,F)$ and*
$$\big|f \otimes u\big| = |f| \otimes |u|\,.$$

Proof. (1) Let $0 \le f \in E^\sim$, and let $u, v \in F$. Clearly, $f \otimes u \le f \otimes (u \vee v)$ and $f \otimes v \le f \otimes (u \vee v)$ both hold. On the other hand, if some $T \in \mathcal{L}(E,F)$ satisfies $f \otimes u \le T$ and $f \otimes v \le T$, then for each $x \in E^+$ we have

$$\begin{aligned} \big[f \otimes (u \vee v)\big](x) &= f(x)(u \vee v) = \big[f(x)u\big] \vee \big[f(x)v\big] \\ &\le T(x) \vee T(x) = T(x)\,. \end{aligned}$$

That is, $f \otimes (u \vee v) \le T$ holds in $\mathcal{L}_b(E,F)$, and so $f \otimes (u \vee v)$ is the least upper bound of $f \otimes u$ and $f \otimes v$ in $\mathcal{L}(E,F)$, as required. The other case can be proven in a similar manner.

(2) Fix $u \in F^+$ and $f, g \in E^\sim$. Clearly, $f \otimes u \le (f \vee g) \otimes u$ and $g \otimes u \le (f \vee g) \otimes u$. Now let $T \in \mathcal{L}(E,F)$ satisfy $f \otimes u \le T$ and $g \otimes u \le T$. Observe that if $y, z \in E^+$ satisfy $y + z = x$, then

$$[f \otimes u](y) + [g \otimes u](z) \le T(y) + T(z) = T(x)$$

holds. Thus, for each $x \in E^+$ we have

$$\begin{aligned}\big[(f \vee g) \otimes u\big](x) &= \big[(f \vee g)(x)\big] \cdot u \\ &= \big[\sup\big\{f(y) + g(z)\colon\ y, z \in E^+ \text{ and } y + z = x\big\}\big] \cdot u \\ &= \sup\big\{f(y)u + g(z)u\colon\ y, z \in E^+ \text{ and } y + z = x\big\} \\ &= \sup\big\{[f \otimes u](y) + [g \otimes u](z)\colon\ y, z \in E^+ \text{ and } y+z=x\big\} \\ &\le T(x)\,.\end{aligned}$$

Therefore, $(f \vee g) \otimes u$ is the least upper bound of $f \otimes u$ and $g \otimes u$ in $\mathcal{L}(E,F)$. The other formula can be proven in a similar fashion.

(3) For each $x \in E^+$ we have

$$\begin{aligned}\pm\big[f \otimes u\big](x) &= \pm[f(x) \cdot u] \le \big|f(x)u\big| = \big|f(x)\big| \cdot |u| \\ &\le |f|(x) \cdot |u| = \big[|f| \otimes |u|\big](x)\,,\end{aligned}$$

and so $\pm[f \otimes u] \le |f| \otimes |u|$. Now assume that some $T \in \mathcal{L}(E,F)$ satisfies

$$f \otimes u \le T \quad \text{ and } \quad -[f \otimes u] \le T\,.$$

Let $x \in E^+$. If $f(x) < 0$, then $[f \otimes |u|](x) \le T(x)$ holds trivially. On the other, if $f(x) \ge 0$, then we have

$$[f \otimes |u|](x) = f(x)|u| = \big[f(x)u\big] \vee \big[-f(x)u\big] \le T(x)\,.$$

Therefore, $f \otimes |u| \le T$ holds. By the symmetry of the situation we have $(-f) \otimes |u| \le T$. Thus, by part (2) we see that

$$|f| \otimes |u| = \big[f \otimes |u|\big] \vee \big[(-f) \otimes |u|\big] \le T\,.$$

Consequently, $|f| \otimes |u|$ is the least upper bound of $f \otimes u$ and $-f \otimes u$. That is, $|f \otimes u| = |f| \otimes |u|$ holds in $\mathcal{L}(E,F)$. ■

