THE BANACH SPACE C(S)

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These notes are based on a course of lectures given during 1969/70. They cover a selection of topics from the extensive literature concerning C(S) and its dual, the space of measures

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CONTENTS

SECTION 1

Preliminaries.

Let S be a compact Hausdorff space and C(S) be the set of all real or complex valued continuous functions defined on S. Then C(S) is a linear algebra under the usual pointwise operations

$$(f_1+f_2)(s) = f_1(s)+f_2(s), \qquad s\in S$$
$$(\alpha f)(s) = \alpha f(s), \qquad s\in S$$
$$(f_1f_2)(s) = f_1(s)f_2(s), \qquad s\in S.$$

There is a natural partial ordering $f_1 \geq f_2$ iff

$$f_1(s)-f_2(s) \geq 0 \qquad \text{all } s\in S.$$

The real continuous functions form a vector lattice under

$$(f_1\vee f_2)(s) = \max|f_1(s),f_2(s)|, \qquad s\in S$$
$$(f_1\wedge f_2)(s) = \min|f_1(s),f_2(s)|, \qquad s\in S.$$

We denote by $|f|$ the function

$$|f|(s) = |f(s)|, \qquad s\in S.$$

With the norm

$$\|f\| = \max_{s\in S}|f(s)|$$

C(S) becomes a Banach space, a commutative Banach algebra with unit 1 (the constant) and real C(S) is a Banach lattice (i.e. $|f|\leq|g| \rightarrow \|f\|\leq\|g\|$).

We denote by $\mathcal{M}(S,\mathcal{B})$, or simply $\mathcal{M}(S)$, the class of all regular countably additive real or complex valued measures on the Borel sets $\mathcal{B}$ of S for which

$$\|\mu\| = \text{total variation } (\mu)$$
$$= \sup\sum_{i=1}^{n}|\mu(E_i)| < \infty,$$

the supremum being taken over all finite disjoint collections $\{E_1, \ldots, E_n\}$ of Borel sets in S. Then, with this norm and the usual setwise operations, $\mathcal{M}(S)$ is a Banach space, and we have the following basic theorem identifying $\mathcal{M}(S)$ with the conjugate space of C(S). The scalars may be real or complex.

Theorem 1.1. (F.Riesz.) For every $f^* \in C(S)^*$ there is a unique measure $\mu \in \mathcal{M}(S)$ such that

$$f^*(f) = \int_S f(s) \,(ds), \qquad \text{all } f \in C(S).$$

The map $f^* \to \mu$ is one to one, onto, linear and isometric. A proof may be found in [12], p.265.

As a way to get some feeling for C(S) and $\mathcal{M}(S)$ we shall begin by examining the unit spheres. Let $B = \{f \in C(S) \mid \|f\| \leq 1\}$.

We recall

Definition 1.2. An element x of a convex set K in a linear space is an extreme point of K if there exist no elements $x_1, x_2 \in K$, $x_1 \neq x_2$ and t, $0 < t < 1$, such that

$$x = t x_1 + (1-t)x_2.$$

The set of extreme points of K will be written ext(K).

Theorem 1.3. Let $B = \{f: f \in C(S), \|f\| \leq 1\}$. A function $f \in B$ is an extreme point iff $|f| \equiv 1$, i.e. $|f(s)| = 1$ for all $s \in S$.

Proof. Suppose $|f| \equiv 1$ and $f = t f_1 + (1-t)f_2$. Then since $|f_i| \leq 1$ we get $|f_i| \equiv 1$. If α, β are complex $|\alpha| = |\beta| = 1$ $|t\alpha + (1-t)\beta| = 1$, then $\alpha = \beta$. This is easy to see. We get $f_i \equiv f$.

Conversely, if $f \in B$ and there exists s_0 with $|f(s_0)| < 1$ we can find a neighbourhood $V \ni s_0$ and a function g with $0 \leq g \leq 1$, $g(s_0) = 1$, $g \equiv 0$ off V, and a $\delta > 0$ such that $|f(s)| < 1-\delta$, for all $s \in V$. If $0 < \varepsilon < \delta$, the functions $f \pm \varepsilon g$ are in B and $f = \frac{1}{2}(f+\varepsilon g) + \frac{1}{2}(f-\varepsilon g)$. In fact $|f+\varepsilon g| = |f|$ outside V and $|f+\varepsilon g| \leq |f| + |\varepsilon| g \leq 1-\delta+\varepsilon \leq 1$ inside V. It follows that f is not extreme.

Corollary 1.4. Let S be connected and $C(S)$ be the real continuous functions. Then 1 and -1 are the only extreme points of the unit ball.

