CHAPTER 6

Radon–Nikodým Theorems

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HANDBOOK OF MEASURE THEORY
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1. Introduction

Suppose λ is the Lebesgue measure on the real line and f is an integrable function. Then the measure ν defined for all the Lebesgue measurable sets

$$\nu(E) = \int_E f\,d\lambda$$

is called the *indefinite integral* of f with respect to λ. It is obvious from the definition of the integral that if $\lambda(E) = 0$, then $\nu(E) = 0$ for any Lebesgue-measurable set E. This is expressed saying that ν is *absolutely continuous* with respect to λ and will be denoted by $\nu \ll \lambda$.

It is not difficult to check (see [27, Section 30, Theorem B]) that under the assumption that ν is a finite measure, this condition is equivalent to another one which can be given in ε–δ terms, which will be denoted by $\nu \ll_\varepsilon \lambda$: given any $\varepsilon > 0$, there exists a $\delta > 0$ such that if $\lambda(E) < \delta$, then $\nu(E) < \varepsilon$, for any Lebesgue-measurable set E.

We always have that $\nu \ll_\varepsilon \lambda$ implies that $\nu \ll \lambda$, but the converse implication does not hold in general, as it can be easily seen considering for instance the measure $\nu(E) = \int_E |x|\,d\lambda(x)$.

In the σ-additive case, absolute continuity of a measure ν with respect to another measure μ implies, under mild conditions on μ and ν, that ν is the *integral measure* of some function $f \in L^1(\mu)$: such function is called *Radon–Nikodým derivative* of ν with respect to μ, and it is denoted by $\frac{d\nu}{d\mu}$.

Nikodým [44] was the first to prove this result in a quite general setting. If μ and ν are two finite measures on the same σ-algebra, and if $\nu \ll \mu$, then there exists a Radon–Nikodým derivative $\frac{d\nu}{d\mu}$, i.e., a μ-integrable function f such that

$$\nu(E) = \int_E f\,d\mu$$

holds, for any measurable set E.

Earlier, Radon [46] proved (using Vitali bases) the same implication under the assumption that the maximal element of the σ-algebra is a measurable subset of the n-dimensional Euclidean space. For the real line, the corresponding result goes back to Lebesgue.

However the situation is not so nice, either when the measures are just *finitely additive*, or when they take values in a Banach space of infinite dimension, and it gets even worse if *both* of the above cases occur. We shall outline a survey of the results known so far, involving the existence of a Radon–Nikodým derivative, in some appropriate sense, and with respect to suitable types of integrals. In this first section, we deal essentially with a general Radon–Nikodým theorem for nonnegative finitely additive scalar measures, from which the basic ideas of further results can be drawn. In the second section, we turn to the σ-additive measures, investigating conditions which permit that even unbounded measures have Radon–Nikodým derivatives. In the third section we face the finitely additive case for scalar measures, indicating how the existence of a Radon–Nikodým derivative $d\nu/d\mu$ is strictly related to some geometrical properties of the *range* of the vector valued measure (μ, ν). The fourth section deals with the most beautiful (and difficult) results concerning those Banach spaces possessing the so-called *Radon–Nikodým Property* (RNP): the geometric concepts met in the third section here become essential tools for describing such Banach spaces. The fifth section is devoted to finitely additive measures taking values in Banach spaces, and to the research of *weaker* types of derivatives. Finally, the sixth section concerns a number of recent results, for locally convex-valued measures, for multimeasures, for Riesz space-valued measures, and finally also a brief outline of the so-called *fuzzy integration*: from this survey one can find that, though hidden by the abstract settings and the different kinds of integrals involved, the leading ideas of the first section are always at work.

We end the chapter with an appendix devoted to some decomposition theorems for measures, which are relevant for the Radon–Nikodým theorem.

We do not give all proofs, some of them being too long and technical: when it is possible, we give an outline of the main steps, trying to clarify the basic ideas. Also, we realize that it would be almost impossible to present here a detailed account of *all* the contributions given to this problem along almost 100 years, so we simply *chose* among the results in our knowledge, sometimes simplifying settings, in order to reach a sufficient variety within a relatively concise treatment.

The general result, from which we start, is due to Greco [25]. Though the integration theory and the results by Greco involve more general set functions, we shall give the proof only for the finitely additive case. It should be noted that the idea to relate the Radon–Nikodým theorem to the existence of a "scaled" Hahn decomposition goes back, for the σ-additive case, to J.L. Kelley [33].

We need some definitions (see also [19]).

DEFINITION 1.1. Let (Ω, Σ) be any measure space, where Σ is a σ-algebra of subsets of Ω, and let μ be any monotone nonnegative set function on Σ, $\mu(\emptyset) = 0$. We say that a measurable function $f : \Omega \to \mathbb{R}^+$ is *integrable* with respect to μ if the following integral is finite:

$$\int_\Omega f\,d\mu := \int_0^\infty \mu(\{x\colon\ f(x) > t\})\,dt.$$

Since $\{x\colon\ f(x) > t\}$ is nonincreasing in t, the second integral has to be understood in the Riemann sense.

If this is the case, for each set $E \in \Sigma$, we put:

$$f\mu(E) := \int_E f\,d\mu := \int_\Omega f 1_E\,d\mu.$$

The set function $f\mu$ is often called the *integral measure* of f, with respect to μ.

The integral defined above is called *Choquet* integral, and coincides with the usual integral, in case f is nonnegative and μ is a σ-additive nonnegative measure, or a finitely additive one.

THEOREM 1.2. *Assume that μ and ν are two monotone nonnegative set functions, defined on the σ-algebra Σ, such that $\mu(\emptyset) = \nu(\emptyset) = 0$, satisfying:*

$$\nu(S) = \mu(S) = 0 \quad \Rightarrow \quad \mu(A \cup S) = \mu(A) \tag{1}$$

for all $A \in \Sigma$.

Then, the following conditions are equivalent:

(a) *There exists a measurable nonnegative function $f : \Omega \to \mathbb{R}$ such that*

$$\nu(E) = \int_E f\, d\mu \tag{2}$$

for all $E \in \Sigma$.

(b) *There exists a (decreasing) family of sets $\{A_r\}_{r>0}$ in Σ, satisfying:*

(b1) $\nu(A) - \nu(B) \geqslant r\mu(A) - r\mu(B)$, $A, B \in \Sigma$, $B \subset A \subset A_r$, $r > 0$;

(b2) $\nu(E) - \nu(E \cap A_r) \leqslant r(\mu(E) - \mu(E \cap A_r))$, $E \in \Sigma$, $r > 0$;

(b3) $\lim \nu(A_r) = 0$, *as* $r \to +\infty$.

It is clear that (1) is satisfied, as soon as μ and ν are additive. Moreover, in case ν and μ are finitely additive, conditions (b1) and (b2) above are equivalent respectively to (b′1) and (b′2) below:

(b′1) $\nu(E) \geqslant r\mu(E)$, for all $E \subset A_r$, $r > 0$;

(b′2) $\nu(F) \leqslant r\mu(F)$, for all $F \subset \Omega \setminus A_r$, $r > 0$.

Moreover, from (b′1) we deduce that $\lim \mu(A_r) = 0$ (since $\nu(A_r) \leqslant \nu(\Omega) < \infty$), hence, in case $\nu \ll_\varepsilon \mu$, (b3) is satisfied, too.

PROOF. As already mentioned, we shall give the proof just in the case of finitely additive, nonnegative measures, such that $\nu \ll_\varepsilon \mu$.

We first assume (a) and prove (b). Set: $A_r := \{x \in \Omega\colon f(x) > r\}$ and fix $E \subset A_r$, $E \in \Sigma$. As $\nu(E) = \int f 1_E\, d\mu$, we get $\nu(E) \geqslant r \int 1_E\, d\mu = r\mu(E)$, so (b′1) is proved.

If F is fixed, $F \in \Sigma$, $F \subset A_r^c$, then $\nu(F) = \int f 1_F\, d\mu \leqslant r\mu(F)$, so (b′2) holds. As already observed, (b3) is satisfied because of the absolute continuity, so the first implication is completely proved.

Now, we assume that (b) holds, and construct a suitable *derivative* f.

For all $x \in \Omega$, set: $f(x) := \sup\{r > 0\colon x \in A_r\}$. To see that f is measurable, fix any $t \in]0, +\infty[$, and observe that $\{x\colon f(x) > t\} = \bigcup\{A_r\colon r \in D,\ r > t\}$, where D denotes the set of all dyadic positive numbers, i.e., $D = \{\frac{h}{2^k},\ h \text{ and } k \text{ positive integers}\}$.

Now, for each element $\frac{h}{2^k} \in D$, set $B_h^k := A_{h/2^k} \setminus A_{(h+1)/2^k}$.

We see that $\nu(B_h^k) \geqslant \frac{h}{2^k}\mu(B_h^k)$, and also $\nu(B_h^k) \leqslant \frac{h+1}{2^k}\mu(B_h^k)$. For each positive integer k, set

$$f_k := \sum_{h=1}^{k2^k} \frac{h}{2^k} 1_{B_h^k}.$$

We can easily see that $\int_E f\,d\mu = \lim_k \int_E f_k\,d\mu$, for all $E \in \Sigma$, and moreover

$$\int_E f_k\,d\mu \leqslant \sum_{h=1}^{k2^k} \frac{h}{2^k}\frac{2^k}{h}\nu\bigl(B_h^k \cap E\bigr) \leqslant \nu(E)$$

for all k and E. Therefore, we get:

$$\int_E f\,d\mu \leqslant \nu(E),$$

for all E. On the other hand, for each k, we have:

$$\int f_k\,d\mu = \int \left(\sum_{h=1}^{k2^k} \frac{h+1}{2^k} 1_{B_h^k}\right) d\mu - \int \frac{1}{2^k}\left(\sum_{h=1}^{k2^k} 1_{B_h^k}\right) d\mu$$
$$\geqslant \sum_{h=1}^{k2^k} \nu\bigl(B_h^k\bigr) - \mu(\Omega)/2^k.$$

As $\sum_{h=1}^{k2^k} \nu(B_h^k) = \nu(A_{1/2^k}) - \nu(A_k)$, from (b3) we deduce that

$$\lim_k \int f_k\,d\mu \geqslant \lim_k \nu(A_{1/2^k})$$

from which we also get $\lim_k \int f_k\,d\mu \geqslant \nu(\Omega)$, because $\nu(A_{1/2^k}) \leqslant 2^{-k}\mu(A_{1/2^k}) \leqslant 2^{-k}\mu(\Omega)$.

So far, we have seen that the finitely additive measures ν and $f\mu$ are in this relation: $\nu(E) \geqslant f\mu(E)$, for all $E \in \Sigma$, and $\nu(\Omega) \leqslant f\mu(\Omega)$. From this it follows immediately that the two measures agree on Σ (simply considering the *complements* of the involved sets), and therefore the theorem is proved. □

COROLLARY 1.3. *If μ and ν are finite and countably additive, and $\nu \ll \mu$ (or equivalently $\nu \ll_\varepsilon \mu$), then there exists a* Radon–Nikodým derivative $\frac{d\nu}{d\mu}$.

PROOF. Conditions (b′1) and (b′2) mean that there exists a Hahn decomposition for the measure $\nu - r\mu$, for all $r > 0$: this is always the case, for σ-additive measures, and this concludes the proof. □

The importance of Theorem 1.2 rests not only on its generality, but also on the relative simplicity of the involved conditions, from which other similar criteria have been obtained. We shall see many of them in the sequel, dealing with finitely additive measures, with Banach- and nuclear-valued measures, and also with different kinds of integrals. Our dissertation however now focuses the σ-additive case for not necessarily bounded measures.

2. The σ-additive case

In this chapter, we mainly deal with countably additive, possibly unbounded measures. In general, the Radon–Nikodým theorem fails to hold, for unbounded measures, as the following example shows.

EXAMPLE 2.1. Let $\Omega = [0, 1]$, let $0 \leqslant i < j \leqslant 1$ and consider the Hausdorff i- and j-dimensional measures $\mathcal{H}^i$ and $\mathcal{H}^j$. Then $\mathcal{H}^j \ll \mathcal{H}^i$ but $d\mathcal{H}^j/d\mathcal{H}^i$ does not exist.

The reason for the failure of the Radon–Nykodým theorem in the previous example has been described already by S. Saks [50] who considered the case $i = 0$ and $j = 1$ (see also Volčič [59]). We will make some additional comments on this example after Theorem 2.12.

However we will see that, under suitable conditions, Radon–Nikodým derivatives do exist, even if both measures, μ and ν, range over $[0, +\infty]$. Such conditions involve properties of the so-called *measure algebra*, which we are now going to introduce.

As usual, (Ω, Σ, μ) denotes a measure space, where Σ is a σ-algebra and μ is any nonnegative σ-additive measure, taking values in $[0, +\infty]$. We shall assume that the Carathéodory extension has already been done, and so Σ is actually the σ-algebra of all μ-measurable sets. In particular, all subsets of μ-null sets belong to Σ. We recall the definition of the *outer measure*, $\mu^* : \mathcal{P}(\Omega) \to [0, +\infty]$:

$$\mu^*(E) = \inf\left\{\sum_{n=1}^{\infty} \mu(F_n):\ F_n \in \Sigma,\ E \subset \bigcup F_n\right\}.$$

DEFINITION 2.2. Given two subsets A and B of Ω, we say that A and B are *equivalent*, if $\mu^*(A \triangle B) = 0$, and write: $A \approx B$ or $A = B$ μ-a.e.

In a similar fashion, if f and g are real functions defined on Ω, we say that f and g are *equivalent* (and write $f \approx g$, or $f = g$ a.e.) if

$$\mu^*\big(\{x \in \Omega:\ f(x) \neq g(x)\}\big) = 0.$$

In the quotient $\mathcal{P}(\Omega)/\approx$ we can introduce a partial order, as follows:

$$[A] \succ [B]$$

if and only if $\mu^*(B \setminus A) = 0$.

Of course, $\mathcal{P}(\Omega)/\approx$ contains $\Sigma/\approx$, which is called the *measure algebra*. One can easily see that $\mathcal{P}(\Omega)/\approx$ and $\Sigma/\approx$ are σ-complete lattices.

We will see that *completeness* of $\Sigma/\approx$ is important for our purposes, however in general the two lattices are not complete. We have the following facts (more details and related results can be found in a series of papers by Volčič [58–60]):

Under CH, if μ is the usual Lebesgue measure on $\Omega := [0, 1]$, then $\mathcal{P}(\Omega)/\approx$ is not complete.

There exist measures such that $\Sigma/\approx$ is not complete (see [27, Section 31, Exercise 9]).

DEFINITION 2.3. Given any measure space (Ω, Σ, μ), we say that μ is *semifinite* if

$$\mu(E) = \sup\{\mu(A)\colon\ A \subset E,\ A \in \Sigma,\ \mu(A) < \infty\},$$

for all sets $E \in \Sigma$.

Such measures are called *essential measures* by N. Bourbaki [8].

PROPOSITION 2.4. *Suppose $\nu \ll \mu$, suppose that ν is semifinite and suppose moreover that $\nu(E) = 0$ whenever $\mu(E) < \infty$. Then $d\nu/d\mu$ exists only if ν is identically zero.*

PROOF. Suppose $d\nu/d\mu$ exists and let E be any set of finite and positive ν measure. Then

$$\{x \in \Omega\colon\ d\nu/d\mu > 0\} \cap E = \bigcup_n \{x \in \Omega\colon\ d\nu/d\mu > \tfrac{1}{n}\} \cap E.$$

Each set $\{x \in \Omega\colon\ d\nu/d\mu > \frac{1}{n}\} \cap E$ has finite μ measure and since $\nu(E) > 0$, at least one of them, A_n say, has positive measure μ, a contradiction. This shows that if $\nu(E) < \infty$, then $\nu(E) = 0$ and from the semifiniteness of the measure we deduce that $\nu \equiv 0$. □

DEFINITION 2.5. A semifinite measure μ on (Ω, Σ) is said to be *strictly localizable*, if there exists a family of sets $\{E_\alpha\}_{\alpha \in A}$, such that

(a) $0 < \mu(E_\alpha) < \infty$
(b) $E_\alpha \cap E_\beta = \emptyset$ when $\alpha \neq \beta$
(c) $\mu(E \cap E_\alpha) = 0\ \forall \alpha \Rightarrow \mu(E) = 0$.