Recall that the **algebraic dual** V^* of a vector space V is the vector space consisting of all linear functionals on V. For an operator $T\colon V \to W$ between two vector spaces its **algebraic adjoint** (or **transpose**) $T^*\colon W^* \to V^*$ is the operator defined by

$$[T^* f](v) = f(Tv)$$

for all $f \in W^*$ and $v \in V$. In standard duality notation this identity is written as

$$\langle T^* f, v\rangle = \langle f, Tv\rangle .$$

Clearly, if $S\colon V \to W$ is another operator and $\alpha \in \mathbb{R}$, then

$$(S+T)^* = S^* + T^* \quad \text{and} \quad (\alpha T)^* = \alpha T^* .$$

When $T\colon E \to F$ is an order bounded operator between two Riesz spaces, then T^* carries $F^\sim$ into $E^\sim$. Indeed, if A is an order bounded subset of E and $f \in F^\sim$, then it follows from $[T^* f](A) = f\big(T(A)\big)$ that $[T^* f](A)$ is a bounded subset of $\mathbb{R}$, and so $T^* f \in E^\sim$. The restriction of T^* to $F^\sim$ is called the (**order**) **adjoint** of T and will be denoted by T'. That is, $T'\colon F^\sim \to E^\sim$ satisfies

$$\langle T' f, x\rangle = \langle f, Tx\rangle$$

for all $f \in F^\sim$ and $x \in E$. Note that if T is a positive operator, then its adjoint T' is likewise a positive operator.

The adjoint of an order bounded operator between two Riesz spaces is always order bounded and order continuous. The details follow.

Theorem 1.73. *If $T\colon E \to F$ is an order bounded operator between two Riesz spaces, then its* (*order*) *adjoint $T'\colon F^\sim \to E^\sim$ is order bounded and order continuous.*

Proof. Assume that $T\colon E \to F$ is an order bounded operator. We shall first establish that $T'\colon F^\sim \to E^\sim$ is order bounded.

To this end, let $0 \le f \in F^\sim$. Consider the set

$$D = \Big\{ \sum_{i=1}^{n} |T' f_i| \colon\ f_i \ge 0 \text{ for each } i \text{ and } \sum_{i=1}^{n} f_i = f \Big\} .$$

We claim that $D \uparrow$ holds in $E^\sim$. To see this, let $f_1, \dots, f_n \in F_+^\sim$ and $g_1, \dots, g_m \in F_+^\sim$ satisfy $\sum_{i=1}^n f_i = \sum_{j=1}^m g_j = f$. By Theorem 1.20 there exist linear functionals $h_{ij} \in F_+^\sim$ $(i = 1, \dots, n;\ j = 1, \dots, m)$ such that

$$f_i = \sum_{j=1}^{m} h_{ij} \text{ for } i = 1, \dots, n \quad \text{and} \quad g_j = \sum_{i=1}^{n} h_{ij} \text{ for } j = 1, \dots, m .$$

Clearly, $\sum_{i=1}^n \sum_{j=1}^m h_{ij} = f$. On the other hand, we have

$$\sum_{i=1}^{n} |T' f_i| = \sum_{i=1}^{n} \Big| \sum_{j=1}^{m} T' h_{ij} \Big| \le \sum_{i=1}^{n} \sum_{j=1}^{m} |T' h_{ij}| ,$$

and similarly

$$\sum_{j=1}^{m} |T' g_j| \le \sum_{i=1}^{n} \sum_{j=1}^{m} |T' h_{ij}| .$$

The above show that $D \uparrow$ holds in $E^\sim$.

Now let $x \in E^+$. Since T is order bounded, there exists some $u \in F^+$ satisfying $|Ty| \le u$ for all $|y| \le x$. Consequently, if $f_1, \ldots, f_n \in F^\sim_+$ satisfy $\sum_{i=1}^n f_i = f$, then we have

$$\begin{aligned}\Big\langle \sum_{i=1}^n |T'f_i|, x \Big\rangle &= \sum_{i=1}^n \sup\big\{\langle T'f_i, y\rangle\colon\ |y| \le x\big\}\\ &= \sum_{i=1}^n \sup\big\{\langle f_i, Ty\rangle\colon\ |y| \le x\big\}\\ &\le \sum_{i=1}^n \langle f_i, u\rangle = f(u)\,, \qquad (\star)\end{aligned}$$

which shows that the set $\big\{\phi(x)\colon\ \phi \in D\big\}$ is bounded above in $\mathbb{R}$ for each $x \in E^+$. By Theorem 1.19 the supremum $h = \sup D$ exists in $E^\sim$. Now if $0 \le g \le f$, then $|T'g| \le |T'g| + |T'(f-g)| \le h$ holds in $E^\sim$, which shows that $T'[0,f] \subseteq [-h,h]$. Therefore, $T'\colon F^\sim \to E^\sim$ is order bounded.