For complex $C(S)$ the situation is vastly different. Indeed for any real $f \in C(S)$, $\exp(if)$ is extreme from Theorem 1.3. In fact we shall show that the unit ball is the closed convex hull of its extreme points - written $B = \overline{co}(\text{ext}B)$. We need

Theorem 1.5. Let X be a real or complex Banach space and $B = \{x \mid \|x\| \leq 1\}$. Let E be a subset of $\partial B = \{x: \|x\| = 1\}$ such that $e \in E$ and $|\alpha| = 1 \Longrightarrow \alpha e \in E$. Then $\overline{co}(E) = B$ iff

$$\|x^*\| = \sup_{e \in E} |x^*(e)| \qquad \text{for each } x^* \in X^*.$$

Proof. We prove the complex case, the real case being similar Suppose $B = \overline{co}(E)$ and $x^* \in X^*$. Given $\varepsilon > 0$, there exists $0 \leq t_i \leq 1$, $e_i \in E$ with $x_o = \sum t_i e_i$, $\sum t_i = 1$, $|x^*(x_o)| > \|x^*\| - \varepsilon$. Then we must clearly have $|x^*(e_i)| > \|x\|^* - \varepsilon$ for some i. Conversely, if $x_o \in B$ and $x_o \notin \overline{co}(E)$, the separation theorem for convex sets [12, p. 417]. yields $x_o^* \in X^*$ such that

$$\|x_o^*\| \geq \text{Re}\, x_o^*(x_o) > \sup\{\text{Re}\; x_o^*(x);\; x \in \overline{co}(E)\}$$

$$\geq \sup\{\text{Re}\; x_o^*(e);\; e \in E\}.$$

If $e \in E$, $x_o^*(e) = |x_o^*(e)|e^{i\varphi}$ and so

$$x_o^*(e^{-i\varphi}e) = |x_o^*(e)|$$

$e^{-i\varphi}e \in E$ by hypothesis. Therefore

$$\|x_o^*\| > \sup_{e \in E} |x_o^*(e)|$$

which is a contradiction.

Theorem 1.6. Let $C(S)$ be the complex continuous functions on S. Then $B = \overline{co}(\text{ext}(B))$.

<u>Proof</u>. Let

$$E = \text{ext}\, B = \{f : |f| \equiv 1\} .$$

By the above theorem it is enough to prove that

$$\|\mu\| = \sup_{f \in E} \left| \int_S f\, d\mu \right|$$

for each $\mu \in \mathcal{M}(S)$. Let $|\mu|$ denote the variation of μ. Let

$$f \in B = \{f : \|f\| \leq 1\}$$

be such that

$$\|\mu\| < \left| \int f\, d\mu \right| + \varepsilon .$$

Let g be a Borel measurable bounded real valued function such that

$$f = |f| \exp(ig).$$

We have

$$\int 1\, d|\mu| = \|\mu\| < \left| \int f\, d\mu \right| + \varepsilon \\ \leq \int |f|\, d|\mu| + \varepsilon ,$$

so that

$$\int (1-|f|)\, d|\mu| < \varepsilon .$$

Therefore

$$\left| \int (e^{ig} - f)\, d\mu \right| \leq \int |e^{ig} - f|\, d|\mu| \\ = \int (1-|f|)\, d|\mu| < \varepsilon.$$

We can find a continuous real function h such that

$$\int |e^{ig} - e^{ih}|\, d|\mu| < \varepsilon .$$

We then have from the above

$$\left| \int e^{ih} d\mu \right| > \|\mu\| - 2\varepsilon .$$

Since $e^{ih} \in \text{ext}\, B$, we are done.

Definition 1.7. S is called totally disconnected if there exists a base for the topology consisting of sets which are both open and closed (clopen sets).

Theorem 1.8. Let $C(S)$ be real continuous functions on S. Then $B = \overline{co}(ext(B)) \iff S$ is totally disconnected.

Proof. Let S be totally disconnected and $\mu \in \mathcal{M}(S)$ be real. Let $S = E_1 \cup E_2$ be a Hahn decomposition ([12], p. 129) of S into disjoint Borel sets so that

$$\|\mu\| = \mu(E_1) - \mu(E_2).$$

Let $\varepsilon > 0$. By regularity we can find K compact, W open with $K \subset E_1 \subset W$ and $|\mu|(W-K) < \varepsilon$. Each point of K has a clopen neighbourhood contained in W. By compactness of K, we get a clopen set V with $K \subset V \subset W$. Let $V_2 = V_1{}^c$ and define $f(s) = 1$ on V_1 and $f(s) = -1$, $s \in V_2$. Then $f \in ext(B)$ and

$$\begin{aligned}\int_S f\,d\mu &= \mu(V_1) - \mu(V_2)\\ &\geq \mu(E_1) - \mu(E_2) - |\mu(V_1) - \mu(E_1)|\\ &\quad - |\mu(V_2) - \mu(E_2)| > \|\mu\| - 4\varepsilon.\end{aligned}$$

Now use Lemma 1.5.

Supposing $B = \overline{co}(ext(B))$ we see that the class of functions taking finitely many values is dense in $C(S)$. Let $s_o \in S$, and $g \in C(S)$, $0 \leq g \leq 1$, $g(s_o) = 1$, $g(V^c) = 0$, where V is a neighbourhood of s_o. Let h be a finitely valued function such that $\|g-h\| < \frac{1}{4}$. Then the set $\{s : h(s) \geq \frac{1}{2}\}$ is clopen neighbourhood of s_o contained in V.

Another proof of Theorem 1.8 can be given as follows:

Total disconnectedness of S implies that functions taking only finitely many values are dense in $C(S)$. If $\|f\| \leq 1$ we may assume that the approximating sequence $\{f_n\}$ (each of which is finitely valued) has norm 1 (if $f_n \to f$ uniformly $\frac{f_n}{\|f_n\|} \to f$ uniformly). Finally a function f with $\|f\| \leq 1$ and taking finitely many values is a convex combination of functions whose modulu are $\equiv 1$ [because every point in the n-dimensional cube is a convex combination of its 2^n-vertices].