THEOREM 2.6. *If $\mu(\Omega) < \infty$ (or more in general if μ is strictly localizable), then $\Sigma/\approx$ is complete.*

PROOF. We will limit the proof to the case of a finite measure. Given any family $\mathcal{G}$ of measurable sets, we shall show that there exists a measurable set G_0 such that:

(1) $\mu^*(G \setminus G_0) = 0$ for all $G \in \mathcal{G}$, and
(2) If $\mu^*(G \setminus G_1) = 0$ for all $G \in \mathcal{G}$, then $\mu^*(G_0 \setminus G_1) = 0$.

Without loss of generality, we may assume that $\mathcal{G}$ is a σ-ideal. Now, set:

$$\alpha := \sup\{\mu(G)\colon\ G \in \mathcal{G}\}.$$

As μ is finite, $\alpha < \infty$. Now, let $(G_n)_{n \in \mathbb{N}}$ be any sequence in $\mathcal{G}$, such that $\mu(G_n) > \alpha - \frac{1}{n}$, and put $G_0 := \bigcup G_n$. As $\mathcal{G}$ is a σ-ideal, $G_0 \in \mathcal{G}$. So, we only have to prove that G_0 satisfies (1) above. If this is not the case, then there exists $H \in \mathcal{G}$, satisfying $\mu(H \setminus G_0) > 0$. This implies that $H \cup G_0 \in \mathcal{G}$, and $\mu(H \cup G_0) > \alpha$, a contradiction. □

DEFINITION 2.7. A semifinite measure μ is said to be *Maharam* (or also *localizable*) if $\Sigma/\approx$ is complete (see [36]).

REMARK 2.8. According to Theorem 1, strictly localizable measures are Maharam. It was a long standing open question whether Maharam measures are strictly localizable. The problem has been stated explicitly for the first time in [34, Problem 1], and is relevant in lifting theory. The question has been solved in the negative by D. Fremlin [24].

DEFINITION 2.9. Given two measures μ and ν on Σ we say that μ and ν are *strongly comparable* if μ and ν are the Carathéodory extensions of $\mu|_{\mathcal{S}}$ and of $\nu|_{\mathcal{S}}$ respectively, where $\mathcal{S}$ is the ideal of all sets $E \in \Sigma$, such that $\mu(E) + \nu(E) < +\infty$.

The following definition is related to the previous one.

DEFINITION 2.10. Given two measures μ and ν on Σ we say that μ and ν are *weakly comparable* if μ and ν are the Carathéodory extensions of $\mu|_{\Sigma}$ and of $\nu|_{\Sigma}$ respectively, where Σ is the intersection of the two σ-algebras of μ- and ν-measurable sets (in the Carathéodory sense).

The measures $\mathcal{H}^i$ and $\mathcal{H}^j$ are not strongly comparable, if $i \neq j$, but they are weakly comparable.

In 1951, I.E. Segal proved the following version of the Radon–Nikodým theorem [51].

THEOREM 2.11. *Given a σ-additive measure μ, the following properties are equivalent:*

(1) *μ is Maharam;*
(2) *for any σ-additive measure ν, strongly comparable with μ and such that $\nu \ll \mu$, there exists a Radon–Nikodým derivative $d\nu/d\mu$. Such function is unique, up to equivalence.*

PROOF. We only prove the implication (1) $\Rightarrow$ (2). So, assume that μ is Maharam, and ν is any strongly comparable, σ-additive measure, $\nu \ll \mu$. We denote by $\mathcal{S}$ the ideal of all sets $E \in \Sigma$, such that $\mu(E) + \nu(E) < \infty$. For every element $E \in \mathcal{S}$ there exists a Radon–Nikodým derivative of $\nu|_E$ with respect to $\mu|_E$: we denote by f_E such derivative, and define f_E on E^c as the null function. Now, for every positive real number t, set:

$$A_t := \bigvee \{A_t(E)\colon\ E \in \mathcal{S}\},$$

where $A_t(E) = \{x \in \Omega\colon\ f_E(x) > t\}$. (The supremum exists as μ is Maharam.) It is clear that (A_t) is a decreasing family in $\Sigma/\approx$. Now, for all $x \in \Omega$ we define:

$$f(x) := \sup\{r > 0\colon\ x \in A_r\}.$$

As in the proof of Theorem 1.2, one can prove that f is measurable. Let us show that f is the required derivative. In view of the strong comparability, it is enough to check that

$$\int_E f\, d\mu = \nu(E)$$

for all $E \in \mathcal{S}$. So, fix any element $E \in \mathcal{S}$, and compute:

$$\int_E f\,d\mu = \int_0^\infty \mu\big(\{x \in E\colon\ f(x) > t\}\big)\,dt. \tag{3}$$

Now, $\mu(\{x \in E\colon\ f(x) > t\}) = \sup\{\mu(E \cap A_{t+\frac{1}{n}})\colon\ n \in \mathbb{N}\}$. For every positive number r, we claim that

$$E \cap A_r \approx A_r(E). \tag{4}$$

Indeed, $A_r(E) \subset E$ by definition, and $A_r(E) \subset A_r$ up to a null-set, hence $A_r(E) \subset E \cap A_r$ up to a null-set; conversely, $E \cap A_r = \vee(E \cap A_r(F))$, where the supremum is taken as $F \in \mathcal{S}$. Now, $E \cap A_r(F) = A_r(E \cap F)$ because of the essential uniqueness of f_E, hence $E \cap A_r \subset A_r(E)$ a.e. and (4) is proved.

From (4), we get immediately

$$\sup\big\{\mu(E \cap A_{t+\frac{1}{n}})\colon\ n \in \mathbb{N}\big\} = \sup\big\{\mu\big(A_{t+\frac{1}{n}}(E)\big)\colon\ n \in \mathbb{N}\big\} = \mu\big(A_t(E)\big)$$

for all positive numbers t, and so from (3) we deduce

$$\int_E f\,d\mu = \int_0^\infty \mu\big(\{x \in \Omega\colon\ f(x) > t\}\big)\,dt = \int_0^\infty \mu\big(A_t(E)\big)\,dt$$
$$= \int f_E\,d\mu = \nu(E),$$

by definition of f_E. This concludes the proof. □

For weakly comparable measures, we have the following result.

THEOREM 2.12. *Given a σ-additive measure μ on (Ω, Σ), the following properties are equivalent:*

(1) *μ is Maharam;*

(2) *for any σ-additive measure ν, weakly comparable with μ and such that $\nu \ll \mu$, there exists a decomposition Ω_1, Ω_2 of Ω, $\Omega_i \in \Sigma$, for $i = 1, 2$, such that:*
- *for any $E \in \Sigma$ and $E \subset \Omega_1$, $\mu(E) < \infty$ implies that $\nu(E) = 0$;*
- *if we denote by ν_2 the restriction of ν to $\Sigma \cap \Omega_2$, the Radon–Nikodým derivative $\frac{d\nu_2}{d\mu}$ exists.*

The decomposition Ω_1, Ω_2 and the function $\frac{d\nu_2}{d\mu}$ are unique, up to equivalence.

The proof follows from the combination of Theorem 2.11 and the decomposition Theorem A.4 from the Appendix.

REMARK 2.13. The previous theorem explains why Radon–Nikodým theorem fails when μ is the counting measure $\mathcal{H}^0$ on $[0, 1]$, and ν is the Lebesgue measure, in spite of the fact that μ is localizable. The measures μ and ν are in this case just weakly and not strongly comparable, and $\nu \ll \mu$. Theorem 3 applies here with $\Omega_1 = \Omega$.

We conclude this section presenting two interesting variants of the Radon–Nikodým theorem.

We need first two definitions.

DEFINITION 2.14. Given a measure space (Ω, Σ, μ), we say that μ admits a *monotone differentiation*, if there exists a mapping $\frac{d}{d\mu}$, defined on the family $\mathcal{N}$ of all the measures which are strictly comparable with μ and absolutely continuous with respect to μ, such that if $\nu_i \in \mathcal{N}$, for $i = 1, 2$, and $\nu_1 \leqslant \nu_2$, then

$$\frac{d\nu_1}{d\mu}(x) \leqslant \frac{d\nu_2}{d\mu}(x),$$

for any $x \in \Omega$.

DEFINITION 2.15. Given a measure space (Ω, Σ, μ), we say that μ admits a *linear differentiation*, if there exists a mapping $\frac{d}{d\mu}$, defined on the family $\mathcal{N}$ of all the measures which are strictly comparable with μ and absolutely continuous with respect to μ, such that if $\nu_i \in \mathcal{N}$, for $i = 1, 2$, and a_1, a_2 are two real numbers, then

$$\frac{d(a_1\nu_1 + a_2\nu_2)}{d\mu}(x) = a_1 \frac{d\nu_1}{d\mu}(x) + a_2 \frac{d\nu_2}{d\mu}(x),$$

for any $x \in \Omega$.

The following result is due to D. Kölzow [34, Theorems 4, 7 and 8].

THEOREM 2.16. *Given a measure space (Ω, Σ, μ), the following properties are equivalent*:

(i) *μ admits a monotone differentiation*;
(ii) *μ admits a linear differentiation*;
(iii) *μ is strictly localizable.*

For the long and sophisticated proof we refer to Kölzow's monograph, which also illustrates in great detail the strong relations between the monotone and linear versions of the Radon–Nikodým theorem and lifting theory.

The Radon–Nikodým theorem for the Daniell integral is discussed by M.D. Carrillo [15], Chapter 11 in this Handbook.

3. The finitely additive case

When the measures (even one of them) are just finitely additive, the Radon–Nikodým theorem does not hold, in general: (we remark that, in this case, absolute continuity is to be intended in the ε–δ sense).

It is appropriate to begin with the oldest result concerning the Radon–Nikodým theorem for finitely additive measures, attributed to Bochner [7].

THEOREM 3.1. *If μ and ν are two finitely additive measures on Σ, such that $\nu \ll_\varepsilon \mu$, then for any $\eta > 0$ there exists a μ-integrable function f_η, such that*

$$\left| \nu(E) - \int_E f_\eta \, d\mu \right| < \eta,$$

for any $E \in \Sigma$.

In order to give an example, we introduce a definition.

DEFINITION 3.2. Let $\mu : \Sigma \to \mathbb{R}_0^+$ be a finitely additive measure, and let $\mathfrak{I}$ denote the ideal of null sets. If $\mathfrak{I}$ is a σ-ideal, we say that μ has the *property* σ.

One can easily see that, if $f \geqslant 0$ and μ has property σ, then

$$\int f \, d\mu = 0$$

implies that $f = 0$ a.e.

Similarly, one can deduce that, whenever μ has property σ, and f and g are two μ-integrable functions, then $f = g$ μ-a.e. as soon as $\int_A f \, d\mu = \int_A g \, d\mu$ for all $A \in \Sigma$ (this entails *uniqueness* of the Radon–Nikodým derivatives, up to a.e. equivalence).

EXAMPLE 3.3 ([10]). Let Δ be the family of all subsets $D \subset [0, 1]$, such that the following limit exists:

$$\lim_{\varepsilon \to 0^+} \frac{\lambda(D \cap [0, \varepsilon[)}{\varepsilon} = \delta(D).$$

Though Δ is not an algebra, the function δ can be extended (using the axiom of choice) to the whole σ-algebra Σ of the Lebesgue measurable sets, in such a way that

$$\liminf_{\varepsilon \to 0^+} \frac{\lambda(B \cap [0, \varepsilon[)}{\varepsilon} \leqslant \delta(B) \leqslant \limsup_{\varepsilon \to 0^+} \frac{\lambda(B \cap [0, \varepsilon[)}{\varepsilon},$$

for all sets $B \in \Sigma$, and so that δ is finitely additive on Σ. Obviously, if $\lambda(B) = 0$ then $\delta(B) = 0$.

Now set $\mu := \lambda + \delta$. It is clear that $\mu(B) = 0$ if and only if $\lambda(B) = 0$, hence μ enjoys property σ. Moreover, $\lambda \ll \mu$ also in the $(\varepsilon$–$\delta)$ sense, however one can easily see that $d\lambda/d\mu$ does not exist. In fact, such a function f should satisfy

$$\lambda(B \cap [\varepsilon, 1]) = \int_{B \cap [\varepsilon, 1]} f \, d\mu = \int_{B \cap [\varepsilon, 1]} f \, d\lambda + \int_{B \cap [\varepsilon, 1]} f \, d\delta = \int_{B \cap [\varepsilon, 1]} f \, d\lambda,$$

for all $\varepsilon > 0$, and all $B \in \Sigma$, i.e., $f = 1$ a.e. in $[\varepsilon, 1]$ for all $\varepsilon > 0$. Thus, by the property σ, it should follow that $f = 1$ μ-a.e., a contradiction.

The property σ has a double face: on one hand, it looks like a strengthening of finite additivity, in the sense of σ-additivity. On the other hand, it prevents in some sense that a finitely additive measure might behave like a σ-additive one, at least with respect to the Radon–Nikodým property. This statement is clarified by the next theorem.

THEOREM 3.4. *Let $\mu : \Sigma \to \mathbb{R}_0^+$ be any finitely additive measure, defined on the σ-algebra Σ, and enjoying property σ. The following are equivalent:*
(1) *μ is countably additive;*
(2) *for every finitely additive measure $\nu : \Sigma \to \mathbb{R}_0^+$, which is ($\varepsilon$–$\delta$) absolutely continuous with respect to μ, there exists $d\nu/d\mu$.*

PROOF. Of course, since (1) and the other assumptions imply that ν is also σ-additive, just (2) $\Rightarrow$ (1) needs to be proved. Let $(A_n)_{n\in\mathbb{N}}$ be any increasing sequence of sets in Σ, and denote by A their union. Set: $\nu(B) := \lim \mu(A_n \cap B)$, for all $B \in \Sigma$. It is clear that ν satisfies the condition (2), hence there exists $f = d\nu/d\mu$.

Now, for every $B \in \Sigma$, we have $\nu(B) = \nu(B \cap A) = \int_B f 1_A \, d\mu$, so f can be replaced by $f 1_A$. On the other hand, it is clear that $f 1_{A_n} = 1_{A_n}$ μ-a.e., hence $f = 1_A$ μ-a.e.

As a consequence, we have $\lim_{n\to\infty} \mu(A_n) = \nu(A) = \int_A f \, d\mu = \int_A 1 \, d\mu = \mu(A)$. As the sequence $(A_n)_{n\in\mathbb{N}}$ was arbitrary, μ turns out to be σ-additive. □

In the spirit of the remark preceding Theorem 3.4, we can give an antithetic example.

EXAMPLE 3.5. Let Ω be any infinite set, and let $\mathfrak{I}$ denote the ideal of all finite subsets of Ω. According to the Axiom of Choice, there exists a maximal ideal $\mathfrak{I}^*$, including $\mathfrak{I}$. Define for $A \in \mathcal{P}(\Omega)$:

$$\theta(A) := \begin{cases} 0 & \text{if } A \in \mathfrak{I}^*, \\ 1 & \text{otherwise.} \end{cases}$$

According to a well-known theorem by Ulam [57], θ is a finitely additive measure, lacking the property σ. However, if $\nu : \mathcal{P}(\Omega) \to \mathbb{R}_0^+$ is any finitely additive measure, (0–0) absolutely continuous with respect to θ, then the only possibility is that $\nu = k\theta$, for some nonnegative constant k. In this case, obviously, $k = d\nu/d\theta$.