Finally, we show that T' is order continuous. To this end, let $f_\alpha \downarrow 0$ in $F^\sim$, and let $x \in E^+$ be fixed. Pick some $u \in F^+$ with $|Ty| \le u$ for all $|y| \le x$. From $(\star)$ and part (3) of Theorem 1.21 we see that $\big[|T'|f\big](x) \le f(u)$ holds for all $0 \le f \in F^\sim$. In particular, we have $\big[|T'|f_\alpha\big](x) \le f_\alpha(u) \downarrow 0$, and so $\big[|T'|f_\alpha\big](x) \downarrow 0$ holds for each $x \in E^+$, i.e., $|T'|f_\alpha \downarrow 0$ holds in $E^\sim$. Therefore, $|T'|$ is order continuous, and so T' is likewise order continuous. The proof of the theorem is now complete. ■

It is interesting to know that the converse of the preceding theorem is false. That is, there are operators $T\colon E \to F$ between Riesz spaces that are not order bounded, while their algebraic adjoints carry $F^\sim$ into $E^\sim$ and are order bounded and order continuous. For instance, the operator $T\colon L_1[0,1] \to c_0$ defined by

$$T(f) = \Big(\int_0^1 f(x)\sin x\,dx, \int_0^1 f(x)\sin 2x\,dx, \ldots\Big)\,,$$

is not order bounded, while

$$T'\colon c_0^\sim = \ell_1 \to L_1^\sim[0,1] = L_\infty[0,1]$$

(where $\langle T'(x_1,x_2,\ldots), f\rangle = \sum_{n=1}^\infty x_n \int_0^1 f(x)\sin nx\,dx$) is order bounded and order continuous. For details see Exercise 10 of Section 5.1.

Consider an order bounded operator $T\colon E \to F$ between two Riesz spaces. By Theorem 1.73 we know that $T'\colon F^\sim \to E^\sim$ is likewise order bounded, and so (since $E^\sim$ is Dedekind complete) the modulus of T' exists. On the other hand, if the modulus of T also exists, then it follows from $\pm T \le |T|$ that $\pm T' \le |T|'$. That is, whenever the modulus of T exists, then

$|T'| \le |T|'$ holds. The strict inequality $|T'| < |T|'$ may very well happen, as the next example shows.

Example 1.74. Consider the operator $T\colon \ell_1 \to \ell_\infty$ defined by

$$T(x_1, x_2, \ldots) = (x_1 - x_2, x_2 - x_3, x_3 - x_4, \ldots)\,.$$

Clearly, T is a regular operator, and an easy argument shows that

$$\begin{aligned} |T|(x_1, x_2, \ldots) &= \sup\big\{T(y_1, y_2, \ldots)\colon\ |(y_1, y_2, \ldots)| \le (x_1, x_2, \ldots)\big\} \\ &= (x_1 + x_2, x_2 + x_3, x_3 + x_4, \ldots) \end{aligned}$$

holds for all $0 \le (x_1, x_2, \ldots) \in \ell_1$.

Next consider the Riesz subspace c of ℓ_∞ consisting of all convergent sequences. Clearly, c majorizes ℓ_∞, and moreover the formula

$$\phi(x_1, x_2, \ldots) = \lim_{n\to\infty} x_n\,, \quad (x_1, x_2, \ldots) \in c\,,$$

defines a positive linear functional on c. By Theorem 1.32 the positive linear functional ϕ has a positive linear extension to all of ℓ_∞, which we denote by ϕ again. Put $e = (1, 1, \ldots)$, and note that

$$\big\langle |T|'\phi, e\big\rangle = \big\langle \phi, |T|e\big\rangle = \phi(2, 2, \ldots) = 2\,.$$

Now let $\psi \in \ell_\infty^\sim$ satisfy $|\psi| \le \phi$. Note that if $(x_1, x_2, \ldots) \in \ell_\infty$ satisfies $\lim_{n\to\infty} x_n = 0$, then the relation

$$\big|\psi(x)\big| \le |\psi|(|x|) \le \phi(|x|) = \lim_{n\to\infty} |x_n| = 0\,,$$

implies $\psi(x) = 0$. Therefore, $[T'\psi](x) = \psi(Tx) = 0$ holds for all $x \in \ell_1$. In other words, $T'\psi = 0$ holds for all $|\psi| \le \phi$, and so by Theorem 1.14 we see that

$$|T'|\phi = \sup\big\{|T'\psi|\colon\ |\psi| \le \phi\big\} = 0\,.$$

Thus, $0 = \big\langle |T'|\phi, e\big\rangle \ne \big\langle |T|'\phi, e\big\rangle = 2$, and consequently the operator T satisfies $|T'| < |T|'$. ■

To continue our discussion we need a simple lemma.