We next examine the extreme points of the unit ball B of $\mathfrak{M}(S)$.

<u>Theorem 1.9</u>. The extreme points of the unit ball in $\mathfrak{M}(S)$ are the measures of the form $\mu = \alpha\nu$, where ν is a unit point mass at a point of S and $|\alpha| = 1$.

<u>Proof</u>. That a unit point mass is extreme is easy to see. If μ is not concentrated at a point, then there exists a Borel set E with $|\mu|(E) > 0$, $|\mu|(E^c) > 0$. Since $\|\mu\| \leq 1$, $|\mu|(E), |\mu|(E^c) \leq 1$ We can write

$$\mu = t\mu_1 + (1-t)\mu_2$$

where

$$\mu_1(A) = \frac{1}{|\mu|(E)}\mu(A \cap E),$$

$$\mu_2(B) = \frac{1}{1-|\mu|(E)}\mu(B \cap E^c)$$

for all Borel sets A and B and $t = |\mu|(E)$. We have clearly $\|\mu_1\| = 1$, $\|\mu_2\| \leq 1$.

Before proceeding we recall some facts about the weak star topology of conjugate Banach spaces. Let X be a Banach space. The weak star topology for X^* is the coarsest topology for which all the maps $x^* \to x^*(x)$, $x \in X$ are continuous. A basic result we need is

<u>Theorem 1.10</u>. (Alaoglu-Bourbaki.). The unit ball

$$B^* = \{x^* : \|x^*\| \leq 1\}$$

is compact in the weak star topology.

Proof. We regard B^* as a subset of a product of discs

$$B^* \subset \prod_{x \in X} \{\lambda_x : |\lambda_x| \leq \|x\|\} = P$$

and give the product its product topology. By Tychonoff's theorem P is compact; the relative topology of B^* as a subset of P is the weak star topology. B^* is a closed subset of P since pointwise limit of linear functionals is linear. It follows that B^* is compact.

We now return to $\mathcal{M}(S)$. The set

$$\hat{S} = \{\delta_s : s \in S\}$$

where δ_s denotes the positive unit mass at s is contained in ∂B^*.

Theorem 1.11. $\hat{S}$ is a weak star compact subset of ∂B^* and the map $\tau : s \rightarrow \delta_s$ is a homeomorphism of S onto $\hat{S}$.

Proof. That the map τ is 1-1 onto is clear. The topology on $\hat{S}$ is the smallest topology for which the map $\delta_s \rightarrow \int f \delta_s = f(s)$ is continuous. This means that τ is continuous.

Thus we have a way of recovering S from $C(S)$. This fact yields the following important result.

Theorem 1.12. (Banach-Stone.) Let S and T be compact Hausdorff spaces. Let $V: C(S) \rightarrow C(T)$ be an isometric linear map onto $C(T)$. Then there exists a homeomorphism τ of T onto S and a function $\alpha \in C(T)$ with $|\alpha(t)| \equiv 1$ and

$$(Vf)(t) = \alpha(t) f(\tau(t)), \qquad f \in C(S), \quad t \in T.$$

Proof. The map V^* defined by

$$(V^*\mu, f) = (Vf, \mu), \qquad f \in C(S), \quad \mu \in \mathcal{M}(T)$$

is an isometric isomorphism of $\mathcal{M}(T)$ onto $\mathcal{M}(S)$. It follows that V^* maps the unit ball B_T^* of $\mathcal{M}(T)$ onto the unit ball B_S^* of

$\mathfrak{M}(S)$. Further V^* maps $\text{ext}(B_T^*)$ onto $\text{ext}(B_S^*)$. Therefore $V^*(\hat{T}) \subset \text{ext}(B_S^*)$. From Theorem 1.9 we can write for each $t \in T$, $V^*(\delta_t) = \alpha(t)\delta_{\tau(t)}$, where $|\alpha(t)| = 1$. Let τ denote the map $t \to \tau(t)$.

τ is a map of T into S. Let us show that τ is one to one. If $\tau(t_1) = \tau(t_2)$ we have

$$\delta_{t_1} = \alpha(t_1)V^{*-1}\delta_{\tau(t_1)},$$

$$\delta_{t_2} = \alpha(t_2)V^{*-1}\delta_{\tau(t_2)}$$

i.e. δ_{t_1} and δ_{t_2} are multiples of the same measure

$$V^{*-1}\delta_{\tau(t_1)} = V^{*-1}\delta_{\tau(t_2)},$$

i.e. $t_1 = t_2$. Let us show that τ is onto. $s \in S \Longrightarrow \delta_s \in \hat{S} \Longrightarrow V^{*-1}\delta_s \in \text{ext}(B_T^*) \Longrightarrow$ there exists $t \in T$ and $\beta(t)$ with $|\beta(t)| = 1$, and $\beta(t)\delta_t = V^{*-1}\delta_s$ i.e. $V^*\delta_t = \alpha(t)\delta_{\tau(t)}$, where $\alpha(t) = \beta(t)^{-1}$, $\tau(t) = s$. Thus we have shown that τ is one to one onto. By definition we have for any $f \in C(S)$

$$(Vf)(t) = (Vf,\delta_t) = (f,V^*\delta_t)$$
$$= (f,\alpha(t)\delta_{\tau(t)}) = \alpha(t)f(\tau(t)).$$

For $f \equiv 1$ we get $V1 = \alpha$, i.e. $\alpha \in C(T)$. We know that $|\alpha| \equiv 1$ and $Vf \in C(T)$. Therefore $f(\tau(t)) \in C(T)$ for all $f \in C(S)$ and therefore τ is continuous [the set of continuous functions on a completely regular space determines its topology].