To find a characterization of the Radon–Nikodým property, we have to introduce the concept of *completeness*. In a recent paper by A. Basile and K.P.S. Bhaskara Rao [1] an excellent presentation is given, mainly concerning finitely additive measures defined on *algebras*. We shall adapt here one of their results, dealing with the Radon–Nikodým property.

DEFINITION 3.6. Let $\mu : \Sigma \to \mathbb{R}_0^+$ be any finitely additive measure, on a σ-algebra Σ. If A and B are elements of Σ, we set: $d_\mu(A, B) = \mu(A \triangle B)$. It is clear that (Σ, d_μ) is a semimetrizable space. We say that (Σ, μ) is *complete*, if the quotient $\Sigma/\approx$ is complete as a metric space, where $\approx$ is the natural equivalence relation in Σ.

The characterization given in [1] can be stated as follows (we recall that in our assumptions Σ is a σ-field, and contains all subsets of sets of measure 0).

THEOREM 3.7. *Let $\mu : \Sigma \to \mathbb{R}_0^+$ be any finitely additive measure, on a σ-algebra Σ. The following are equivalent:*

(1) *(Σ, d_μ) is complete.*

(2) *For every increasing sequence of sets $(F_n)_{n\in\mathbb{N}}$ in Σ, there exists a sequence of μ-null sets $(H_n)_{n\in\mathbb{N}}$ in Σ, such that*

$$\lim_{n\to\infty} \mu(F_n) = \lim_{n\to\infty} \mu(F_n \setminus H_n) = \mu\left(\bigcup_{n\in\mathbb{N}} (F_n \setminus H_n)\right)$$

From Theorem 3.7 one can easily deduce that if $\nu \ll_\varepsilon \mu$ and (Σ, d_μ) is complete, then (Σ, d_ν) is complete, too.

COROLLARY 3.8. *If (Σ, d_μ) is complete, and ν is any* signed *finitely additive bounded measure such that $\nu \ll_\varepsilon \mu$, then ν admits a Hahn decomposition.*

PROOF. We only have to show that there exists a set $P \in \Sigma$, such that $\nu(P) = \sup_{A\in\Sigma} \nu(A)$.

As ν is bounded, the number $S := \sup_{A\in\Sigma} \nu(A)$ is finite, and there exists a sequence $(A_n)_{n\in\mathbb{N}}$ in Σ, such that $S = \lim_{n\to\infty} \nu(A_n)$. Setting now

$$\nu^+(A) = \sup_{E\in\Sigma} \nu(A \cap E) \quad \text{and} \quad \nu^-(A) = -\inf_{E\in\Sigma} \nu(A \cap E), \ \forall A \in \Sigma,$$

we readily see that $\nu^+(A_n \triangle A_m) \to 0$ and $\nu^-(A_n \triangle A_m) \to 0$ as both m and n diverge. Now, $\nu^+$and ν^- are both (ε–δ) absolutely continuous with respect to μ, and so is $|\nu| := \nu^+ + \nu^-$, hence there exists a set $P \in \Sigma$, such that $\lim_{n\to\infty} |\nu|(A_n \triangle P) = 0$. The set P has the required property. □

COROLLARY 3.9. *The following are equivalent:*

(1) *(Σ, d_μ) is complete;*

(2) *For every nonnegative finitely additive measure $\nu \ll_\varepsilon \mu$, there exists $d\nu/d\mu$.*

PROOF. We only prove the implication (1) ⇒ (2). As $\nu \ll_\varepsilon \mu$, we see that the signed measure $\nu - r\mu$ is ε–δ absolutely continuous with respect to μ, and therefore it has a Hahn decomposition. Due to Corollary 1.3, this implies the existence of $d\nu/d\mu$. □

The corollary above has been proved in [1].

A different approach to the problem consists in finding additional conditions on the two measures μ and ν, besides absolute continuity, which ensure the existence of $d\nu/d\mu$.

One of the first consequences of Theorem 1.2 and Corollary 1.3 is the following (see also [3,12,13]):

THEOREM 3.10. *Let μ, ν be two nonnegative finitely additive measures, defined on the same σ-algebra Σ, and such that $\nu \ll_\varepsilon \mu$. If the range of the pair (μ, ν) is a closed subset of $\mathbb{R}^2$, then there exists $d\nu/d\mu$.*

PROOF. In view of Corollary 1.3, it is enough to show that, for every real number $r > 0$, there exists a Hahn decomposition for $\nu - r\mu$. But it is clear from the assumption on the range of (μ, ν) that the range of $\nu - r\mu$ is a closed subset of $\mathbb{R}$, hence there exists an element $A \in \Sigma$ in which $\nu - r\mu$ attains its maximum: so (A, A^c) is a Hahn decomposition for $\nu - r\mu$, and we are done. □

Of course, closedness of the range of (μ, ν) is just a sufficient condition. In [3] and [12] necessary and sufficient conditions are given on the range of (μ, ν) in order that $d\nu/d\mu$ exists. In both papers, the condition involves the so-called *exposed points* of the range, according with the following definition.

DEFINITION 3.11. Let R be any convex, bounded subset of $\mathbb{R}^n$. Let us denote by $\overline{R}$ its closure. We say that a point $Q \in \mathbb{R}^n$, $Q \in \partial R$, is an *exposed* point for R if for every hyperplane H, supporting R at Q, we have $\overline{R} \cap H = \{Q\}$.

THEOREM 3.12. *Let (μ, ν) be a pair of nonnegative finitely additive measures, defined on a σ-algebra Σ and such that $\nu \ll_\varepsilon \mu$, and let R denote the range of (μ, ν). Then the following are equivalent:*

(1) *There exists $d\nu/d\mu$.*
(2) *The convex hull of R contains its exposed points.*

The theorem above has been proved in [3].

In [12] a similar theorem is stated, for the case of *continuous* measures.

A measure $\mu : \Sigma \to \mathbb{R}_0^+$ is said to be *continuous* if for every $\varepsilon > 0$ it is possible to decompose Ω into a finite number of subsets $A_1, A_2, \ldots, A_n$ belonging to Σ, such that $\mu(A_i) < \varepsilon$ for all i.

If μ and ν are both continuous, finitely additive and nonnegative, it is well-known that the range of (μ, ν) is a bounded convex subset of the plane (see also [11]), hence condition (2) of Theorem 3.12 can be expressed in a simpler way. However, in [12] the following result has been proved, which gives a full description of those bounded convex sets $R \subset \mathbb{R}^2$ that are the range of some pair (μ, ν) for which $d\nu/d\mu$ exists.

THEOREM 3.13. *Let R be any bounded convex subset of $[0, \infty[^2$. Then the following properties are equivalent:*

(1) *There exists a pair (μ, ν) of nonnegative continuous finitely additive measures, defined on a suitable σ-algebra, such that there exists $d\nu/d\mu$, and such that R is the range of (μ, ν).*

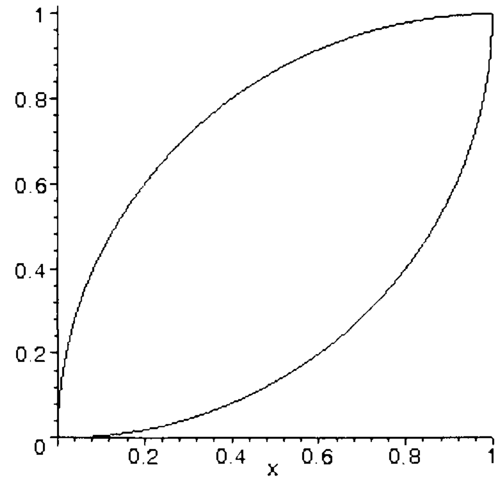

Fig. 1. Equation of the lower curve: $y = 1 - \sqrt{(1 - x^2)}$.

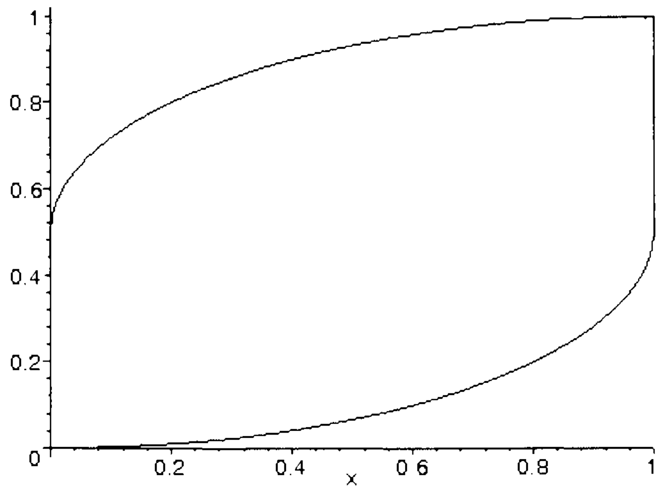

Fig. 2. Equation of the lower curve: $y = (1 - \sqrt{(1 - x^2)})/2$.

(2) *R contains its exposed points, and for every segment $L \subset \partial R$, at least a (possibly degenerate) closed subsegment $I \subset L$ is contained in R.* (See Figure 5 (a) and (b).)

Usually, if (μ, ν) is a pair of nonnegative continuous finitely additive measures, the range looks like in Figure 1, or in Figure 3, when $\nu \ll_\varepsilon \mu$. In Figure 2 a particular situation is shown, where ν is obtained by adding to μ some measure which is singular with respect to μ.

We observe that the range is always symmetric with respect to the midpoint $(\mu(\Omega)/2, \nu(\Omega)/2)$: hence in Figure 5 (a) and (b) just half of ∂R has been drawn.

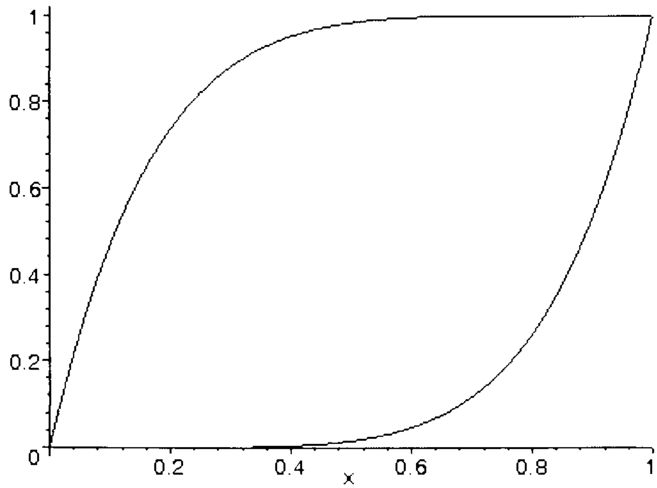

Fig. 3. Equation of the lower curve: $y = x^6$.

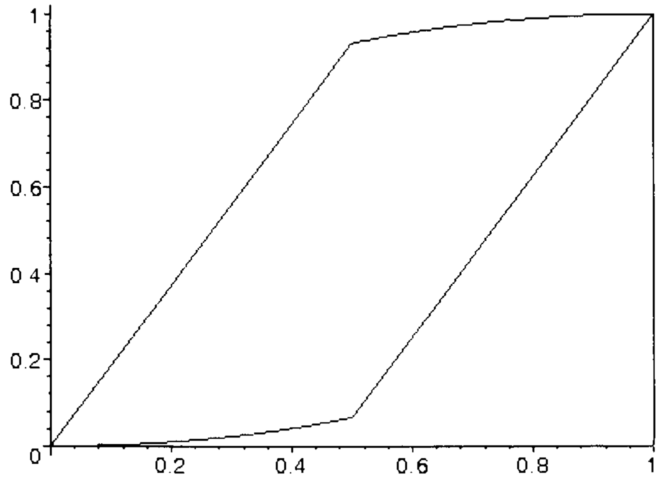

Fig. 4. Lower curve: $y = (1 - \sqrt{(1 - x^2)})/2$, for $x < 0.5$, then linear.

4. The Banach-valued case

The results from the previous sections can be easily extended to bounded signed measures, since the Jordan decomposition (even in the finitely additive case) always exists. Similarly a complex measure or function can be studied investigating separately its real and imaginary part. In the same way we can also extend the previous results to measures or functions taking values in a finite-dimensional space.

A really different situation occurs with infinite-dimensional measures, as we shall see in this section. The research on this subject led, mainly in the seventies, to a variety of quite

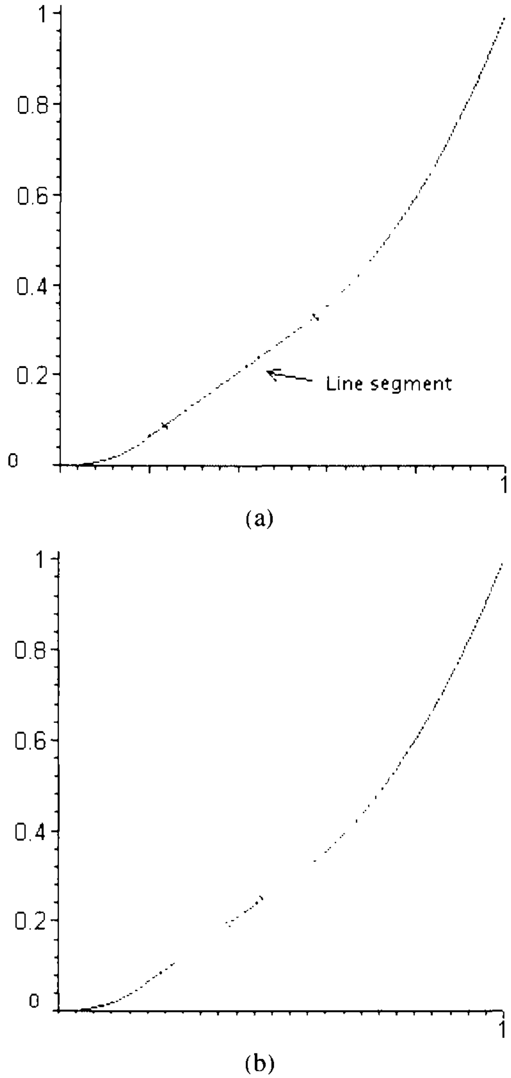

Fig. 5. (a) Range closed. (b) Range not closed. Part of the segments is missing.

interesting results, relating the Radon–Nikodým property to topological and geometric properties of the range space. Here, we shall confine ourselves to the Banach-valued σ-additive measures, and with respect to the Bochner integral. (As to absolute continuity, for the σ-additive case, the (ε–δ) definition is still equivalent to the (0–0) one.) Maybe the richest survey on this subject is [20], so we refer the reader to that paper, in order to find the proofs missing here, and an exhaustive historical account.

We start, recalling the definition of Bochner integral [6,21,22].

DEFINITION 4.1. Given a finite measure space (Ω, Σ, μ), where Σ is a σ-algebra, and a Banach space X, a function $f: \Omega \to X$ is said to be *simple* if it is of the form: $f = \sum_{i=1}^{N} x_i 1_{A_i}$, where $x_i \in X$ and $A_i \in \Sigma$ for all $i = 1, \ldots, N$. If this is the case, then the *integral* of f is the element:

$$\int f \, d\mu := \sum_{i=1}^{N} x_i \mu(A_i) \in X.$$

In the same framework, if $(f_n)_{n \in \mathbb{N}}$ is any sequence of simple functions, we say that f_n *converges in measure* to some function f, if the following holds:

$$\lim_{n \to \infty} \mu^*\left(\left\{\omega \in \Omega: \left\| f_n(\omega) - f(\omega) \right\| > \varepsilon\right\}\right) = 0$$

for all $\varepsilon > 0$ (here, $\mu^*(A)$ is the outer measure defined in Section 2).

In case $f : \Omega \to X$ is the limit in measure of some sequence $(f_n)_{n \in \mathbb{N}}$ of simple functions, then f is said to be *measurable* (also, *strongly measurable*).