Lemma 1.75. *If $T\colon E \to F$ is an order bounded operator between two Riesz spaces, then for each $0 \le f \in F^\sim$ and each $x \in E^+$ we have*

$$\big\langle f, |Tx|\big\rangle \le \big\langle |T'|f, x\big\rangle\,.$$

Proof. Fix $0 \le f \in F^\sim$ and $x \in E^+$. Then by Theorem 1.23 there exists some $g \in F^\sim$ with $|g| \le f$ and $\big\langle f, |Tx|\big\rangle = \big\langle g, Tx\big\rangle$. Thus,

$$\big\langle f, |Tx|\big\rangle = \big\langle g, Tx\big\rangle = \big\langle T'g, x\big\rangle \le \big\langle |T'||g|, x\big\rangle \le \big\langle |T'|f, x\big\rangle\,,$$

as desired. ■

Although $|T'|$ and $|T|'$ need not be equal, they do agree on the order continuous linear functionals. This important result is due to U. Krengel [**105**] and J. Synnatzschke [**181**] and is stated next.

Theorem 1.76 (Krengel–Synnatzschke)**.** *If $T\colon E \to F$ is an order bounded operator between two Riesz spaces with F Dedekind complete, then*

$$\big|T'\big|f = |T|'f$$

holds for all $f \in F_{\mathrm{n}}^{\sim}$.

Proof. Let $0 \le f \in F_{\mathrm{n}}^{\sim}$ be fixed. We already know that $\big|T'\big|f \le |T|'f$ holds. On the other hand, if $0 \le x \in E$, then from Theorem 1.21 and Lemma 1.75 we see that

$$\begin{aligned}
\big\langle |T|'f, x\big\rangle &= \big\langle f, |T|x\big\rangle \\
&= \Big\langle f, \sup\Big\{\sum_{i=1}^{n} |Tx_i| \colon\ x_i \in E^+ \text{ and } \sum_{i=1}^{n} x_i = x\Big\}\Big\rangle \\
&= \sup\Big\{\sum_{i=1}^{n} \big\langle f, |Tx_i|\big\rangle \colon\ x_i \in E^+ \text{ and } \sum_{i=1}^{n} x_i = x\Big\} \\
&\le \Big\{\sum_{i=1}^{n} \big\langle |T'|f, x_i\big\rangle \colon\ x_i \in E^+ \text{ and } \sum_{i=1}^{n} x_i = x\Big\} \\
&= \big\langle \big|T'\big|f, x\big\rangle ,
\end{aligned}$$

and so $|T|'f \le \big|T'\big|f$. Therefore, $\big|T'\big|f = |T|'f$ holds for all $f \in F_{\mathrm{n}}^{\sim}$. ∎

When is every order bounded linear functional on a Riesz space σ-order continuous?

As we shall see, this question is closely related to the following question regarding a σ-order continuity property of the map $T \mapsto T^2$, from $\mathcal{L}_{\mathrm{b}}(E)$ to $\mathcal{L}_{\mathrm{b}}(E)$. *When does $0 \le T_n \uparrow T$ in $\mathcal{L}_{\mathrm{b}}(E)$ imply $T_n^2 \uparrow T^2$?*

In general, $0 \le T_n \uparrow T$ does not imply $T_n^2 \uparrow T^2$, even if T and all the T_n are rank one operators.

Example 1.77. Let $E = \ell_\infty$, the Dedekind complete Riesz space of all bounded real-valued sequences, and consider the Riesz subspace c of E consisting of all convergent sequences. Clearly, c majorizes E and the formula $f(x) = \lim x_n$ defines a positive linear functional on c. By Theorem 1.32 the positive linear functional has a positive linear extension to all of E (which we denote by f again.)

Now let $u_n = (1, 1, \ldots, 1_n, 0, 0, \ldots)$ and $e = (1, 1, \ldots)$. Put $T_n = f \otimes u_n$, $T = f \otimes e$, and note that $0 \le T_n \uparrow T$ holds in $\mathcal{L}_{\mathrm{b}}(E)$. On the other hand, it is not difficult to see that $T_n^2 = 0$ for each n and $T^2 = T$. So, $T_n^2 \not\uparrow T^2$. ∎

In contrast to the preceding example, observe that $T_n \downarrow 0$ in $\mathcal{L}_b(E)$ implies $T_n^2 \downarrow 0$. (To see this, note that $0 \le T_n^2(x) \le T_n(T_1 x)$ for all $x \in E^+$.)