SECTION 2

Characterizations of C(S).

In this section we shall characterize $C(S)$ among systems of various types. We start with characterizations of $C(S)$ among Banach algebras. In this case of real scalars an important tool will be the Krein-Milman Theorem:

Theorem 2.1. (Krein-Milman.) Let L be a locally convex linear topological space and K be a compact convex subset. Then K is the closed convex hull of its extreme points:

$$K = \overline{\text{co}}(\text{ext}(K)).$$

Proof. We say that a subset S is a support of K if

(a) S is a compact convex subset of K.

(b) If an interior point of a line segment in K is in S, then the entire segment lies in S.

Remarks. 1) The intersections of supports is a support.

2) If S supports K and T supports S, then T supports K.

3) If $f \in L^*$, then

$$\{k \in K: \quad f(k) = \max_{k' \in K} f(k')\}$$

is a support of K.

Let S be a support of K. By Zorn's Lemma there exists a collection Q of supports of K maximal with respect to the properties:

(i) $S \in Q$

(ii) Q is closed under finite intersections.

The intersection T of all members of Q is a support which contains properly no support of K. T must contain only one point, since if $p, q \in T$ the separation theorem yields a functional f

taking different values at p and q. The subset of T where f attains its maximum is a support of T, thus of K by (2), contradicting the minimality of T. Thus we have proved: Every support of K contains a one point support, i.e. an extreme point of K. Finally if

$$C = \overline{co}(\mathrm{ext}(K)) \neq K,$$

let $x \in K-C$. There exists $f \in L^*$ with

$$f(x) > \max_{y \in C} f(y).$$

Then the maximum set for f contains an extreme point for K which is not in C. This contradiction completes the proof.

Our first task is to characterize C(S) among real Banach algebras.

Theorem 2.2. (Arens.) Let $\mathfrak{A}$ be a commutative Banach algebra with unit e over the reals. Then $\mathfrak{A}$ is isometrically isomorphic to the algebra $C(\Omega)$ of all real continuous functions on a compact Hausdorff space Ω iff

(1) $\|x^2\| = \|x\|^2, \qquad x \in \mathfrak{A}.$

(2) $\|x^2+y^2\| \geq \|x^2\|, \qquad x,y \in \mathfrak{A}.$

The proof is in steps. Consider the set of squares

$$P = \{x^2 : \quad x \in \mathfrak{A}\}.$$

We will show that P is a closed cone in $\mathfrak{A}$, i.e. $x,y \in P \Longrightarrow x+y \in P$ and $\lambda x \in P$ if $\lambda > 0$. Note $P-P = \mathfrak{A}$ because

$$x = \left(\frac{e+x}{2}\right)^2 - \left(\frac{e-x}{2}\right)^2, \qquad x \in \mathfrak{A}.$$

Lemma 2.3. If $x \in \mathfrak{A}$, $\|x\| \leq 1$, then $e-x \in P$. Also for $x \in \mathfrak{A}$ the following are equivalent:

(a) $\|x\| \leq 1$ and $\|e-x\| \leq 1$.

(b) $x \in P$ and $e-x \in P$.

Proof. Recall $\sqrt{1-t} = \sum_0^{\infty} a_n t^n$, with $\sum |a_n| < \infty$. Hence if $\|x\| \leq 1$, $\sum a_n x^n$ converges absolutely and is in $\mathfrak{A}$. So

$$e-x = (\sum a_n x^n)^2 \in P.$$

This proves the first statement.

(a) $\Longrightarrow$ (b) is proved as follows: $\|x\| \leq 1 \Longrightarrow$ by above $e-x \in P$ and $\|e-x\| \leq 1 \Longrightarrow e-(e-x) = x \in P$.

(b) $\Longrightarrow$ (a): If $x = u^2$, $e-x = v^2$ we have

$$1 = \|e\| = \|u^2+v^2\| \geq \|u\|^2 \text{ and } \|v^2\|$$

i.e. $\|u^2\| \leq 1$, $\|v^2\| \leq 1$.

Lemma 2.4. P is a closed cone.

Proof. $x \in P$, $t \geq 0 \Longrightarrow tx \in P$ is clear.

Let $x, y \in P$ and $\|x\| \leq 1$, $\|y\| \leq 1$. By Lemma 2.3, $e-x \in P$, $e-y \in P$. Since $x \in P$ and $e-x \in P$, by Lemma 2.3 $\|e-x\| \leq 1$ and similarly $\|e-y\| \leq 1$ implies that $\|\frac{1}{2}(e-x) + \frac{1}{2}(e-y)\| \leq 1$, i.e. $\|e - \frac{x+y}{2}\| \leq 1$ implies that $e - (e - \frac{x+y}{2}) \in P$ by Lemma 2.3. Thus P is a cone. $x_n \in P$, $\|e-x_n\| \leq 1$, $e-x_n \rightarrow e-x_o$, $\|e-x_o\| \leq 1$. Since $\|x_o\| \leq 1$ and $\|e-x_o\| \leq 1$, by Lemma 2.3 again $x_o \in P$. Thus P is a closed cone.