We say that $f : \Omega \to X$ is *(Bochner)-integrable*, if there exists a sequence $(f_n)_{n \in \mathbb{N}}$ of simple functions, converging in measure to f, and such that

$$\lim_{(n,m) \to \infty} \int \| f_n - f_m \| \, d\mu = 0.$$

It can be proved that, if $f : \Omega \to X$ is integrable, then the sequence $(\int f_n \, d\mu)_{n \in \mathbb{N}}$ is convergent in X, and the limit is independent of the particular sequence $(f_n)_{n \in \mathbb{N}}$. Of course, the limit $\lim \int f_n \, d\mu$ is called the *integral* of f.

Also, as in the real-valued case, one can prove that convergence in measure implies convergence a.e. for some subsequence, hence a strongly measurable function is essentially *separably valued*, i.e., there exists a separable subspace $Y \subset X$, such that $f(\omega) \in Y$ for all but a μ-null set of elements ω. Moreover, a strongly measurable function is also *weakly measurable*, i.e., $\langle x^*, f \rangle$ is measurable for all x^* belonging to X^*.

Like in the scalar case, it can be proved that integrability of f implies integrability of $\|f\|$, and that $\| \int f \, d\mu \| \leqslant \int \|f\| \, d\mu$. When f is integrable with respect to μ, we also say that f is in $L^1_X(\mu)$. The latter is a Banach space if we identify functions which coincide a.e. with the norm $\|f\| = \int \|f\| \, d\mu$. Moreover, if f is integrable, then $f 1_E$ is also integrable, for all $E \in \Sigma$, and the integral becomes a function of E: we set

$$f\mu(E) := \int_E f \, d\mu := \int f 1_E \, d\mu.$$

One can see that $f\mu$ is an X-valued measure, which is separably valued, whose *variation* is $\|f\|\mu$. The *variation* of an X-valued measure ν is denoted by $|\nu|$ and is defined as:

$$|\nu|(E) = \sup \sum \left\| \nu(E_i) \right\|,$$

where the supremum is taken over all finite partitions of E: one can also regard $|\nu|$ as the *least upper bound* of $\{|\langle x^*, \nu\rangle|: x^* \in X^*, \|x^*\| = 1\}$, in the lattice of measures.

Moreover, $f\mu \ll \mu$.

The converse question leads, of course, to the problem of the existence of a Radon–Nikodým derivative. The problem can be stated in various different ways:

(1) Given a measure $\nu : \Sigma \to X$, which is absolutely continuous with respect to a scalar positive measure μ, does there exist a Bochner-integrable function f, such that $\nu = f\mu$?

(2) Given a measure $\nu : \Sigma \to X$, with finite variation, $|\nu|$, does there exist a function f, Bochner integrable with respect to $|\nu|$, and such that $\nu = f|\nu|$?

Of course, a necessary condition for an affirmative answer to (1), is that ν has finite variation. Moreover, if $|\nu|$ is finite and $\nu \ll \mu$, then it is clear that $|\nu| \ll \mu$: hence, an affirmative answer to (2) would yield an affirmative answer to (1). Conversely, assuming the existence of $d\nu/d\mu$, one can decompose μ into the sum $\mu_1 + \mu_2$, where $\mu_1 \ll |\nu|$ and $\mu_2 \perp |\nu|$. (See Theorem A4.) Hence $|\nu| \ll \mu_1$, $d\nu/d\mu = d\nu/d\mu_1$, and $d\nu/d|\nu| = (d\nu/d\mu_1)(d\mu_1/d|\nu|)$.

In conclusion, we can see that (1) and (2) are essentially equivalent.

Moreover, in [17], Chatterji remarks that in (1) above the σ-algebra Σ can be equivalently replaced by the Borel σ-algebra on the unit interval, and (up to irrelevant renorming) the measure μ by the usual Lebesgue measure.

Now, we give a couple of examples showing that the answer is not always affirmative.

EXAMPLE 4.2. Let X be the L^1space over the unit interval in $\mathbb{R}$. Also, let $([0, 1], \Sigma, \lambda)$ be the measure space, where Σ denotes the Borel σ-field, and λ the Lebesgue measure. For all $A \in \Sigma$, set $\nu(A) = 1_A$, i.e., the *characteristic function* of the set A.

It is easy to see that ν is an X-valued countably additive measure, defined on Σ, and that $|\nu| = \lambda$. However, if a Radon–Nikodým derivative $f := d\nu/d\lambda$ existed, we would have:

$$\int_A g(x)\,dx = \langle g, 1_A\rangle = \int_A \langle f(x), g\rangle\,dx$$

for all $A \in \Sigma$, and all $g \in L^\infty$ $(\equiv (L^1)^*)$. Therefore, for every fixed $g \in L^\infty$, we would have:

$$g(x) = \langle f(x), g\rangle \text{ a.e.}$$

Hence, there would exist a λ-null set $N \in \Sigma$ such that, for all α, β in $[0, 1] \cap \mathbb{Q}$, $\alpha < \beta$, we would have

$$\left\langle f(x), 1_{[\alpha,\beta]}\right\rangle = 1$$

for all $x \in N^c \cap [\alpha, \beta]$. So, if we fix $x \notin N$, and choose $(\alpha_n)_{n\in\mathbb{N}}$, $(\beta_n)_{n\in\mathbb{N}}$ in such a way that $\alpha_n < x < \beta_n$, $\alpha_n, \beta_n \in \mathbb{Q}$, and $\beta_n - \alpha_n \to 0$, we get:

$$\lim_{n\to\infty} \left\langle f(x), 1_{[\alpha_n,\beta_n]}\right\rangle = 1$$

which is impossible, since $f(x) \in L^1$.

EXAMPLE 4.3. The next example concerns the same measure space, $([0,1], \Sigma, \lambda)$, but with a different range: here, X is c_0, the space of all real sequences, vanishing at infinity. Let us define:

$$\mu(A) := \big(\mu_1(A), \ldots, \mu_n(A), \ldots\big).$$

where

$$\mu_n(A) = \int_A \sin(2\pi n x)\, dx$$

for all $A \in \Sigma$, and $n \in \mathbb{N}$.

As $|\mu_n(A)| \leqslant \lambda(A)$ for all A and all n, it is clear that μ has bounded variation, and $|\mu|([0,1]) \leqslant 1$. Obviously, $\mu \ll \lambda$, however $d\lambda/d\mu$ does not exist. Indeed, such a derivative f, $f = (f_1, \ldots, f_n, \ldots)$, should satisfy:

$$\mu_n(A) = \int_A \sin(2\pi n x)\, dx = \int_A f_n(x)\, dx$$

for all A and n. Thus, it would be $f_n(x) = \sin(2\pi n x)$ a.e., which is impossible, because $\sin(2\pi n x)$ is not vanishing, as $n \to \infty$.

The first affirmative answer to the Radon–Nikodým problem in infinite-dimensional spaces is due to Birkhoff.

THEOREM 4.4 ([4]). *If X is a Hilbert space, then every σ-additive measure $\mu : \Sigma \to X$ with bounded variation admits a Radon–Nikodým derivative with respect to $|\mu|$.*

We do not give the proof now, because we will see later some generalizations of this result.

REMARK 4.5. What happens if we try to adapt Theorem 4.4 to the example given in Example 4.2? One can always think of 1_A as an element of L^2, and of ν as an L^2-valued σ-additive measure, with $\nu \ll \lambda$. However, by the same argument, one can still show that $d\nu/d\lambda$ does not exist. What is wrong? A simple calculation shows that μ does *not* have bounded variation, when considered with values in L^2. This remark also shows the importance for a measure of having bounded variation. (However, more can be said on this example, see Remark 4.18.) From now on, we shall write BV to mean "bounded variation".

DEFINITION 4.6. We say that a Banach space X has the *Radon–Nikodým Property* (RNP) if (1) or (2) above holds, i.e., every countably additive X-valued BV measure μ, defined on the measure space (Ω, Σ), admits a Radon–Nikodým derivative with respect to $|\mu|$.

Hence, according to Theorem 4.4, all Hilbert spaces have the RNP.

Other spaces with RNP can be derived from another important sufficient condition, due to Dunford and Pettis [23].

THEOREM 4.7. *If X is a Banach space, whose* dual space *X^* is separable, then X^* has RNP.*

PROOF (*Sketch*). Let $\nu : \Sigma \to X^*$ be any BV measure, with $\nu \ll \mu$. For any $x \in X$, and any $A \in \Sigma$, set $\nu_x(A) = \langle \nu(A), x \rangle$. It is clear that $\nu_x \ll \mu$. Of course, there exists $d\nu_x/d\mu$. The idea now is to set $f_x = d\nu_x/d\mu$, and to show that f_x defines, as x varies in X, a linear continuous functional f; however one must take into account that f_x is defined a.e., hence a "good" function f must be defined carefully, and this can be done, if X^* is separable. □

For example, the space L_1 is not a dual space, because we know from Example 4.2 that it has not RNP, however l_1 is a separable dual, hence it enjoys RNP.

We will see in the sequel other classes of spaces with that property, but we will introduce first a very important definition.

DEFINITION 4.8. Given a bounded nonempty subset B of a Banach space X, we say that B is *dentable* if for every $\varepsilon > 0$ there exists a point $x_\varepsilon \in B$ such that x_ε does not belong to the closed convex hull of $B \setminus B(x_\varepsilon, \varepsilon)$. In case it is possible to choose the same x for all ε, then x is said to be a *denting point* of B.

Dentable sets are closely related to the existence of Radon–Nikodým derivatives, as Rieffel showed in [47,48].

We first remark that B is dentable whenever every countable subset of B has this property. This was proved by Maynard [39], and can be seen as follows: assume that B is not dentable; then there exists $\varepsilon > 0$ such that $x \in \overline{co}(B \setminus B(x, \varepsilon))$, $\forall x \in B$. So, choose any element $x \in B$, and points $y_1, \ldots, y_n \in B$, such that $\|y_i - x\| > \varepsilon$ for all i, and some convex combination z_1 of these points belongs to $B(x, \varepsilon/2)$. For the initial point x, and each of these points, say y_j, one can choose a finite subset $F_j \subset B$, such that $\|y - y_j\| > \varepsilon$, for any $y \in F_j$, and such that some convex combination z_2 of F_j belongs to $B(y_j, \varepsilon/4)$. Now, the union of the sets F_j, together with x and the points y_j, gives a finite subset $E_1 \subset B$. For each element $s \in E_1$ the same construction is possible, giving rise to a finite subset E_2. We continue by induction. The union of all the sets E_j is then a countable subset of B, which is not dentable.

Now, let us state a lemma, where the construction of a Radon–Nikodým derivative is shown, under suitable assumptions.

LEMMA 4.9. *Let (Ω, Σ, μ) be any finite measure space, and $\nu : \Sigma \to X$ be any BV measure, $|\nu| \ll \mu$. Then $d\nu/d\mu$ exists, whenever the following condition holds:*

(a) *For every $\varepsilon > 0$ there exist sequences $(x_n^\varepsilon)_{n \in \mathbb{N}}$ in X, and $(H_n^\varepsilon)_{n \in \mathbb{N}}$ in Σ, such that the sets H_n^ε are pairwise disjoint, have positive measure, $\mu(\Omega) = \sum \mu(H_n^\varepsilon)$, and*

$$\left\{ \frac{\nu(A)}{\mu(A)} : A \in \Sigma,\ A \subset H_n^\varepsilon \right\} \subset B\left(x_n^\varepsilon, \varepsilon\right).$$

PROOF. Assume that (a) holds, and fix $\varepsilon > 0$. Consequently, a sequence $(x_n^\varepsilon)_{n\in\mathbb{N}}$ exists in X, and a corresponding sequence $(H_n^\varepsilon)_{n\in\mathbb{N}}$ in Σ, according to (a). Now choose n_ε large enough, so that $\mu(\Omega\setminus\bigcup_{j\leqslant n_\varepsilon} H_j^\varepsilon) < \varepsilon$, and set

$$f_\varepsilon := \sum_{j\leqslant n_\varepsilon} x_j^\varepsilon 1_{H_j^\varepsilon}.$$

It is easy (though tedious) to see that (f_ε) is Cauchy in $L_X^1(\mu)$, hence converges to some function f in this space. Now, since $\int_E f\,d\mu = \lim \int_E f_\varepsilon\,d\mu$ as $\varepsilon \to 0$, for all $E \in \Sigma$, it is enough to compare $\nu(E)$ with $\int_E f_\varepsilon\,d\mu$. Of course, up to ε, we can assume $E \subset \bigcup_{j\leqslant n_\varepsilon} H_j^\varepsilon$, and so $\nu(E) = \sum_{j\leqslant n_\varepsilon} \nu(E\cap H_j^\varepsilon)$, while $\int_E f_\varepsilon\,d\mu = \sum_{j\leqslant n_\varepsilon} x_j^\varepsilon \mu(E\cap H_j^\varepsilon)$.
Now, as $E\cap H_j^\varepsilon \subset H_j^\varepsilon$, we have $\|\nu(E\cap H_j^\varepsilon) - x_j^\varepsilon\mu(E\cap H_j^\varepsilon)\| \leqslant \varepsilon\mu(E\cap H_j^\varepsilon)$ by (a), therefore $\|\nu(E) - \int_E f_\varepsilon\,d\mu\| \leqslant \varepsilon\mu(\Omega)$, and this proves the lemma. □

THEOREM 4.10 (Rieffel). *Let (Ω, Σ, μ) be any finite measure space, and let $\nu : \Sigma \to X$ be any BV measure, with $|\nu| \ll \mu$. The following are equivalent*:

(1) *There exists $d\nu/d\mu$.*

(2) *For each set $E \in \Sigma$, $\mu(E) > 0$, there exists $D \subset E$, $D \in \Sigma$, $\mu(D) > 0$, such that*

$$\mathcal{A}(D) = \left\{\frac{\nu(A)}{\mu(A)} \colon A \subset D,\ A \in \Sigma,\ \mu(A) > 0\right\} \quad \textit{is dentable.}$$

PROOF (*Sketch*). (1) ⇒ (2) We first observe that the result is obvious, if $d\nu/d\mu$ is simple: for every set $E \in \Sigma$, having positive measure μ, there exists $D \subset E$, with $D \in \Sigma$, and $\mu(D) > 0$, such that $d\nu/d\mu$ is constant in D, and hence $\{\frac{\nu(A)}{\mu(A)} \colon A \subset D, A \in \Sigma, \mu(A) > 0\}$ is a singleton. In the general case, let f denote the derivative $d\nu/d\mu$, and let $(s_n)_{n\in\mathbb{N}}$ be any sequence of simple functions, converging to f in the L^1-norm, and also almost uniformly. For any set $E \in \Sigma$, such that $\mu(E) > 0$, there exists a set $D \subset E$, with $D \in \Sigma$, such that $\mu(D) > 0$, and $s_n \to f$ uniformly on D. Now, it is easy to apply the previous argument.

(2) ⇒ (1) We shall use Lemma 4.9. To prove (a), we proceed as follows: first, we show that, for every set $E \in \Sigma$, such that $\mu(E) > 0$, and for any $\varepsilon > 0$, there exists $D \subset E$, with $D \in \Sigma$, and $\mu(D) > 0$, such that the set $\mathcal{A}(D)$ is contained in $B(x, \varepsilon)$ for some element $x \in X$.

Once we have proved this, it is possible to replace E with D^c, and iterate the procedure, constructing a disjoint sequence as in (a).

So fix $E \in \Sigma$, with $\mu(E) > 0$, and let $\varepsilon > 0$. We know that there exists $E_0 \subset E$, with $E_0 \in \Sigma$, and $\mu(E_0) > 0$, such that $\mathcal{A}(E_0)$ is dentable. Let x be an element of $\mathcal{A}(E_0)$, such that $x \notin \overline{co}(\mathcal{A}(E_0)\setminus B(x,\varepsilon))$. Write $x = \nu(D_0)/\mu(D_0)$, for suitable $D_0 \subset E_0$. If $\mathcal{A}(D_0) \subset B(x, \varepsilon)$, we have finished. Otherwise, there exists $E_1 \subset D_0$, with $\mu(E_1) > 0$, such that

$$\left\|\frac{\nu(E_1)}{\mu(E_1)} - x\right\| > \varepsilon.$$

As $\frac{\nu(E_1)}{\mu(E_1)} \in \mathcal{A}(D_0)$, we also have $\frac{\nu(E_1)}{\mu(E_1)} \in \overline{co}(\mathcal{A}(D_0)\setminus B(x,\varepsilon))$.