Example 1.77 can be used to establish the existence of a Dedekind complete Riesz space E with the property that for each k, there exists a sequence $\{T_n\}$ of positive operators on E such that $0 \le T_n^i \uparrow_n T^i$ holds for each $i = 1, \ldots, k$ and $T_n^{k+1} \not\uparrow T^{k+1}$. The next example is taken from [**18**].

Example 1.78. Let f, u_n, and e be as they were defined in Example 1.77, and let $E = (\ell_\infty)^{\mathbb{N}}$ (= the Dedekind complete Riesz space of all ℓ_∞-valued sequences).

Now let k be fixed, and define the positive operators

$$T_n(x_1, x_2, \ldots) = \big(f(x_k)u_n, x_1, \ldots, x_{k-1}, 0, 0, \ldots\big)\,,$$

and

$$T(x_1, x_2, \ldots) = \big(f(x_k)e, x_1, \ldots, x_{k-1}, 0, 0, \ldots\big)\,.$$

Then it is a routine matter to verify that

$$0 \le T_n^i \uparrow T^i \ \text{ for each } i = 1, \ldots, k \quad \text{and} \quad T_n^{k+1} \not\uparrow T^{k+1}$$

hold in $\mathcal{L}_b(E)$. ■

The next result of C. D. Aliprantis, O. Burkinshaw and P. Kranz [**18**] characterizes the Riesz spaces on which every positive linear functional is σ-order continuous.

Theorem 1.79 (Aliprantis–Burkinshaw–Kranz)**.** *For a Riesz space E whose order dual separates the points of E the following statements are equivalent:*

(a) $E_c^\sim = E^\sim$, *i.e., every positive linear functional on E is σ-order continuous.*

(b) *Whenever $T\colon E \to E$ is a positive operator and a sequence $\{T_n\}$ of positive operators from E to E satisfies $T_n(x) \uparrow T(x)$ in E for each $x \in E^+$, then $T_n^2(x) \uparrow T^2(x)$ likewise holds in E for each $x \in E^+$.*

Proof. (1) $\Longrightarrow$ (2) Let $0 \le T_n(x) \uparrow T(x)$ for each $x \in E^+$, and let $y \in E^+$ be fixed. Clearly, $0 \le T_n^2(y) \uparrow \le T^2(y)$ holds in E. To see that $T^2(y)$ is the least upper bound of the sequence $\{T_n^2(y)\}$, let $T_n^2(y) \le z$ hold in E for all n. Then for each $0 \le f \in E^\sim$ we have $f\big(T_n^2(y)\big) \le f(z)$ for all n.

On the other hand, it follows that for each $0 \le f \in E^\sim$ the sequence $\{f \circ T_n\} \subseteq E^\sim = E_c^\sim$ satisfies $0 \le f \circ T_n \uparrow f \circ T$ in $E^\sim$. Thus,

$$f\big(T_n^2(y)\big) = [f \circ T_n](T_n y) \uparrow [f \circ T](Ty) = f\big(T^2(y)\big)\,,$$

and so $f\big(T^2(y)\big) \le f(z)$ holds for all $0 \le f \in E^\sim$. Since $E^\sim$ separates the points of E, it follows from Theorem 1.66 that $T^2(y) \le z$. Therefore, $T_n^2(y) \uparrow T^2(y)$ holds in E for each $y \in E^+$.

(2) $\Longrightarrow$ (1) Fix $0 \le f \in E^\sim$, and let $0 \le x_n \uparrow x$ in E. Then we have $0 \le [f \otimes x_n](y) \uparrow [f \otimes x](y)$ for all $y \in E^+$, and so by our hypothesis

$$[f \otimes x_n]^2(y) = f(x_n)\big[f(y)x_n\big] \uparrow [f \otimes x]^2(y) = f(x)\big[f(y)x\big]$$

also holds for all $y \in E^+$. Now an easy argument shows that $f(x_n) \uparrow f(x)$, and hence f is σ-order continuous. Therefore, $E_c^\sim = E^\sim$ holds. ■

Since $E^\sim$ is Dedekind complete, every band of $E^\sim$ is a projection band (see Theorem 1.42). The rest of the section is devoted to deriving formulas for the order projections of $E^\sim$.