Lemma 2.5. If $x \in P$, $y \in P$, $\|x\| \leq 1$, $\|y\| \leq 1$, then $\|x-y\| \leq 1$.

Proof. From Lemma 2.3, $e-x \in P$, $e-y \in P$. Since $(x-y)^2 \in P$ and

$$e - (x-y)^2 = [(e-x)+y][(e-y) + x]$$

is also in P [because $(e-x)$, $y \in P$ implies that $e-x+y \in P$ so $e-x+y$ is a square and product of squares is a square]. Thus from Lemma 2.3

$$\|(x-y)^2\| \leq 1, \quad \text{i.e. } \|x-y\| \leq 1.$$

We denote elements of $\mathfrak{A}^*$ by $\xi, \eta,$ etc.

The <u>dual cone</u> P^* is defined by

$$P^* = \{\xi \in \mathfrak{A}^* \mid \xi(x) \geq 0, \quad x \in P\}.$$

Let

$$\Sigma = P^* \cap B^*,$$

where B^* is the unit ball in $\mathfrak{A}^*$ and denote by Ω the set of all multiplicative linear functionals, i.e.

$$\xi(xy) = \xi(x)\xi(y), \quad x,y \in \mathfrak{A}, \quad \xi \in \Omega.$$

We have to see that $\Sigma \neq \emptyset$. First let us show that $P^* \neq \emptyset$. P is a closed convex set in $\mathfrak{A}$ and $-e \notin P$. [Indeed if $-e \in P$, then $-e = x^2$ for some $x \in \mathfrak{A}$. We then have $0 = e+(-e) = e^2+x^2 \Longrightarrow \|0\| \geq \|e^2\| = 1$.] The separation theorem gives a $\xi \in \mathfrak{A}^*$ such that $\xi(-e) < \inf_{y \in P} \xi(y)$. Now $\lambda > 0$, $y \in P \Rightarrow \lambda y \in P$, so $\xi(\lambda y) = \lambda \xi(y) > \xi(-e) \Longrightarrow \xi(y) \geq 0$, all $y \in P$.

That Σ is not empty follows from the following lemma:

<u>Lemma 2.6</u>. (a) Every linear functional on $\mathfrak{A}$ which is positive on P is continuous, i.e. is in P^* and

$$\|\xi\| = \xi(e), \quad \xi \in P^*.$$

(b) P^* is weak star closed and Σ is a weak star compact convex subset of P^*.

(c) Ω is a weak star compact subset of Σ.

<u>Proof</u>. Suppose $F(p) \geq 0$, $p \in P$. If $\|y\| \leq 1$, then

$$e \pm y \in P \Longrightarrow F(e \pm y) \geq 0 \Longrightarrow F(e) \geq |F(y)|.$$

This proves (a).

Part (b) is clear. Multiplicative functionals are positive on P and hence continuous and one at e. Thus $\xi \in \Omega \Longrightarrow \xi \in \Sigma$. That Ω is weak star compact is easy.

It is true that Ω is precisely the set of non-zero extreme points of Σ. [We know Σ has extreme points by Krein-Milman.] We prove part of this fact, but first we need an observation about extreme points.

Lemma 2.7. Suppose ξ is a non-zero extreme point of Σ, that $\eta \in \Sigma$ with $\xi - \eta \in P^*$. Then

$$\eta = \eta(e)\xi .$$

Proof. It is true if $\eta = \xi$ or $\eta = 0$. So suppose $0 < \eta(e) < 1$. [If $\eta(e) = 1$, then, since $\xi - \eta$ is positive on P, we would have $\eta \equiv \xi$.] We may write

$$\xi = \eta(e)\frac{\eta}{\eta(e)} + [\xi(e) - \eta(e)]\frac{\xi - \eta}{\xi(e) - \eta(e)} .$$

(Note $\xi(e) = 1$, so we have written ξ as a convex combination of elements of Σ.) Since ξ is extreme

$$\xi = \frac{\eta}{\eta(e)} .$$

Lemma 2.8. If ξ is a non-zero extreme point of Σ, then $\xi \in \Omega$.

Proof. Suppose $x \in P$, $\|x\| \leq 1$. Define

$$\xi_x(y) = \xi(xy), \qquad y \in \mathfrak{A} .$$

Then $\xi_x \in \Sigma$, since

$$|\xi_x(y)| = |\xi(xy)| \leq \|\xi\| \; \|x\| \|y\| \leq \|y\| ,$$

for $y \in \mathfrak{A}$ and clearly $\xi_x \in P^*$. Now if $y \in P$, $y(e-x) \in P$. So

$$(\xi - \xi_x)(y) = \xi((e-x)y) \geq 0, \quad \text{i.e.} \quad \xi - \xi_x \in P^* .$$

Thus

$$\xi_x = \xi_x(e)\xi , \quad \text{i.e.} \quad \xi_x(y) = \xi(x)\xi(y) .$$

$x \in P$ and $\|x\| \leq 1$ is clearly no restriction because $P - P = \mathfrak{A}$.

The next lemma is crucial.