Let us denote by k_1 the smallest positive integer for which such an element E_1 exists, satisfying $\mu(E_1) \geqslant \frac{1}{k_1}$, and choose in this way E_1. Set $D_1 = D_0 \setminus E_1$. It is impossible that $\mu(D_1) = 0$, because this would imply

$$x = \frac{\nu(D_0)}{\mu(D_0)} = \frac{\nu(E_1)}{\mu(E_1)},$$

a contradiction.

Now, if $\mathcal{A}(D_1) \subset B(x, \varepsilon)$, we are done. Otherwise, continuing that way, we can produce a disjoint sequence $(E_n)_{n \in \mathbb{N}}$ in Σ, an increasing sequence $(k_n)_{n \in \mathbb{N}}$ in $\mathbb{N}$, such that $\mu(E_n) \leqslant \frac{1}{k_n}$, and $\frac{\nu(E_n)}{\mu(E_n)} \in \overline{co}(\mathcal{A}(D_0) \setminus B(x, \varepsilon))$. Of course, we have $k_n \uparrow \infty$. Setting $E_\infty = \bigcup E_n$, and $D = D_0 \setminus E_\infty$, we can deduce that D is the desired set, i.e., $\mu(D) > 0$ and $\mathcal{A}(D) \subset B(x, \varepsilon)$. Indeed, if it were $\mu(D) = 0$, we would have

$$x = \frac{\nu(D_0)}{\mu(D_0)} = \frac{\nu(E_\infty)}{\mu(E_\infty)} = \frac{\sum \nu(E_n)}{\mu(E_\infty)} = \sum \frac{\nu(E_n)}{\mu(E_n)} \frac{\mu(E_n)}{\mu(E_\infty)} \in \overline{co}\big(\mathcal{A}(E_0) \setminus B(x, \varepsilon)\big),$$

which is impossible.

Now, to show that $\mathcal{A}(D) \subset B(x, \varepsilon)$, fix $D' \subset D$, with $D' \in \Sigma$ and $\mu(D') > 0$. If $\frac{\nu(D')}{\mu(D')} \notin B(x, \varepsilon)$, then $\frac{\nu(D')}{\mu(D')} \in \overline{co}(\mathcal{A}(E_0) \setminus B(x, \varepsilon))$, because $D' \subset D_0 \setminus \bigcup_{i=1}^{n} E_i$ for all n. But then, by the choice of $(k_n)_{n \in \mathbb{N}}$, we must have $\mu(D') < \frac{1}{k_n}$ for all n, and hence $\mu(D') = 0$, a contradiction. □

Let us see some consequences of Theorem 4.10.

For instance, we can deduce that if X has the RNP then every subspace of X has the same property. Indeed, the construction of Lemma 4.9 shows that, if ν takes values in a subspace $Y \subset X$, then the derivative can be constructed as a limit of Y-valued simple functions.

Another consequence is that X enjoys RNP as soon as every bounded subset of X is dentable. However, the converse is also true, as Huff in [29], and Davis and Phelps in [18], independently proved.

THEOREM 4.11. *Let X be any Banach space. The following are equivalent.*

(1) *X enjoys RNP.*

(2) *Every bounded subset of X is dentable.*

We refer the reader to [20] for a proof of the implication (1) ⇒ (2).

Another consequence is that *Radon–Nikodým Property is separably determined*, i.e., a Banach space X possesses RNP if and only if every separable subspace $Y \subset X$ has RNP. This is now an easy consequence of Theorem 4.11, and the remark following Definition 4.8: a set is dentable as soon as its countable subsets are dentable.

As weakly compact sets are dentable, it is now clear that every *reflexive* Banach space has the RNP.

An interesting consequence of Theorems 4.7 and 4.11 is that X enjoys RNP as soon as each of its separable subspaces are duals. In [52], Stegall showed a stronger result:

THEOREM 4.12. *The following are equivalent, for any Banach space X:*
(1) *X^* has RNP;*
(2) *every separable subspace of X has separable dual;*
(3) *every separable subspace of X is embedded in some separable dual.*

A consequence of Theorem 4.12 is that *the dual of a separable Banach space possesses RNP if and only if it is separable.*

Some of the previous results are also included in a "martingale-type result", obtained by Chatterji in [17]. For the sake of completeness, we give some definitions.

DEFINITION 4.13. Let (Ω, Σ, P) be any probability space, i.e., any measure space, with $P(\Omega) = 1$. Given any Bochner-integrable function $f : \Omega \to X$ (here, X is any Banach space), and given any sub-σ-algebra $\Sigma_0 \subset \Sigma$, the *conditional expectation* of the function f with respect to Σ_0 is the Bochner-integrable function (defined P-a.e.), denoted by $E(f|\Sigma_0)$, which has the following two properties:
(1) $E(f|\Sigma_0)$ is strongly Σ_0-measurable;
(2) $\int_F f\,dP = \int_F E(f|\Sigma_0)\,dP$ for any $F \in \Sigma_0$.

The existence of $E(f|\Sigma_0)$ is *independent* of the RNP, and is proved first directly, for *simple* functions f, and then by approximation, in the general case.

DEFINITION 4.14. Let X, and (Ω, Σ, P) be as above. An X-valued *martingale* is a sequence $(f_n, \Sigma_n)_{n \in \mathbb{N}}$ of Bochner-integrable functions f_n and sub-σ-algebras $\Sigma_n \subset \Sigma$, such that:
(1) $\Sigma_n \subset \Sigma_{n+1}$, $\forall n \in \mathbb{N}$;
(2) $f_n = E(f_{n+1}|\Sigma_n)$, $\forall n \in \mathbb{N}$.

The martingale $(f_n, \Sigma_n)_{n \in \mathbb{N}}$ is said to be *convergent* if there exists a Bochner-integrable function $f_\infty : \Omega \to X$, such that $f_n = E(f_\infty|\Sigma_n)$ for all $n \in \mathbb{N}$.

A typical way to obtain martingales is the following: assume $\Omega = [0, 1]$, Σ is the Borel σ-algebra, and λ the Lebesgue measure. Construct a sequence of decompositions $(D_n)_{n \in \mathbb{N}}$ of the unit interval, first splitting $[0, 1]$ into two sub-intervals of the same length, thus obtaining D_1, and then, by induction, dividing each interval from D_{n-1} into two disjoint sub-intervals of the same length in order to get D_n. Let Σ_n be the (σ-) algebra generated by D_n. Now, if ν is any measure defined on Σ, set: $f_n(x) = \nu(I_n(x))/\lambda(I_n(x))$, where $I_n(x)$ is the unique interval of D_n containing x. It is clear that f_n is constant on each interval of D_n, hence it is Σ_n-measurable. It is also easy to see that $\int_{I_n} f_{n+1}\,d\lambda = \int_{I_n} f_n\,d\lambda$ $(= \nu(I_n))$, which implies that $(f_n, \Sigma_n)_{n \in \mathbb{N}}$ is a martingale.

If this martingale is convergent, and if ν is σ-additive, then the function f_∞ satisfies the identity:

$$\int_F f_\infty\,d\lambda = \nu(F),$$

for any $F \in \bigcup \Sigma_n$, and therefore for any $F \in \Sigma$, i.e., we have $f_\infty = d\nu/d\lambda$ (and of course $\nu \ll \lambda$).

In case ν is real-valued, a classical condition for the convergence of a martingale is particularly simple: as soon as $\sup_{n\in\mathbb{N}} \int_\Omega |f_n|\, d\lambda < \infty$, the functions f_n are uniformly integrable, and pointwise converging to f_∞.

When the martingale (or the measure ν) takes values in a general Banach space X, this is no longer true, in general. Chatterji's result clarifies the situation.

THEOREM 4.15 ([17]). *Let (Ω, Σ, P) be any probability space, and assume that X is a Banach space. The following are equivalent:*

(1) *X has the RNP with respect to (Ω, Σ, P).*

(2) *Every X-valued martingale $(f_n, \Sigma_n)_{n\in\mathbb{N}}$, satisfying $\sup_{n\in\mathbb{N}} \int_\Omega |f_n|\, dP < \infty$, is convergent, in the sense that f_n converges P-a.e. and strongly in X to a Bochner-integrable function $f_\infty : \Omega \to X$, such that $E(f_\infty|\Sigma_n) = f_n$, $\forall n \in \mathbb{N}$.*

We shall not give the proof here: for the implication (2) $\Rightarrow$ (1), the construction above gives an idea of the main steps. The converse is less elementary. As a consequence of this result, one can easily deduce, besides the already mentioned classes of reflexive spaces, and separable dual spaces, another class of Banach spaces with the RNP, i.e., the *weakly complete spaces with separable dual.*

We conclude this section with an analytic result, due to Moedomo and Uhl [41], connecting "weak" derivatives with "strong" ones. We need a further definition. (See Chapter 12.)

DEFINITION 4.16. Let (Ω, Σ, μ) be any finite measure space, and X any Banach space. Given a strongly measurable function $f : \Omega \to X$ (see Definition 4.1), we say that f is *Pettis integrable* if for every set $A \in \Sigma$ there exists an element $J(A) \in X$, such that

$$\langle x^*, J(A)\rangle = \int_A \langle x^*, f\rangle\, d\mu$$

for any $x^* \in X^*$. The element $J(A)$ is called the *Pettis Integral* of f in A, and is denoted by $(P)\int_A f\, d\mu$.

THEOREM 4.17 ([41]). *Let $\nu : \Sigma \to X$ be any σ-additive measure, $\nu \ll \mu$. The following are equivalent:*

(1) *There exists a Pettis integrable function $f : \Omega \to X$ such that*

$$(\mathrm{P}) \int_A f\, d\mu = \nu(A)$$

for all $A \in \Sigma$. (f is the Pettis derivative *of ν.)*

(2) *For every $\varepsilon > 0$ there exists a set $H_\varepsilon \in \Sigma$, such that $\mu(H_\varepsilon^c) < \varepsilon$, and the set $\{\frac{\nu(A)}{\mu(A)}: A \subset H_\varepsilon,\ \mu(A) > 0\}$ is relatively compact.*

(3) *For every $A \in \Sigma$, with $\mu(A) > 0$, there exists $B \in \Sigma$, $B \subset A$, with $\mu(B) > 0$, such that $\{\frac{\nu(E)}{\mu(E)}: E \subset B,\ \mu(E) > 0\}$ is relatively compact.*

(*In items* (2) *and* (3) *above, "relatively compact" can be equivalently replaced by "weakly relatively compact".*)

Moreover, f turns out to be Bochner integrable, if and only if ν is BV.

We shall not give a proof of Theorem 4.17, however we emphasize again the fact that weak compactness of bounded sets ensures that they are dentable, so at least part of this theorem is a consequence of Theorem 4.10.

REMARK 4.18. In some sense, Theorem 4.17 tells us that bounded variation is not an essential requirement for ν to be an integral measure: if one of the properties (2) or (3) of Theorem 4.17 is satisfied, ν is at least the *Pettis* integral of some function f, with respect to μ. For example, if X is reflexive, the only requirement is that (2) or (3) above hold, with "relatively compact" replaced by "bounded"; however, if ν is not BV, this is not automatic, even for Hilbert spaces: if we consider the Remark 4.5, the example outlined there deals with an L^2-valued measure ν, absolutely continuous with respect to the Lebesgue measure λ, lacking even a Pettis derivative: indeed, the "averages" $\frac{\nu(A)}{\lambda(A)}$ fail to be bounded, as soon as $\lambda(A)$ decreases to 0.

5. Finitely additive Banach-valued measures

Of course, the problems concerning the existence of a Radon–Nikodým derivative are still harder, when one allows also finitely additive measures into consideration. As we already observed, even for real-valued finitely additive measures, there are examples in which the derivative does not exist, hence it makes no sense to look for spaces with a property which would take the place of RNP.

This is clarified by the next theorem, which is a Banach-valued version of the approximate Radon–Nikodým–Bochner Theorem 3.1. We need first a definition.

DEFINITION 5.1. If X is a Banach space, we shall say that it has the "approximate finitely additive Radon–Nikodým property" (AFARNP), if for any measure space (Ω, Σ, μ), where μ is a finitely additive X-valued BV measure, and for any $\eta > 0$, there exists a μ-integrable function f_η (which may be taken simple), such that

$$\left\| \mu(E) - \int_E f_\eta \, d\mu \right\| < \eta,$$

for any $E \in \Sigma$.

The following result has been communicated to us by Anna Martellotti.

THEOREM 5.2. *A Banach space X has AFARNP if and only if it has RNP.*

PROOF. Assume X has RNP and let μ be a BV finitely additive measure on (Ω, Σ), with values in X. Since μ is BV, it follows from [11] that it admits a Stone extension, i.e., there exists a countably additive measure $\tilde{\mu}$ on $\mathcal{G}_\delta$, the Baire σ-algebra of the Stone space S associated to the quotient of Σ with respect to $\mathcal{N}$, the family of all $|\mu|$-null sets.

It is also known that $\tilde{\mu}$ is BV and that

$$|\tilde{\mu}| = \widetilde{|\mu|}.$$

By RNP there exists a Bochner-integrable function $f : S \to X$ such that

$$\tilde{\mu}(F) = \int_F f \, d|\tilde{\mu}|,$$

for any $F \in \mathcal{G}_\delta$.

A density argument shows that, given $\eta > 0$, there exists a $\mathcal{G}_\delta$-simple function $g : S \to X$ such that

$$\int_S \|g - f\| \, d|\tilde{\mu}| < \eta.$$

Let $g = \sum_{i=1}^n x_i 1_{G_i}$, with G_i pairwise disjoint. Since $G_i \in \mathcal{G}_\delta$, there exist pairwise disjoint sets $E_1, \ldots, E_n$ in Σ, such that $h([E_i]) = G_i$, where h is the natural embedding of $\Sigma/\mathcal{N}$ in S.

Put now $\gamma = \sum_{i=1}^n x_i 1_{E_i}$, and let $\nu = \gamma|\tilde{\mu}|$. It is easy to check that $\tilde{\nu} = \int g \, d|\tilde{\mu}|$ and from [11] we have

$$|\tilde{\mu} - \tilde{\nu}| = \int \|f - g\| \, d\widetilde{|\mu|}.$$

and moreover $\widetilde{\mu - \nu} = \tilde{\mu} - \tilde{\nu}$ and $|\mu - \nu|(\Omega) = |\widetilde{\mu - \nu}|(S)$, so we can conclude that $|\mu - \nu|(\Omega) < \eta$.

Conversely, assume X has AFARNP. Let $\mu : \Sigma \to X$ be a countably additive BV measure. For each $\eta > 0$ there exists $f_\eta : \Omega \to X$ such that

$$\big|\mu - f_\eta|\mu|\big| < \eta.$$

Consider $\eta_n = \frac{1}{2^n}$ and let $f_n = f_{\eta_n}$. We want to show that $(f_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $L^1(|\mu|)$. Indeed, for any n and m, if we put $\nu_n = f_n|\mu|$, we have

$$\|f_n - f_m\|_1 = \int_\Omega \|f_n - f_m\|_X \, d|\mu| = |\nu_n - \nu_m|.$$

Since the total variation is a norm in the space of all countably additive BV measures on (Ω, Σ), we have

$$|\nu_n - \nu_m| \leqslant |\nu_n - \mu| + |\mu - \nu_m|.$$

hence $\|f_n - f_m\|_1 < \eta_n + \eta_m$, and so $(f_n)_{n\in\mathbb{N}}$ is Cauchy.