Theorem 1.80. *Let E be a Riesz space and let $\phi \in E^\sim$. If P_ϕ denotes the order projection of $E^\sim$ onto the band generated by ϕ, then for each $x \in E^+$ and each $0 \le f \in E^\sim$ we have*

$$[P_\phi f](x) = \sup_{\epsilon > 0} \inf\big\{ f(y)\colon\ 0 \le y \le x \ \text{ and } \ |\phi|(x-y) < \epsilon \big\}\,.$$

Proof. We can assume that $0 \le \phi \in E^\sim$. Fix $x \in E^+$ and $0 \le f \in E^\sim$, and put

$$r = \sup_{\epsilon > 0} \inf\big\{ f(y)\colon\ 0 \le y \le x \ \text{ and } \ |\phi|(x-y) < \epsilon \big\}\,.$$

Fix $\epsilon > 0$. Since $f \wedge n\phi \uparrow P_\phi f$ (Theorem 1.47), there exists some k with $(P_\phi f - f \wedge k\phi)(x) < \epsilon$. Now let $0 < \delta < \epsilon$, and let $0 \le y \le x$ satisfy $\phi(x-y) < \delta$. Then we have

$$\begin{aligned} [P_\phi f](x) &= (P_\phi f - f \wedge k\phi)(x) + (f \wedge k\phi)(x) < \epsilon + (f \wedge k\phi)(x) \\ &\le \epsilon + k\phi(x-y) + f(y) < \epsilon + k\delta + f(y)\,, \end{aligned}$$

and consequently

$$\begin{aligned} [P_\phi f](x) &\le \epsilon + k\delta + \inf\big\{ f(y)\colon\ 0 \le y \le x \ \text{ and } \ \phi(x-y) < \delta \big\} \\ &\le \epsilon + k\delta + r \end{aligned}$$

holds for all $0 < \delta < \epsilon$. Thus, $[P_\phi f](x) \le \epsilon + r$ holds for all $\epsilon > 0$, and therefore $[P_\phi f](x) \le r$.

For the reverse inequality, let $\epsilon > 0$. Since $(f - P_\phi f) \wedge \phi = 0$, for each $0 < \delta < \epsilon$ there exists some $0 \le z \le x$ with $\big(f - P_\phi f\big)(z) + \phi(x-z) < \delta$. This implies $f(z) < \delta - \phi(x-z) + [P_\phi f](z) < \delta + [P_\phi f](x)$. In particular, we have

$$\inf\big\{ f(y)\colon\ 0 \le y \le x \ \text{ and } \ \phi(x-y) < \epsilon \big\} \le f(z) < \delta + [P_\phi f](x)$$

for all $\epsilon > 0$. This implies that $r \le [P_\phi f](x)$, and hence $[P_\phi f](x) = r$. ■

The next theorem presents a formula for P_ϕ in terms of increasing sequences and is due to W. A. J. Luxemburg [**125**, Note XV].

Theorem 1.81 (Luxemburg)**.** *Let E be a Riesz space and let $\phi \in E^\sim$. Then for each $x \in E^+$ and $0 \le f \in E^\sim$ we have*

$$[P_\phi f](x) = \inf\bigl\{\sup f(x_n)\colon\ 0 \le x_n \uparrow\le x \ \text{ and }\ |\phi|(x - x_n) \downarrow 0\bigr\}\,.$$

Proof. We can assume that $0 \le \phi \in E^\sim$. Fix $x \in E^+$ and $0 \le f \in E^\sim$ and put

$$r = \inf\bigl\{\sup f(x_n)\colon\ 0 \le x_n \uparrow\le x \ \text{ and }\ |\phi|(x - x_n) \downarrow 0\bigr\}\,.$$

Let $0 \le x_n \uparrow\le x$ satisfy $\phi(x - x_n) \downarrow 0$. Then for each n and k we have

$$\begin{aligned} [P_\phi f](x) - f(x_n) &\le [P_\phi f](x - x_n) \\ &\le (P_\phi f - f \wedge k\phi)(x) + (f \wedge k\phi)(x - x_n) \\ &\le (P_\phi f - f \wedge k\phi)(x) + k\phi(x - x_n)\,, \end{aligned}$$

and so, taking limits with respect to n, we get

$$[P_\phi f](x) - \sup f(x_n) \le (P_\phi f - f \wedge k\phi)(x)$$

for all k. Since $f \wedge k\phi \uparrow P_\phi f$, it follows that $[P_\phi f](x) \le \sup f(x_n)$, and from this we see that $[P_\phi f](x) \le r$.