Lemma 2.9. If $x \in P$, then there exists $\xi_0 \in \Omega$ with

$$\xi_0(x) = \|x\|.$$

Proof. Let $x \in P$. We show first that there is an element $\eta \in \Sigma$ for which $\eta(x) = \|x\|$. We may assume $\|x\| = 1$. Define η on the ray $\{\alpha x: \alpha \text{ real}\}$ by $\eta(\alpha x) = \alpha$, and extend it to all of $\mathfrak{A}$ by Hahn-Banach theorem, so $\|\eta\| = 1$. We assert $\eta \in P^*$ Let $y \in P$, $\|y\| \leq 1$. If $\eta(y) < 0$, then

$$\eta(x-y) = \eta(x) - \eta(y) > \eta(x) = 1.$$

But $\|x-y\| \leq 1$ by Lemma 2.5. Thus $\eta \in P^*$. Now

$$H = \{\xi \in \Sigma; \xi(x) = 1 = \|x\|\}$$

is a support of Σ, and hence contains an extreme point [see the proof of Theorem 2.1] and now use Lemma 2.8.

Corollary 2.10. If $x \in \mathfrak{A}$, there exists $\xi \in \Omega$ with $|\xi(x)| = \|x\|$.

Proof. If $x \in \mathfrak{A}$, $x^2 \in P$. Pick $\xi \in \Omega$ with

$$\xi(x^2) = \|x^2\| = \|x\|^2 \quad \text{(Lemma 2.9.)}.$$

Then

$$(\xi(x))^2 = \xi(x^2) = \|x\|^2.$$

Proof of Theorem 2.2. Define $\mathcal{G}: x \to \hat{x}$ of $\mathfrak{A}$ into $C(\Omega)$ by

$$\hat{x}(\xi) = \xi(x), \qquad \xi \in \Omega, \quad x \in \mathfrak{A}.$$

Then $\mathcal{G}$ is linear and

$$\widehat{xy} = \hat{x}\,\hat{y}.$$

$\hat{e}=1$; $\|\hat{x}\|_\infty \leq \|x\|$. By Corollary 2.10 $\mathcal{G}$ is isometric, so $\mathcal{G}(\mathfrak{A})$ is a closed subalgebra of $C(\Omega)$. $\mathcal{G}(\mathfrak{A})$ separates points of Ω [if $\xi_1 \neq \xi_2$, there exists $x \in \mathfrak{A}$ with $\xi_1(x) \neq \xi_2(x)$, i.e. $\hat{x}(\xi_1) \neq \hat{x}(\xi_2)$]. The Stone-Weierstrass theorem shows that $\mathcal{G}(\mathfrak{A}) = C(\Omega)$.

Remark. The isomorphism $\mathcal{G}$ carries P onto the positive cone in $C(\Omega)$. $\mathcal{G}^*$: $C(\Omega)^* \to \mathfrak{A}^*$ maps the positive measures in $\mathcal{M}(\Omega)$ onto P^*, and the extreme points of the (compact convex) set of probability measures onto the extreme points of Σ. Because

$$Q = \{\mu \mid \mu \geq 0,\ \|\mu\| = 1\} = \{\mu : \|\mu\| = 1 = \mu(1)\}$$

is convex and is a support to the unit ball in $\mathcal{M}(\Omega)$. It follows that the set Ω of non-zero multiplicative linear functionals coincides with $\operatorname{ext}(\Sigma)$.

We now do the complex case.

Definition 2.11. A commutative complex Banach algebra with unit e is a B*-algebra if there exists an operation $*$ in $\mathfrak{A}$ with the properties

$$(x^*)^* = x, \qquad (\alpha x)^* = \bar{\alpha} x^*$$

$$(xy)^* = x^*y^*, \quad (x+y)^* = x^*+y^*$$

$$\|xx^*\| = \|x\|^2.$$

Lemma 2.12.

(a) $e^* = e$

(b) $\|x\| = \|x^*\|$, $\quad x \in \mathfrak{A}$.

Proof.

(a) $ee^* = e^*$ (because e is unit). Since

$$e^{**} = e, \quad ee^* = e^{**}e^* = (e^*e)^* = (e^*)^* = e.$$

(b) $\|xx^*\| \leq \|x\|\,\|x^*\|$, so $\|x\|^2 \leq \|x\|\,\|x^*\| \implies \|x\| \leq \|x^*\|$ since $(x^*)^* = x$ we get the opposite inequality.

Theorem 2.13. (Gelfand-Naimark.) A complex commutative Banach algebra is isometrically isomorphic to the algebra of all continuous complex functions on a compact Hausdorff space iff it is a B*-algebra.

Proof. Necessity is clear, since we may define $g^* = \bar{g}$, where $\bar{g}(\omega) = \overline{g(\omega)}$, $\omega \in \Omega$.

Sufficiency. Let

$$R = \{ x: \quad x^* = x \} .$$

Then R is a closed, real subalgebra containing e. For $x, y \in R$

$$\|x^2+y^2\| = \|(x+iy)(x-iy)\| = \|(x+iy)(x+iy)^*\|$$
$$= \|(x+iy)\|^2,$$

because

$$(x+iy)^* = x^* + (iy)^* = x-iy.$$

But $\|x+iy\| \geq \|x\|$, since $\|x\| > \|x+iy\|$ would mean

$$\|x\| > \|(x+iy)^*\| = \|x-iy\| \Rightarrow 2\|x\| > \|x+iy\| + \|x-iy\| \geq 2\|x\| .$$

Thus

$$\|x^2+y^2\| \geq \|x^2\| .$$

Since $x^* = x$, also

$$\|x^2\| = \|xx^*\| = \|x\|^2.$$

Thus the conditions of Theorem 2.2 are satisfied, so $R = C_r(\Omega) =$ the set of real continuous functions on Ω . If $x \in \mathfrak{A}$, then

$$x = \frac{x+x^*}{2} + i\,\frac{x-x^*}{2i} = u + iv,$$

where u and v belong to R and this representation is unique. Also $\|x\|^2 = \|u+iv\|^2$. It follows that $\mathfrak{A}$ is the algebra of all complex continuous functions on Ω .