Since $|\mu|$ is countably additive, $L^1(|\mu|)$ is complete and therefore there exists $f \in L^1(|\mu|)$ such that $f_n \to f$ (in $L^1(|\mu|)$). By definition, $(f_n)_{n\in\mathbb{N}}$ is a defining sequence for f, namely $\int f_n \, d\mu \to \int f \, d\mu$. From the definition of f_n it follows now the conclusion, namely that $\mu = f \, d|\mu|$. □

From the second part of the proof above, it follows also that an approximate Radon–Nikodým–Bochner theorem for Banach-valued σ-additive measures does not hold, unless the range space has RNP.

From now on, we shall limit ourselves to look for conditions, necessary and/or sufficient, to ensure the existence of *some kind* of weaker Radon–Nikodým derivatives.

To be able to do that, besides the Bochner and Pettis-type integral, we shall also deal with the so-called *Gel'fand* integral, and later on with the *Bartle–Dunford–Schwartz* integral.

DEFINITION 5.3. Let $\mu : \Sigma \to \mathbb{R}_0^+$ be a finitely additive measure, and let X be any Banach space. Given a function $f : \Omega \to X^{**}$, we say that f is *Gel'fand integrable* with respect to μ, if:

(a) $\langle f, x^* \rangle$ is μ-integrable, for all $x^* \in X^*$.

(b) For every $A \in \Sigma$ there exists an element $J_A \in X$ such that

$$\langle x^*, J_A \rangle = \int_A \langle f, x^* \rangle \, d\mu.$$

for any $x^* \in X^*$.

(c) The function $A \to J_A$ is a μ-continuous finitely additive measure.

(d) For every $\varepsilon > 0$ there exists $H \subset \Omega$, such that $\mu(H^c) < \varepsilon$, and $\sup_{\omega \in H} \|f(\omega)\| < \infty$.

When a finitely additive measure $\nu : \Sigma \to X$ is given, with $\nu \ll \mu$ (throughout this section we use always the ε–δ definition of absolute continuity), we say that ν has a *Gel'fand-type derivative*, if there exists a Gel'fand integrable function $f : \Omega \to X^{**}$, for which $J_A = \nu(A)$ for all $A \in \Sigma$, i.e.,

$$\int_A \langle f, x^* \rangle \, d\mu = \langle x^*, \nu(A) \rangle,$$

for any $A \in \Sigma$, and $x^* \in X^*$.

An example of such a derivative can be seen in Example 4.3, where the function f takes values in l_∞, while the space X is c_0.

Another interesting situation is described in Example 4.2, where the existence of a Gel'fand derivative is a consequence of the Lifting Theorem [43,36,30,31] (see also Chapter 28 in this Handbook [53]), in particular for the Lebesgue measure. Let us denote by ϕ the lifting map. For any $\omega \in \Omega = [0, 1]$, $f(\omega)$ can be defined as the element of L_∞^*, which associates $\phi(h)(\omega)$ to every $h \in L_\infty$. So we see that $\langle f, h \rangle$ turns out to be simply the function $\phi(h)$, which is bounded and measurable, for each $h \in L_\infty \equiv X^*$, hence (a) is satisfied, and $\int_A \langle f, h \rangle \, d\lambda = \int_A h \, d\lambda = \langle h, 1_A \rangle = \langle h, \nu(A) \rangle$, for any $A \in \Sigma$ and $h \in X^*$. This proves (b), (c), and the fact that f is the Gel'fand derivative $d\nu/d\lambda$.

Dealing with finitely additive measures, the existence of Gel'fand derivatives can be derived from the following theorems, which include Example 4.3 as a particular case.

THEOREM 5.4 ([13]). *Let Y be any separable Banach space, and assume that (Ω, Σ, μ) is any finitely additive measure space, with μ nonnegative (and finite). Let us suppose that a bounded finitely additive scalar measure $T(y)$ is associated to every $y \in Y$, such that there exists $dT(y)/d\mu$, and moreover $|T(y)|(A) \leqslant \|y\|\mu(A)$ for every $A \in \Sigma$ and $y \in Y$, where $|T(y)|$ denotes as usual the total variation of $T(y)$. Then, there exists a function $f: \Omega \to Y^*$ such that:*

(1) *$f(\cdot)(y) = dT(y)/d\mu$ for every $y \in Y$.*

The proof of Theorem 5.4 is in some sense reminiscent of the sketch of proof given for Theorem 4.11. We skip the details, because they are too technical.

A direct consequence is the existence of *bounded* Gel'fand derivatives, under some conditions.

THEOREM 5.5 ([13]). *Assume that X is any Banach space, with separable dual, and let (Ω, Σ, μ) be as above. Suppose that $\nu: \Sigma \to X$ is a finitely additive measure, such that $d\langle x^*, \nu\rangle/d\mu$ exists, for every $x^* \in X^*$. Assuming moreover that the set*

$$S := \left\{ \frac{\nu(A)}{\mu(A)} \colon A \in \Sigma,\ \mu(A) > 0 \right\}$$

is bounded, then there exists a bounded Gel'fand type derivative $d\nu/d\mu$.

PROOF. It is sufficient to apply Theorem 5.4, setting $Y := X^*$, and $T(x^*) = \langle x^*, \nu\rangle$. □

COROLLARY 5.6 ([13]). *Let X, (Ω, Σ, μ), ν be as above. For the existence of a Gel'fand derivative $d\nu/d\mu$ it is necessary and sufficient that the following two conditions hold:*

(i) *$d\langle x^*, \nu\rangle/d\mu$ exists, for every $x^* \in X^*$, and*

(ii) *for every $\varepsilon > 0$ there exists $H \in \Sigma$, $\mu(H^c) < \varepsilon$, such that the set*

$$S(H) := \left\{ \frac{\nu(A)}{\mu(A)} \colon A \in \Sigma,\ A \subset H,\ \mu(A) > 0 \right\}$$

is bounded.

PROOF. The necessity is clear, by definition of Gel'fand integrability. To see the converse, let us take an increasing sequence of sets $(H_n)_{n\in\mathbb{N}}$, such that $\mu(H_n^c) \downarrow 0$, and with $S(H_n)$ bounded for each n: then, applying Theorem 5.5 to H_n one gets a Gel'fand derivative f_n on H_n, and *pasting* together the functions f_n gives the required derivative. □

It is interesting to compare Corollary 5.6 with the result due to Moedomo–Uhl, quoted in Theorem 4.17: in some sense, if *compactness* for the average sets is replaced by *boundedness*, then Gel'fand derivatives take the place of Pettis derivatives.

The requirement concerning the existence of $d\langle x^*, \nu\rangle/d\mu$ in the previous theorems may look too restrictive; however, we can see that it is satisfied under rather mild conditions. One of these has been outlined in [13], where the following well-known theorem by Rybakov is used [49].

Let us recall that a measure μ on Σ is said to be s-bounded, if for every disjoint sequence of sets $E_n \in \Sigma$, we have $\lim_n \mu(E_n) = 0$.

THEOREM 5.7 ([49]). *Given an s-bounded finitely additive measure $\nu : \Sigma \to X$ (here, Σ may be just an algebra), there always exists some element $x^* \in X^*$, such that $\nu \ll |\langle x^*, \nu\rangle|$.*

DEFINITION 5.8. If $\nu : \Sigma \to X$ is s-bounded and $x^* \in X^*$ is such that $\nu \ll |\langle x^*, \nu\rangle|$, then the measure $|\langle x^*, \nu\rangle|$ is called a *Rybakov control* for ν. We can (and do) assume that $\|x^*\| = 1$.

THEOREM 5.9 ([13]). *Let X be any Banach space with separable dual, and let (Ω, Σ) be any measurable space. Assume that $\nu : \Sigma \to X$ is any s-bounded finitely additive measure, and let μ be any Rybakov control for ν. If ν has weakly closed range, then a Gel'fand derivative $d\nu/d\mu$ exists if and only if condition* (ii) *of Corollary* 5.6 *holds.*

PROOF. Having in mind Corollary 5.6, it only remains to show that $d\langle x^*, \nu\rangle/d\mu$ exists, as soon as $x^* \in X^*$. Let us denote by y^* the element of X^*, such that $\mu = |\langle y^*, \nu\rangle|$. As the range of ν is weakly closed, we see that the measure $\langle sx^* - ry^*, \nu\rangle = s\langle x^*, \nu\rangle - r\langle y^*, \nu\rangle$ has closed range, hence it admits a Hahn decomposition, for all real numbers s and r, and all elements $x^* \in X^*$. Now, an easy consequence of Corollary 1.3 gives the desired derivative. □

A more general formulation can be found in the next statement.

THEOREM 5.10 ([13]). *Assume that X^* has the RNP, and let $\mu : \Sigma \to X$ be any s-bounded measure, with separable weakly closed range. Then* (ii) *of Corollary* 5.6 *is a necessary and sufficient condition for the existence of a Gel'fand derivative $d\nu/d\mu$, where μ is any Rybakov control for ν.*

PROOF. If Y denotes the separable subspace of X, generated by the range of ν, from Theorem 4.12 we find that Y^* is separable, hence the conclusion follows from the previous results. □

In order to obtain Bochner (i.e., strong) derivatives in the finitely additive case, one has to make different assumptions, as Hagood in [26] showed. The definition of Bochner integrable function in the finitely additive setting is formally the same as in the σ-additive one (see also [22]). For the sake of simplicity, we shall give here just a particular form of Hagood's result, also related to a previous work by Maynard [40].

THEOREM 5.11. *Let (Ω, Σ, μ) be any nonnegative finite and finitely additive measure space, and let $\nu : \Sigma \to X$ be any Banach-valued finitely additive measure, $\nu \ll \mu$. Then the following are equivalent*:

(1) *there exists $d\nu/d\mu$ in the Bochner sense*;

(2) *for every $\varepsilon > 0$ and $\delta > 0$, there exist $C \in \Sigma$, $\mu(C) > 0$, and $\alpha \in]0, 1[$ such that*:

(2i) $\mu(C^c) < \delta$,
(2ii) *the set* $S(C) := \{\frac{\nu(A)}{\mu(A)}: A \in \Sigma, A \subset C, \mu(A) > 0\}$ *is bounded,*
(2iii) *for all* $E \subset C$, $E \in \Sigma$, $\mu(E) > 0$, *there exists* $F \subset E$, $F \in \Sigma$, *such that* $\mu(F) > \alpha\mu(E)$ *and* $\operatorname{diam}(S(F)) < \varepsilon$.

PROOF. First, let us prove that (1) ⇒ (2). We denote by f the derivative $d\nu/d\mu$, and fix $\varepsilon, \delta > 0$. By strong measurability of f, there exists a simple function $g = \sum_{i=1}^{n} \beta_i 1_{B_i}$, such that the sets B_i form a finite partition of Ω, and $\mu^*(\{\omega \in \Omega: \|f(\omega) - g(\omega)\| > \varepsilon/4\}) < \delta$. Let us choose a set $H \in \Sigma$, such that $\{\omega \in \Omega: \|f(\omega) - g(\omega)\| > \varepsilon/4\} \subset H$, and $\mu(H) < \delta$. Set now: $C = H^c$. Then, (2i) is satisfied. Moreover, if $A \in \Sigma$, $A \subset C$, we have $\|f(\omega) - g(\omega)\| \leqslant \varepsilon/4$, for all $\omega \in A$, hence $\|\nu(A)\| \leqslant \int_A \|g\|\, d\mu + \frac{\varepsilon}{4}\mu(A) \leqslant M\mu(A)$, where $M = \max\{\|\beta_i\| + \varepsilon/4\}$. So, also (2ii) is satisfied. Finally, choose $\alpha = \frac{1}{2n}$: if E is any fixed element of Σ contained in C, denote by B^* one of the elements B_i such that $\mu(E \cap B^*) > \frac{1}{2n}\mu(E)$, and put $F = B^* \cap E$. So, $\mu(F) > \alpha\mu(E)$. Now, let us prove that the diameter of $S(F)$ is less than ε. Let β^* denote the value of g in B^*, and choose any set $A \in \Sigma$, with $A \subset F$, and $\mu(A) > 0$. We get

$$\frac{\nu(A)}{\mu(A)} = \frac{\int_A f\, d\mu}{\mu(A)} = \frac{\beta^*\mu(A)}{\mu(A)} + \frac{\int_A (f-g)\, d\mu}{\mu(A)},$$

from which we deduce that $\|\frac{\nu(A)}{\mu(A)} - \beta^*\| < \varepsilon/2$, and therefore $\operatorname{diam}(S(F)) < \varepsilon$.

We now turn to the converse, i.e., (2) ⇒ (1). Fix $\varepsilon > 0$, and $\delta = \varepsilon$. Then, let C and α be the corresponding elements, obtained from (2). Now, apply (2iii) to $E = C$: we find a set $F_1 \in \Sigma$, with $F_1 \subset C$, and $\mu(F_1) > \alpha\mu(C)$, and such that $\operatorname{diam}(S(F_1)) < \varepsilon$. If $\mu(F_1) = \mu(C)$ we are finished; otherwise, apply again (2iii) to $E = C \setminus F_1$, thus finding $F_2 \subset C \setminus F_1$, with $F_2 \in \Sigma$, such that $\mu(F_2) > \alpha\mu(C \setminus F_1)$, and $\operatorname{diam}(S(F_2)) < \varepsilon$. So, F_1 and F_2 are disjoint members of Σ, both satisfying $\operatorname{diam}(S(F_i)) < \varepsilon$. By an exhaustion argument, it is possible to get a (finite or countable) family $(F_n)_{n \in \mathbb{N}}$ of subsets of C of positive measure, each satisfying $\operatorname{diam}(S(F_n)) < \varepsilon$, and such that $\sum \mu(F_n) = \mu(C)$. Now define $f_\varepsilon = \sum \beta_n 1_{F_n}$, where β_n is, for each n, any element of $S(F_n)$. It is now easy to show that f_ε is Bochner-integrable (indeed, f_ε is bounded, by (2ii)), and

$$\left\| \nu(A) - \int_A f_\varepsilon\, d\mu \right\| < \varepsilon,$$

for any $A \in \Sigma$, $A \subset C$.

A routine argument now gives an increasing sequence $(C_k)_{k \in \mathbb{N}}$, with $\mu(C_k^c) \downarrow 0$, and a corresponding convergent sequence $(f_k)_{k \in \mathbb{N}}$, whose limit is the required derivative. □

To conclude the section, we turn to the so-called *Bartle–Dunford–Schwartz* integral, which allows to integrate a *scalar* function with respect to a *vector-valued* measure. We shall refer to the paper [37], which in turn was inspired at Musiał's paper [42].

Actually, in [37] locally convex-valued measures were considered, but here we shall limit ourselves to the special case of Banach-valued ones, for simplicity.

For the rest of this section, X will denote a Banach space, $\mu, \nu : \Sigma \to X$ will be s-bounded finitely additive measures.

DEFINITION 5.12. We denote by $\mathcal{P}$ the space of all measurable functions $f : \Omega \to \mathbb{R}$, such that $f \in L_1(|\langle x^*, \mu\rangle|)$, for any $x^* \in X^*$.

Given a function $f \in \mathcal{P}$, we say that f is *μ-integrable* if for every $A \in \Sigma$ there exists an element $\nu(A) \in X$ such that

$$\langle x^*, \nu(A)\rangle = \int_A f \, d(x^*\mu)$$

for every $x^* \in X^*$. We then set: $\nu(A) = \int_A f \, d\mu$.

In general, when f is μ-integrable, the function ν is a finitely additive measure, but it satisfies a much stronger condition than absolute continuity with respect to μ. This is evident even in two-dimensional spaces: one can choose the measure space $([0,1], \mathcal{B}, \lambda)$, $X := \mathbb{R}^2$, and then define: $\mu(A) = (\int_A x \, dx, \lambda(A))$, $\nu(A) = (\lambda(A), \int_A x \, dx)$. It is clear that μ and ν are both equivalent to λ, hence $\nu \ll \mu$, but there is no function $f : [0,1] \to \mathbb{R}$ such that $\nu(A) = \int_A f \, d\mu$: indeed, such a function should satisfy:

$$\lambda(A) = \int_A x f(x) \, dx \quad \text{and} \quad \int_A x \, dx = \int_A f(x) \, dx$$

for any $A \in \Sigma$, thus $f(x) = x$ a.e. and $x f(x) = 1$ a.e., which is clearly impossible.