Now let $\epsilon > 0$. Since $(f - P_\phi f) \wedge \phi = 0$ holds, for each n there exists some $0 \le y_n \le x$ with $(f - P_\phi f)(y_n) + \phi(x - y_n) < \epsilon 2^{-n}$. Put $x_n = \bigvee_{i=1}^n y_i$, and note that $0 \le x_n \uparrow\le x$. From $0 \le \phi(x - x_n) \le \phi(x - y_n) \to 0$, we see that $\phi(x - x_n) \downarrow 0$. Also, note that $0 \le (f - P_\phi f)(x_n) \le \sum_{i=1}^n (f - P_\phi f)(y_i) < \epsilon$ holds. Therefore,

$$r \le \sup f(x_n) \le \sup(f - P_\phi f)(x_n) + \sup[P_\phi f](x_n) \le \epsilon + [P_\phi f](x)$$

holds for all $\epsilon > 0$, and so $r \le [P_\phi f](x)$. Consequently, $[P_\phi f](x) = r$ holds, and the proof is finished. ■

A formula, due to the authors [**16**], describing the order projection onto an arbitrary band of $E^\sim$ is presented next.

Theorem 1.82 (Aliprantis–Burkinshaw)**.** *Let E be a Riesz space and let B be a band of $E^\sim$. If P_B denotes the order projection of $E^\sim$ onto B, then for each $x \in E^+$ and $0 \le f \in E^\sim$ we have*

$$[P_B f](x) = \sup_{\substack{\epsilon > 0 \\ \phi \in B^+}} \inf\bigl\{f(y)\colon\ 0 \le y \le x \ \text{ and }\ \phi(x - y) < \epsilon\bigr\}\,.$$

Proof. Fix $x \in E^+$ and $0 \le f \in E^\sim$, and put

$$r = \sup_{\substack{\epsilon > 0 \\ \phi \in B^+}} \inf\bigl\{f(y)\colon\ 0 \le y \le x \ \text{ and }\ \phi(x - y) < \epsilon\bigr\}\,.$$

Note that for each $\phi \in B^+$ we have $P_\phi \le P_B$. Thus from Theorem 1.80 it easily follows that $r \le [P_B f](x)$. Now let $\psi = P_B f$. Then $\psi \in B^+$, and so by Theorem 1.80 we have

$$\begin{aligned}[P_B f](x) &= [P_\psi \psi](x) \le [P_\psi f](x) \\ &= \sup_{\epsilon>0} \inf\bigl\{ f(y)\colon\ 0 \le y \le x \text{ and } \psi(x-y) < \epsilon \bigr\} \le r\,.\end{aligned}$$

Thus, $[P_B f](x) = r$ holds, as desired. ■

In view of Theorem 1.81, it might be expected that the following formula also holds:

$$[P_B f](x) = \inf\bigl\{ \sup f(x_n)\colon\ 0 \le x_n \uparrow\le x \text{ and } \phi(x - x_n) \downarrow 0\ \ \forall\, \phi \in B^+ \bigr\}\,.$$

Unfortunately, such formula is not true. For an example, let E be the Riesz space of all Lebesgue integrable (real-valued) functions on $[0,1]$ with the pointwise ordering. (Note that two functions differing at one point are considered to be different.) Since $x_\alpha \downarrow 0$ in E implies $x_\alpha(t) \downarrow 0$ for each $t \in [0,1]$, it follows that the point evaluations $x \mapsto x(t)$ are all order continuous positive linear functionals on E. This implies that $E_n^\sim$ separates the points of E. Now consider the positive linear functional $f\colon E \to \mathbb{R}$ defined by

$$f(x) = \int_0^1 x(t)\,dt\,.$$

According to Example 1.55, the linear functional f is σ-order continuous but not order continuous. If $B = E_n^\sim$, then

$$\begin{aligned}&\inf\bigl\{ \sup f(x_n)\colon\ 0 \le x_n \uparrow\le \mathbf{1} \text{ and } \phi(\mathbf{1} - x_n) \downarrow 0 \text{ for all } \phi \in B^+ \bigr\} \\ &\qquad = \inf\bigl\{ \sup f(x_n)\colon\ 0 \le x_n \uparrow \mathbf{1} \bigr\} = f(\mathbf{1}) = 1\,.\end{aligned}$$

On the other hand, it is not difficult to see that $[P_B f](\mathbf{1}) < 1$ must hold.

Finally, we close this section by presenting necessary and sufficient conditions for a linear functional to belong to a principal band of $E^\sim$.