Examples.

1) Let S be any set. $M(S)$ = all bounded scalar functions on S with sup norm

$$\|x\| = \sup_{s\in S} |x(s)|, \qquad x^* = \bar{x}.$$

Conditions of Theorem 2.13 are satisfied and $M(S)$ is then $C(\Omega)$ for some Ω. Ω is called the Stone-Cech compactification of the discrete set S. The basic property of Stone-Cech compactification is that all bounded continuous functions have extensions to the compactification. See pp. 149-154 of Kelley for definitions etc.

2) Let (S,Σ,μ) be a σ-finite measure space and $M(S,\Sigma,\mu)$ = all bounded measurable scalar functions.

Again Theorem 2.13 shows that M is some $C(\Omega)$. Let $E \in \Sigma$, then the indicator function $1_E \in M$. Since $(1_E)^2 = 1_E$ we see that the corresponding function on Ω must be zero-one valued. Thus the set E corresponds to an open and closed subset of Ω. Ω can be obtained as follows: Let X be the set consisting of 0 and 1, X is a compact Hausdorff space $\Longrightarrow$ by Tychonoff's theorem X^{Σ} is compact Hausdorff. For each $s \in S$, $E \longrightarrow 1_E(s)$ defines a function of Σ with zero one values. We may thus map S onto a subset $\tilde{S}$ of X^{Σ}. The closure of $\tilde{S}$ is exactly Ω. [We have here ignored the measure μ.]

Similarly $L_\infty(S,\Sigma,\mu)$ is some $C(\Omega)$; the norm in L_∞ is defined as

$$\|x\| = \operatorname{ess.sup}_{s\in S}|x(s)|.$$

3) The following algebras all satisfy the conditions of Theorem 2.13. Let $I = [0,1]$.

$B_0 = C(I)$
B_1 = all bounded pointwise limits of B_0-functions
.
.
.
B_α = all bounded pointwise limits of $\bigcup_{\beta<\alpha} B_\beta$ functions
.
.
.

The above defines B_α for all $\alpha < \Omega$ = first uncountable ordinal.

It can be shown that each B_α is a Banach algebra. B_Ω is the class of all Borel measurable functions.

4) Let $\mathbb{R}$ = real numbers and $BUC(\mathbb{R})$ = the set of bounded uniformly continuous functions on $\mathbb{R}$. $BUC(\mathbb{R})$ satisfies the conditions of Theorem 2.13 and hence is $C(\Omega)$ for some Ω. Ω will be a certain compact Hausdorff space containing $\mathbb{R}$ as a dense subset. The nature of Ω is not clear.

We next use Theorem 2.13 to construct the Stone-Cech compactification of an arbitrary completely regular space.

Theorem 2.14. (Stone-Cech.) Every completely regular space S may be homeomorphically imbedded as a dense subspace of a compact space S such that each bounded continuous function on S has a unique continuous extension in C(S). The space S is unique up to homeomorphisms.

Proof. Let $\mathfrak{A}$ = BC(S) = bounded continuous functions on S. Clearly $\mathfrak{A}$ satisfies the conditions of Theorem 2.13 and hence $\mathfrak{A} \sim C(\hat{S})$, i.e. $\mathfrak{A}$ is isometrically isomorphic to $C(\hat{S})$ where $\hat{S}$ is compact Hausdorff; $\hat{S}$ may be identified with the set of multiplicative linear functionals on $\mathfrak{A}$. Each $s \in S$ defines a multiplicative linear functional on $\mathfrak{A}$ by the map

$$\mathfrak{A} \ni f \longrightarrow f(s).$$

Denote this functional by $\hat{s} = \tau(s)$. Since $\mathfrak{A}$ separates points of S, $s_1 \neq s_2 \Rightarrow \tau(s_1) \neq \tau(s_2)$. The map τ is clearly continuous because the weak star topology restricted to $\hat{S}$ gives the topology of $\hat{S}$. Also τ is a homeomorphism, because of complete regularity the topology on S is the smallest topology with respect to which all functions in $\mathfrak{A}$ are continuous. If $\tau(S)$ were not dense in $\hat{S}$, we could find $h \in C(\hat{S})$ which vanished on $\tau(S)$. Since there exists a g such that $\hat{g} = h$ we see by isometry that $g \equiv 0$ which is a contradiction.

Uniqueness of $\hat{S}$ follows from the Banach-Stone theorem (Theorem 1.12), because for any compact space $\tilde{S}$ which satisfies the requirement of the theorem, $C(\tilde{S})$ must be isometrically isomorphic to $\mathfrak{A}$.

We proceed to characterize Banach lattices which are $C(S)$ for some S.

Definition 2.15. A vector lattice is a vector space E over the reals together with a partial ordering $\leq$ such that

(1) $x \leq y, \quad y \leq z \implies x \leq z$

$x \leq x$

$x \leq y, \quad y \leq x \implies x = y.$

(2) $x + z \leq y + z$ whenever $x \leq y$ and $z \in E$.