The key property is introduced in the following definition.

DEFINITION 5.13. We say that ν is *scalarly uniformly absolutely continuous* with respect to μ, and write: $\nu \lll \mu$, if for every $\varepsilon > 0$ there exists $\delta > 0$ such that for every $x^* \in X^*$ and every $A \in \Sigma$ one has the implication:

$$\big|\langle x^*, \mu\rangle\big|(A) < \delta \quad \Rightarrow \quad \big|\langle x^*, \nu\rangle\big|(A) < \varepsilon.$$

We say that ν is *scalarly dominated* by μ if there exists a positive number $M > 0$ such that

$$\big|\langle x^*, \nu\rangle\big|(A) \leqslant M \big|\langle x^*, \mu\rangle\big|(A)$$

holds, for any $x^* \in X^*$ and any $A \in \Sigma$.

It is easy to see that, in case $f : \Omega \to \mathbb{R}$ is bounded and μ-integrable, the measure ν, defined in Definition 5.12 above, is scalarly dominated by μ, and satisfies $\nu \lll \mu$.

The Radon–Nikodým theorem stated in [37] asserts that, under certain conditions, depending essentially on the finite additivity of the involved measures, scalar domination is equivalent to scalar uniform absolute continuity, and that each of the two conditions is also sufficient for the existence of a bounded derivative $d\nu/d\mu$. The conditions we shall mention here are somewhat stronger, for the sake of simplicity.

THEOREM 5.14 ([37]). *Let λ be any Rybakov control for μ. Assume that for every $x^* \in X^*$, the set*

$$S^{x^*} := \left\{ \frac{\langle x^*, \mu\rangle(A)}{\lambda(A)}: A \in \Sigma, \lambda(A) > 0 \right\}$$

is bounded. Assume also that the ranges of the measures μ and ν are closed. Then the following are equivalent:

(1) $\nu \lll \mu$;

(2) *ν is scalarly dominated by μ;*

(3) *there exists a bounded μ-integrable function $f : \Omega \to \mathbb{R}$, such that*

$$\langle x^*, \nu\rangle(A) = \int_A f \, d\langle x^*, \mu\rangle,$$

for any $A \in \Sigma$, and $x^* \in X^*$.

6. Further results

In this section, we shall see some recent results, concerning different situations: one of them concerns Radon–Nikodým derivatives for measures taking their values in locally convex spaces, and more particularly in nuclear spaces; another situation concerns multivalued measures, taking values in Banach or locally convex spaces; yet a different result concerns Riesz space-valued measures; finally, we shall mention some results concerning Radon–Nikodým derivatives for a different kind of integral, the so-called *Sugeno integral*.

As usual, (Ω, Σ, μ) denotes a measure space, where Σ is a σ-algebra, and μ is any nonnegative, finitely additive measure, taking values in $[0, +\infty[$.

We start with some results concerning (finitely additive) measures, taking values in a locally convex Hausdorff space X. Some definitions are needed, in order to put appropriate conditions on X. Throughout this section, X^* denotes the *strong* dual space of X.

DEFINITION 6.1. Let X be any locally convex space, and let B denote any bounded, nonempty subset of X. B is said to be *bipolar* if $B = B^{00}$, according with the duality $\langle X, X^*\rangle$. Given any bipolar set $B \subset X$, we denote by X_B the subspace of X which can be *absorbed* by B (i.e., the space of those elements $x \in X$ such that $rx \in B$ for some positive scalar r); thus the *Minkowski functional* p_B of B can be defined on X_B (we recall that $p_B(x) = \inf\{r > 0: \frac{x}{r} \in B\}$, for $x \in X_B$). We can endow X_B with the semi-norm p_B and then consider the normed space $X_B / \approx$, where the equivalence relation is defined by: $x \approx x' \iff p_B(x - x') = 0$. We shall denote by $X(B)$ the *completion* of such normed space. We say that X satisfies the *property (SP)* if $X(B)$ is separable, for every bipolar set $B \subset X$.

Obviously, a separable locally convex space X satisfies (SP).

DEFINITION 6.2. We say that a locally convex space X has the *property (SP)'* if $X^*(B^0)$ is separable, for all bounded subsets $B \subset X$. If X^* is separable, then X has (SP)'.

When ν takes values in a space X with property (SP)', then a Radon–Nikodým theorem in the Pettis sense has been proved in [9]. The proof is too long and technical to be presented here.

THEOREM 6.3. *Suppose X has property (SP)', and $\nu : \Sigma \to X$ is any μ-continuous finitely additive measure, satisfying the following two conditions:*

(1) *there exists $d\langle x^*, \nu\rangle/d\mu$, for any $x^* \in X^*$;*

(2) *the set $S := \{\frac{\nu(A)}{\mu(A)}: A \in \Sigma, \mu(A) > 0\}$ is weakly relatively compact in X.*

Then there exists a bounded weakly measurable function $g : \Omega \to X$, such that:

$$\int_A \langle x^*, g(\cdot)\rangle \, d\mu = \langle x^*, \nu(A)\rangle,$$

for any $x^* \in X^*$ and all $A \in \Sigma$.

This theorem is applied in [9] to the case of dual-nuclear spaces. In order to state other results, we need some more definitions (see also [45]).

DEFINITION 6.4. Let X and Y denote two locally convex Hausdorff spaces, and let $\phi : X \to Y$ be any continuous linear map. We say that ϕ is *nuclear* if there exist:

(a) a sequence $(\lambda_n)_{n\in\mathbb{N}}$ in l_1;

(b) an equibounded sequence $(x_n^*)_{n\in\mathbb{N}}$ in X^*,

(c) a bounded sequence $(y_n)_{n\in\mathbb{N}}$ in Y, such that

$$\phi(x) = \sum \lambda_n y_n \langle x_n^*, x\rangle$$

for any $x \in X$.

DEFINITION 6.5. A locally convex Hausdorff space X is said to be *nuclear* if every linear continuous map $\phi : X \to Y$ is nuclear, for every Banach space Y.

This definition is not the original one, due to Grothendieck, but we have chosen this equivalent property, because we think it is easier to work with.

Nuclear spaces enjoy very interesting properties: for instance, it follows from the classical definition of a nuclear space that it is the projective limit of Hilbert spaces. Another important property is that every bounded set in a nuclear space is pre-compact. A useful condition for nuclear spaces is *quasi-completeness*, i.e., every closed bounded subset is complete. Thus, if a nuclear space X is quasi-complete, all closed bounded subsets of X are compact (hence, X is Montel). If the strong dual X^* is nuclear, then the locally convex space X is said to be *dual-nuclear*. It turns out easily that a quasi-complete dual-nuclear space X is *semi-reflexive*, i.e., the canonical embedding $c : X \to X^{**}$ is onto.

From all these remarks, one easily realizes that many interesting results can be found on measures taking values in spaces of this kind. In [9] there are listed some, for measures taking values in a dual-nuclear space. We just recall those results, which are strictly connected with Radon–Nikodým derivatives.

THEOREM 6.6 ([9]). *Let X be a quasi-complete and dual-nuclear space. Then every bounded finitely additive measure $\nu : \Sigma \to X$ is s-bounded, and admits a Rybakov control.*

THEOREM 6.7 ([9]). *Let X be as above, and assume that $\nu : \Sigma \to X$ is any bounded finitely additive measure, such that:*

(1) *$\langle x^*, \nu \rangle \ll \mu$, and there exists $\frac{d\langle x^*, \nu\rangle}{d\mu}$ in L_∞, for all $x^* \in X^*$;*

(2) *the set $S := \{\frac{\nu(A)}{\mu(A)} : A \in \Sigma, \ \mu(A) > 0\}$ is bounded.*

Then there exists a Bochner-*type derivative $\frac{d\nu}{d\mu}$.*

We remark here that "Bochner" means here that the function $\frac{d\nu}{d\mu}$ is the limit in measure of a sequence of simple functions, and the integral is the limit of the corresponding integrals. This looks like a strong conclusion, and it is worth mentioning how it is derived: from the assumptions we see that the mapping $T : X^* \to L_\infty$, defined as $T(x^*) = \frac{d\langle x^*, \nu\rangle}{d\mu}$, is nuclear. (For more details we refer the reader to [9], where also completeness of $L_\infty(\mu)$ is proved.) Therefore one can write:

$$T(x^*) = \sum \lambda_n \langle x^*, e_n \rangle y_n \quad \forall x^* \in X^*,$$

where $(\lambda_n)_{n\in\mathbb{N}} \in l_1$, $(e_n)_{n\in\mathbb{N}}$ is a bounded sequence in $X^{**} = X$, and $(y_n)_{n\in\mathbb{N}}$ is a bounded sequence in L_∞. Choosing a bounded representative f_n from the class of y_n, and putting $g_n(\omega) := \lambda_n e_n f_n(\omega)$ for all $\omega \in \Omega$, the series $\sum g_n$ converges strongly to the desired derivative.

An even stronger result holds for σ-additive measures.

THEOREM 6.8 ([9]). *Let X be as above, and assume that μ is σ-additive. If $\nu : \Sigma \to X$ is any measure, $\nu \ll \mu$, there exists a Bochner-type derivative $d\nu/d\mu$.*

We now turn to multimeasures, according with the definitions of [16] and [38]: we will just recall the notations and the most relevant results in that setting.

DEFINITION 6.9. Given a Hausdorff locally convex space X, the family of all nonempty, closed, convex, bounded subsets of X will be denoted by $C_c(X)$; if Q denotes the set of all continuous seminorms on X, for every $p \in Q$ and every pair of elements $A, B \in C_c(X)$, we set:

$$e_p(A, B) = \sup_{x\in A} \inf_{y\in B} p(x - y).$$

and

$$d_p(A, B) = \max\{e_p(A, B), e_p(B, A)\}.$$

For any $A \in C_c(X)$, we set also: $h_p(A) = d_p(A, \{0\})$.

As d_p is a pseudo-distance, the family $\{h_p: p \in Q\}$ defines a topology on $C_c(X)$, which is compatible with the following *addition*: $A + B := \overline{\{a + b: a \in A, b \in B\}}$.

We remark that the same topology arises if Q is replaced by some *absorbing* subset Q_0. Also, if X is complete, so is $C_c(X)$ with this topology. We shall assume, from now on, that X is complete. Given an absorbing subset $Q_0 \subset Q$, a subset $B \subset C_c(X)$ is Q_0-*uniformly bounded* if $\sup_{p \in Q_0} \sup_{A \in B} h_p(A) < \infty$.

Any finitely additive measure $\nu : \Sigma \to C_c(X)$ is said to be a *multimeasure* in X.

DEFINITION 6.10. Given a multimeasure $\nu : \Sigma \to C_c(X)$, for every $p \in Q$ the *p-variation* of ν is defined for all $E \in \Sigma$ as:

$$\nu_p(E) = \sup_{(A_i) \in P(E)} \sum_{i \in I} h_p(\nu(A_i)),$$

where $P(E)$ denotes the family of all finite partitions of E into sets $A_i \in \Sigma$.

We say that ν has *bounded variation* if $\nu_p(\Omega) < \infty$, $\forall p \in Q$.

DEFINITION 6.11. Given a multimeasure $\nu : \Sigma \to C_c(X)$, and a finitely additive measure $\mu : \Sigma \to \mathbb{R}_0^+$, we say that ν is *absolutely continuous* w.r.t. μ ($\nu \ll \mu$) if for every $\varepsilon > 0$ and every $p \in Q$ there exists a $\delta(\varepsilon, p) > 0$ such that $\mu(E) < \delta$ implies $\nu_p(E) < \varepsilon$.

We now turn to integration of multifunctions. Our presentation is necessarily concise, and restricted to the Radon–Nikodým problem; we refer to Chapter 14 in this Handbook by C. Hess [28], for a more detailed exposition.

DEFINITION 6.12. Let $\mu : \Omega \to \mathbb{R}_0^+$ be any finitely additive measure. A map $F : \Omega \to C_c(X)$ is *simple* if it can be written as:

$$F = \sum_{i=1}^{n} 1_{A_i} C_i,$$

where the C_i's are elements of $C_c(X)$, and the A_i's are disjoint elements of Σ.

The *integral* of F with respect to μ is defined as:

$$\int F \, d\mu = \sum_{i=1}^{n} \mu(A_i) C_i.$$

As each C_i is convex, the definition of integral does not depend on the representation of F.

In case F is not simple, F is said to be *totally measurable* if there exists a sequence $(F_n)_{n\in\mathbb{N}}$ of simple multifunctions such that

(a1) the function $d_p(F_n, F)$ is measurable for every $n \in \mathbb{N}$ and every $p \in Q$,

(a2) the sequence $(d_p(F_n, F))$ μ-converges to 0 for every $p \in Q$.

Moreover, we say that F is *μ-integrable* if there exists a sequence $(F_n)_{n\in\mathbb{N}}$ of simple functions, satisfying (a1) and (a2) above, such that

(a3) $\lim_{m,n\to\infty} \int d_p(F_n, F_m)\, d\mu = 0$ for all $p \in Q$.

Such a sequence will be called a *defining sequence* for F.

In case Q_0 is any absorbing subset of Q, and the conditions (a2) and (a3) above hold *uniformly* with respect to $p \in Q_0$, then F will be said to be *strongly integrable* (with respect to Q_0, if confusion can arise).

In case F is integrable, and $(F_n)_{n\in\mathbb{N}}$ is any defining sequence for F, then for every $E \in \Sigma$ the sequence $(\int_E F_n\, d\mu)_{n\in\mathbb{N}}$ is Cauchy in $C_c(X)$, and hence convergent to some element of $C_c(X)$, which will be called the *integral* of F on E, and denoted by $\int_E F\, d\mu$. This integral does not depend on the defining sequence (see [38] for details).

PROPOSITION 6.13. *If $F : \Omega \to C_c(X)$ is μ-integrable, then $E \to \int_E F\, d\mu$ defines a multimeasure, which is absolutely continuous with respect to μ.*

In order to state a Radon–Nikodým theorem, we need one more definition.

DEFINITION 6.14. Given a multimeasure $\nu : \Sigma \to C_c(X)$, we say that ν is *bounded* if

$$\sup_{A\in\Sigma} h_p\big(\nu(A)\big) = M_p < \infty$$

for every $p \in Q$.

If this is the case, set: $Q_\nu := \{\frac{p}{M_p}\colon p \in Q,\ M_p > 0\}$.

If $\mu : \Sigma \to \mathbb{R}_0^+$ is finitely additive, for every set $E \in \Sigma$, and every $\varepsilon > 0$, we put:

$$S(E) := \left\{\frac{\nu(F)}{\mu(F)}\colon F \in \Sigma,\ \mu(F) > 0\right\};$$

$$S_p(E, \varepsilon) := \big\{C \in C_c(X)\colon d_p\big(\nu(F), C\mu(F)\big) \leqslant \varepsilon\mu(F)\ \forall F \in \Sigma,\ F \subset E\big\};$$

$$S(E, \varepsilon) := \bigcap_{p\in Q_\nu} S_p(E, \varepsilon).$$

Finally we can state one of the Radon–Nikodým theorems from [38]. The proof is based upon the technique of Hagood, see Theorem 5.11.

THEOREM 6.15 ([38]). *Let $\nu : \Sigma \to C_c(X)$ be a multimeasure, $\nu \ll \mu$, where μ is any nonnegative finitely additive measure on Σ. Assume that*

(1) *$S(\Omega)$ is Q_ν-uniformly bounded;*

(2) *for every $\varepsilon > 0$ and every $E \in \Sigma$, $\mu(E) > 0$, there exists a sequence of pairwise disjoint subsets $(E_n)_{n\in\mathbb{N}}$ of E, with $E_n \in \Sigma$, such that $\mu(E) = \sum_n \mu(E_n)$, and such that $S(E_n, \varepsilon) \neq \emptyset$ for any n.*

Then there exists a Q_ν-strongly measurable multifunction $G:\Omega \to C_c(X)$, with Q_ν-uniformly bounded range, such that:

$$\int_E G\,d\mu = \nu(E),$$

for all $E \in \Sigma$.