Theorem 1.83. *Let E be a Riesz space and let $f \in E^\sim$. Then for an order bounded linear functional $g \in E^\sim$ the following statements are equivalent.*

(1) *g belongs to the principal band generated by f in $E^\sim$.*

(2) *For each $x \in E^+$ and $\epsilon > 0$ there exists some $\delta > 0$ such that whenever $|y| \le x$ satisfies $|f|(|y|) < \delta$, then $|g|(|y|) < \epsilon$ holds.*

(3) *If an order bounded sequence $\{x_n\}$ of E satisfies $\lim |f|(|x_n|) = 0$, then $\lim g(x_n) = 0$.*

(4) *If $0 \le x_n \uparrow\le x$ and $\lim |f|(x - x_n) = 0$, then $\lim g(x - x_n) = 0$.*

Proof. (1) $\Longrightarrow$ (2) Let $x \in E^+$ and $\epsilon > 0$. Since $|g| \wedge k|f| \uparrow |g|$ holds in $E^\sim$ (Theorem 1.47), there exists some k with $(|g| - |g| \wedge k|f|)(x) < \epsilon$. If $|y| \le x$ satisfies $|f|(|y|) \le \frac{\epsilon}{k}$, then we have

$$\begin{aligned} |g|(|y|) &= \big(|g| - |g| \wedge k|f|\big)(|y|) + \big(|g| \wedge k|f|\big)(|y|) \\ &\le \big(|g| - |g| \wedge k|f|\big)(x) + k|f|(|y|) < \epsilon + \epsilon = 2\epsilon\,. \end{aligned}$$

(2) $\Longrightarrow$ (3) and (3) $\Longrightarrow$ (4) are obvious.

(4) $\Longrightarrow$ (1) Write $g = \phi + \psi$, with $\phi \in B_f$ and $\psi \perp f$. Fix $x \in E^+$ and $\epsilon > 0$. Now let $0 \le y \le x$. Since $\psi \perp f$ holds, for each n there exists some $0 \le y_n \le y$ with $|\psi|(y_n) + |f|(y - y_n) < 2^{-n}\epsilon$. Then $x_n = \bigvee_{i=1}^n y_i$ satisfies $0 \le x_n \uparrow\le y$ and $|\psi|(x_n) \le \sum_{i=1}^n |\psi|(y_i) < \epsilon$. On the other hand, the inequalities $|f|(y - x_n) \le |f|(y - y_n) \le 2^{-n}\epsilon$ imply $|f|(y - x_n) \downarrow 0$. Hence, by our hypothesis $\lim g(y - x_n) = 0$. In particular, note that

$$\begin{aligned} g(y) &= \lim_{n\to\infty} g(x_n) = \lim_{n\to\infty} \big[\phi(x_n) + \psi(x_n)\big] \\ &\le \limsup_{n\to\infty} \big[|\phi|(x) + |\psi|(x_n)\big] \le |\phi|(x) + \epsilon\,. \end{aligned}$$

Since $\epsilon > 0$ is arbitrary, we see that $g(y) \le |\phi|(x)$ holds for all $0 \le y \le x$. Therefore,

$$g^+(x) = \sup\big\{g(y)\colon\ 0 \le y \le x\big\} \le |\phi|(x)$$

holds for all $x \in E^+$. Hence, $g^+ \in B_f$. Similarly, $g^- \in B_f$, and therefore $g = g^+ - g^- \in B_f$, and the proof is finished. ■

Exercises

1. Show that if $f\colon E \to \mathbb{R}$ is a σ-order continuous linear functional on an Archimedean Riesz space, then f is order bounded.

2. Consider an Archimedean Riesz space E. If $f \in E_n^\sim$ and $g \in E^\sim$, then show that the following statements are equivalent.
  (a) $f \perp g$.
  (b) $C_g \subseteq N_f$.
  (c) $C_g \perp C_f$.

3. Establish the following properties of perfect Riesz spaces.
  (a) Every band of a perfect Riesz space is a perfect Riesz space in its own right.
  (b) If F is a perfect Riesz space, then $\mathcal{L}_b(E, F)$ is likewise a perfect Riesz space for each Riesz space E. (In particular, the order dual of every Riesz space is a perfect Riesz space.)
  (c) If E is a perfect Riesz space, then $E^{\sim\sim}$ is retractable on E.

4. Let E and F be two Riesz spaces such that $E_n^\sim = E^\sim$ and $F^\sim$ separates the points of F. Then show that every positive operator from E to F