A quite different problem arises when the measures are both vector-valued. An example was given in the last part of Section 5, with Banach-valued measures; but sometimes it is interesting to consider a richer structure, and to assume that the vector measure takes values in a Riesz space: this is convenient, when the linear spaces have a natural order structure, and the "order" convergence is not topological, like in the space L_0, with the a.e. convergence.

We shall present here a very *simplified* version of the results, and refer to [5] for further details or generalizations.

Assume that R is an order-complete Riesz space (here, *order-complete* means that every nonempty subset $F \subset R$ which is bounded from above has a least upper bound in R). Order-completeness allows to define sup, inf, lim sup and lim inf.

We first mention the so-called *Riemann*-type integral for bounded functions with values in a Riesz space.

DEFINITION 6.16. Given a bounded function $f:[a,b] \to R$, we say that f is *Riemann-integrable* (with respect to Lebesgue measure) if

$$\sup_{s\in S_f}\int s(x)\,dx = \inf_{t\in T_f}\int t(x)\,dx$$

where S_f (T_f, respectively) is the class of all R-valued *step* functions $s \leqslant f$ ($t \geqslant f$, respectively), and the *elementary integral* of a step function is defined in the obvious way.

This integral is well-defined, and is a linear monotone R-valued functional. Moreover, one can see (by usual techniques) that monotone functions are integrable.

DEFINITION 6.17. Given a positive finitely additive measure $\mu:\Sigma \to R$, we say that μ is *σ-additive* if, for every decreasing sequence $(E_n)_{n\in\mathbb{N}}$ in Σ, $E_n \downarrow E$ implies $\inf\{\mu(E_n)\colon n \in \mathbb{N}\} = \mu(E)$.

Given a positive finitely additive measure $\mu:\Sigma \to R$, and a bounded measurable function $f:\Omega \to \mathbb{R}_0^+$, we say that f is *integrable* with respect to μ if the following Riemann-type integral is finite:

$$\int_\Omega f\,d\mu := \int_0^\infty \mu\big(\{x \in \Omega\colon\ f(x) > t\}\big)\,dt.$$

This means that the set $\{\int_0^M \mu(\{x \in \Omega\colon\ f(x) > t\})\,dt\colon\ M > 0\}$ is bounded in R, and $\int_\Omega f\,d\mu$ is its least upper bound; we observe that the function $g(t) = \mu(\{x \in \Omega\colon\ f(x) > t\})$ is monotone decreasing, and hence Riemann integrable in every interval $[0, M]$.

If f is integrable, then $f1_E$ is integrable too, for every $E \in \Sigma$, and $E \to \int f1_E\, d\mu := \int_E f\, d\mu$ defines a measure ν, which is *absolutely continuous* with respect to μ, in the sense that $\limsup \nu(A_n) = 0$ as soon as $(A_n)_{n\in\mathbb{N}}$ is any sequence from Σ, such that $\limsup \mu(A_n) = 0$.

The Radon–Nikodým theorem for Riesz space-valued measures can be formulated exactly like Theorem 1.2, with (b1) and (b2) replaced respectively by (b′1) and (b′2), and (b3) replaced by absolute continuity.

THEOREM 6.18. *Let μ and ν be positive finitely additive measures defined on Σ and taking values in R, with $\nu \ll \mu$. Then the following are equivalent:*

(a) *There exists a measurable and μ-integrable function $f : \Omega \to [0, \infty[$, such that*

$$\int_E f\, d\mu = \nu(E)$$

for all $E \in \Sigma$.

(b) *there exists a family of sets (A_r) in Σ, for $r > 0$, such that, for any $r > 0$:*
  (b1) $\nu(E) \geqslant r\mu(E)$, *for any $E \in \Sigma$, $E \subset A_r$;*
  (b2) $\nu(E) \leqslant r\mu(E)$, *for any $E \in \Sigma$, $E \subset A_r^c$.*

Though the formulation (and the proof) of this theorem is quite similar to Theorem 1.2, the consequences are not so strong: in fact the same example given before Definition 5.13 shows that even for σ-additive measures, absolute continuity is not sufficient to ensure the existence of the derivative.

A different approach to the idea of integration was presented by Sugeno in [54]. We shall give here quite a short outline of the involved concepts, and also a very simple Radon–Nikodým theorem for this integral. For similar topics, see also Chapter 33 in this Handbook [2].

DEFINITION 6.19. Let (Ω, Σ) be any measurable space. A *fuzzy measure* on this space is a mapping $m : \Sigma \to [0, +\infty[$, such that:

(1) $m(\emptyset) = 0$;
(2) $m(A) \leqslant m(B)$, whenever $A, B \in \Sigma$, $A \subset B$;
(3) if (A_n) is any monotone sequence in Σ, then $m(\lim F_n) = \lim m(F_n)$.

If m is a fuzzy measure on (Ω, Σ), then the triple (Ω, Σ, m) is called a *fuzzy measure space*.

DEFINITION 6.20. Let (Ω, Σ, m) be a fuzzy measure space, and let $h : \Omega \to \mathbb{R}_0^+$ be any measurable function. For each $A \in \Sigma$, define:

$$(\mathrm{S}) \int_A h\, dm := \sup_{\alpha > 0} \{\alpha \wedge m(A \cap F_\alpha)\},$$

where $F_\alpha := \{\omega \in \Omega\colon\ h(\omega) \geqslant \alpha\}$.

The number $(\mathrm{S}) \int_A h\, dm$ is called the *Sugeno integral* of h on the set A.

(Notice that the definition of Sugeno's integral is reminiscent of the *Choquet* integral, one just replaces $\wedge$ with ordinary multiplication, and $\vee$ with addition.)

From this definition, it follows immediately that

$$(\mathrm{S})\int_A h\,dm \leqslant m(A), \quad \forall A \in \Sigma.$$

We list a number of results, concerning this integral.

THEOREM 6.21 ([14]). *Let (Ω, Σ, m) be any fuzzy measure space, and let $h : \Omega \to \mathbb{R}_0^+$ be any measurable function.*

(1) *If h is a constant function, $h(x) = \alpha$, then*

$$(\mathrm{S})\int_A h\,dm = \alpha \wedge m(A) \quad \textit{for any } A \in \Sigma.$$

(2) *If h and $h' : \Omega \to \mathbb{R}_0^+$ are measurable functions, with $h' \leqslant h$, then*

$$(\mathrm{S})\int_A h'\,dm \leqslant (\mathrm{S})\int_A h\,dm \quad \textit{for any } A \in \Sigma.$$

(3) *If $h = 1_A$, for some $A \in \Sigma$, then*

$$(\mathrm{S})\int_\Omega h\,dm = m(A).$$

(4) *If $(h_n)_{n\in\mathbb{N}}$ is any monotone sequence of measurable functions, such that $h_n \to h$, then*

$$\lim (\mathrm{S})\int_A h_n\,dm = (\mathrm{S})\int_A h\,dm.$$

for any $A \in \Sigma$.

From part (4) of the previous theorem, it follows that $A \to (\mathrm{S})\int_A h\,dm$ is a fuzzy measure, not greater than m. A useful characterization of the Sugeno integral is the following:

THEOREM 6.22 ([14]). *Given a measurable function $h : \Omega \to \mathbb{R}_0^+$, define, for $\alpha > 0$:*

$$A_\alpha := \{\omega \in \Omega:\ h(\omega) > \alpha\}. \qquad F_\alpha := \{\omega \in \Omega:\ h(\omega) \geqslant \alpha\}.$$

If (B_α) is any decreasing family in Σ, for $\alpha \geqslant 0$, such that $A_\alpha \subset B_\alpha \subset F_\alpha$, then

$$(\mathrm{S})\int_A h\,dm := \sup_{\alpha > 0}\{\alpha \wedge m(A \cap B_\alpha)\}$$

for all $A \in \Sigma$.

Moreover, if I denotes the set of all $\alpha \in \mathbb{R}_0^+$, such that $\alpha \leqslant m(F_\alpha)$, then

$$(\mathrm{S})\int_\Omega h\,dm = \max I.$$

This result allows us to state a Radon–Nikodým theorem.

THEOREM 6.23 ([14]). *Let m and γ be two fuzzy measures on (Ω, Σ), with $\gamma \leqslant m$. The following are equivalent:*

(a) *There exists a measurable function $h : \Omega \to \mathbb{R}_0^+$, such that*

$$\gamma(A) = (\mathrm{S})\int_A h\,dm$$

for any $A \in \Sigma$.

(b) *There exists a decreasing family (B_α) in Σ, with $\alpha \in \mathbb{R}_0^+$, such that:*
 (b1) $\gamma(A \cap B_\alpha) \geqslant \alpha \wedge m(A \cap B_\alpha)$, *and*
 (b2) $\gamma(A) \leqslant m(A \cap B_{\gamma(A)})$,
 for all $A \in \Sigma$, and all $\alpha \in \mathbb{R}_0^+$.

For a richer treatment of Sugeno integral, and its properties, we refer the reader to [54] and [55].

Appendix. Singularity and decomposition theorems

In this appendix we will present some known facts about singularity of measures, the Lebesgue decomposition theorem and some related decomposition theorems, which are relevant in connection with the Radon–Nikodým theorem.

We assume that all the measures have been already completed in the Carathéodory sense.

DEFINITION A.1. We say that a measure μ is *degenerate*, if it takes no finite, nonzero values, i.e., its range is contained in $\{0, \infty\}$.

In connection to this definition, let us first mention the following result, due to N.Y. Luther [35].

THEOREM A.2. *Let μ be a measure on the σ-algebra Σ. Then there exists a unique decomposition $\mu = \mu_1 + \mu_2$, where μ_1 is semifinite and μ_2 is degenerate, with the property that if $\mu' = \mu'_1 + \mu'_2$, with μ'_1 semifinite and μ'_2 degenerate, then $\mu_1 \leqslant \mu'_1$ and $\mu_2 \leqslant \mu'_2$.*

This result and the fact that degenerate measures are not very interesting from the point of view of their representation as integrals, justify the restriction to semifinite measures in Section 2 of this chapter. We will make the same assumption in the Appendix.

If μ is a measure on (Ω, Σ), we say that μ is *concentrated* on $A \in \Sigma$, if $\mu(\Omega \setminus A) = 0$.

DEFINITION A.3. Suppose μ and ν are two measures on Σ, and suppose there exists a pair of disjoint sets A and B in Σ, such that μ is concentrated on A and ν is concentrated on B. Then we say that ν and μ are *mutually singular* (or that ν is *singular* with respect to μ) and write

$$\nu \perp \mu. \tag{5}$$

Obviously, this relation between measures is symmetric, so (5) holds if and only if $\mu \perp \nu$.

The following properties are quite obvious:

- If $\nu_1 \perp \mu$ and $\nu_2 \perp \mu$, then $\nu_1 + \nu_2 \perp \mu$.
- If $\nu_1 \ll \mu$ and $\nu_2 \perp \mu$, then $\nu_1 \perp \nu_2$.
- If $\nu \ll \mu$ and $\nu \perp \mu$, then $\nu \equiv 0$.

THEOREM A.4. *If μ and ν are two measures on (Ω, Σ) and ν is σ-finite, then there exist $\nu_0 \ll \mu$ and $\nu_1 \perp \mu$ such that $\nu = \nu_0 + \nu_1$. The decomposition is unique.*

Let us sketch the proof. Suppose first that $\nu(\Omega) < \infty$, and let $\alpha = \sup\{\nu(B)\colon B \in \Sigma, \mu(B) = 0\}$. It is easy to prove, using arguments introduced in Section 2, that there exists $C \in \Sigma$, such that $\nu(C) = \alpha$ and $\mu(C) = 0$. Define now $\nu_1(E) = \nu(E \cap C)$ and $\nu_0(E) = \nu(E \setminus C)$. It is easy to check that $\nu_0 \ll \mu$ and $\nu_1 \perp \mu$.

If ν is σ-finite, we consider a measurable decomposition $\{A_n\}$ of Ω such that $\mu(A_n) < \infty$ for all n, and apply the argument above to each A_n.

If ν is not σ-finite, the conclusion of the previous theorem may fail. If for instance μ is the Lebesgue measure on $[0, 1]$ and ν is the counting measure, such a decomposition does not exist.

We need now two definitions. The first is due to R.A. Johnson [32], the second one is taken from [59].

DEFINITION A.5. We say that ν is *S-singular* with respect to μ, denoted $\nu \mathrm{S} \mu$, if given $E \in \Sigma$, there exists $F \in \Sigma$, $F \subset E$, such that $\nu(E) = \nu(F)$ and $\mu(F) = 0$.

DEFINITION A.6. We say that ν is *quasi-singular* (*Q-singular*) with respect to μ, denoted $\nu \mathrm{Q} \mu$, if there exists $A \in \Sigma$ such that ν is concentrated on A and moreover if $\nu(E) < \infty$ then $\mu(E \cap A) = 0$.

Singularity implies Q-singularity and the latter implies S-singularity. The three concepts are not on the other hand equivalent. The counting measure on $[0, 1]$ is Q-singular with respect to the Lebesgue measure, but it is not singular.

To show that S-singularity does not imply Q-singularity we need a more elaborate example. Let us consider the measure space defined in [27, Exercise 31.9], which provides a non-Maharam measure μ as sum of two measures ν and σ. As noted by R.A. Johnson, $\nu \mathrm{S} \sigma$ and $\sigma \mathrm{S} \nu$, but it is not $\nu \perp \sigma$. On the other hand it can not be $\nu \mathrm{Q} \sigma$ (or $\sigma \mathrm{Q} \nu$) because of the following proposition [59].

PROPOSITION A.7. *If* $\nu \, \mathrm{Q} \, \mu$ *and* $\mu \, \mathrm{S} \, \nu$, *then* $\nu \perp \mu$.

R.A. Johnson proved the following decomposition theorem [32].

THEOREM A.8. *If* μ *and* ν *are two measures on* (Ω, Σ), *then there exist* $\nu_0 \ll \mu$ *and* $\nu_1 \, \mathrm{S} \mu$ *such that* $\nu = \nu_0 + \nu_1$. *Always* ν_1 *is unique. Moreover, if* $\nu = \nu_0' + \nu_1$ *is another such decomposition,* $\nu_0 \leqslant \nu_0'$.

The advantage of the previous theorem, compared to Theorem A.2 is that no assumption is required on μ and ν (not even semifiniteness). The disadvantage lies in the weakness of the concept of S-singularity.

Q-singularity stays in between and shares some good and bad aspects of both concepts. A reasonable assumption (specially if we deal with the Radon–Nikodým theorem) provides the following decomposition theorem (Theorem 2.1 in [59]).

THEOREM A.9. *If* $\mu + \nu$ *is Maharam, then* Ω *admits a* $(\mu + \nu)$*-unique decomposition* Ω_i, $i = 1, 2, 3$, *such that if we put* $\nu_i(E) = \nu(E \cap \Omega_i)$ *and* $\mu_i(E) = \mu(E \cap \Omega_i)$ *for* $i = 1, 2, 3$, *then*

$$\nu_1 \, \mathrm{Q} \, \mu, \tag{6}$$

$$\mu_2 \, \mathrm{Q} \, \nu, \tag{7}$$

$$\nu_3 \ll \mu_3 \ll \nu_3. \tag{8}$$

Moreover, ν_3 *and* μ_3 *are strongly comparable.*

The set Ω_1 is defined as the least upper bound in the measure algebra associated to $(\Omega, \Sigma, \mu + \nu)$ of the family of sets

$$\{E\colon\ E \in \Sigma,\ \nu(E) < \infty,\ \mu(E) = 0\}.$$

Similarly we define Ω_2. The major difficulty in the proof is to show that Ω_i, for $i = 1, 2$ are both μ- and ν-measurable.

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