(3) $x \leq y, \ \alpha \geq 0 \implies \alpha x \leq \alpha y$.

(4) For all $x,y \in E$

$x \vee y =$ l.u.b. $\{x,y\}$ and

$x \wedge y =$ g.l.b. $\{x,y\}$

exist in E.

We shall say $x \in E$ is positive if $x \geq 0$.

A great number of simple identities may now be proved. A few we need are:

a) $x + y = x \vee y + x \wedge y$.

b) $(x \wedge y) + z = (x+z) \wedge (y+z)$.

c) The positive elements form a cone P and $P-P = E$.

d) If $x^+ = x \vee 0$, $x^- = (-x) \vee 0$ and $|x| = x^+ + x^-$, then $|x| = x \vee (-x)$, $|x+y| \leq |x| + |y|$ etc.

e) If $0 \leq z \leq x+y$, where $0 \leq x$, $0 \leq y$, then there exists elements a,b with

$$0 \leq a \leq x, \qquad 0 \leq b \leq y$$

and

$$a+b = z.$$

In fact put $a = z \wedge x$, $b = z - z \wedge x$. Clearly $0 \leq a \leq x$, $0 \leq a \leq z$. Since $z \leq x+y$, $z-y \leq x$, also $z-y \leq z$, because $y \geq 0$. It follows that $z-y \leq$ g.l.b. $\{z,x\} = a$, i.e. that $b = z-a \leq y$.

Definition 2.16. A Banach lattice is a vector lattice which is a Banach space under a norm $\| \ \|$ for which

$$|x| \leq |y| \implies \|x\| \leq \|y\|.$$

An element e of a Banach lattice is an order unit if

$$\|e\| = 1 \quad \text{and} \quad \|x\| \leq 1 \implies x \leq e.$$

An M-space is a Banach space in which the norm satisfies the relation

$$\|x \vee y\| = \|x\| \vee \|y\|, \qquad x, y \in E, \quad x \geq 0, \ y \geq 0.$$

Definition 2.17. If E and F are vector lattices, a linear map $T: E \to F$ is a lattice homomorphism if

$$T(x \vee y) = T(x) \vee T(y), \quad x, y \in E.$$

A real valued lattice homomorphism of E is a lattice homomorphism of E into the lattice of real numbers.

Remarks.

1) A linear map $T: E \to F$ is positive if $x \geq 0 \implies Tx \geq 0$. We write $T \geq 0$.

2) A lattice homomorphism is positive and

$$T(x \wedge y) = T(x) \wedge T(y).$$

Indeed

$$x \geq 0 \iff x = x \vee 0$$

so that

$$T(x) = T(x) \vee T(0) = T(x) \vee 0 \implies T(x) \geq 0.$$

Now

$$x + y = x \vee y + x \wedge y$$

so

$$T(x \wedge y) = T(x) + T(y) - T(x \vee y) =$$
$$= T(x) + T(y) - T(x) \vee T(y) = T(x) \wedge T(y).$$

3) If E is an M-space with order unit e, a real lattice homomorphism h is continuous and $\|h\| = h(e)$. In fact

$$\|x\| \leq 1 \implies \pm x \leq e \implies |h(x)| \leq h(e).$$

4) E^* is partially ordered by $f \geq g$ iff $f-g \geq 0$. The positive elements of E^* form a cone P^* which is weak star closed. [It can be shown that E^* is a Banach lattice, but we will not need this.] If E has order unit and f is a positive linear map, then $f \in E^*$. Indeed,

$$\|x\| \leq 1 \implies \pm x \leq e \implies (e \pm x) \geq 0 \implies |f(x)| \leq f(e).$$

Our aim is to prove

Theorem 2.18. (Kakutani.) If E is an M-space with order unit, there exists a compact Hausdorff space Ω and an isometric linear map $T: E \to C(\Omega)$ of E onto $C(\Omega)$ which is a lattice isomorphism.

Hereafter E will be an M-space with order unit e. We denote by Ω the set of all real lattice homomorphisms of E which satisfy $h(e) = 1$. Then Ω is clearly a weak star compact subset of Σ, where

$$\Sigma = \{ g \in E^* : \quad g \geq 0, \quad \|g\| \leq 1 \}.$$

We now define $T: E \to C(\Omega)$ by

$$(T(x))(h) = h(x), \quad x \in E, \quad h \in \Omega.$$

If $C(\Omega)$ has its natural ordering we have

$$T(x \vee y)(h) = h(x \vee y) = h(x) \vee h(y)$$
$$= [T(x) \vee T(y)](h),$$

i.e. T is a lattice homomorphism. Also

$$|T(x)(h)| = |h(x)| \leq \|h\| \|x\| = \|x\|, \quad x \in E.$$

It remains to prove

(A) T is an isometric map.

(B) T is onto all of $C(\Omega)$.

We first work on (A). We will prove that if $x \in E$, $x \geq 0$, then there exists $h \in \Omega$ with $|h(x)| = \|x\|$. It is true that Ω is precisely the set of extreme points of Σ . We shall prove part of this in proving (A).

Lemma 2.19. Let $x \geq 0$. Then there exists $h \in \operatorname{ext}(\Sigma)$ such that

$$h(x) = \|x\|.$$

Proof. We shall show that the set

$$B = \{ g \in \Sigma : \quad g(x) = \|x\| \}$$

is non empty. Since B is a support to Σ it must then contain an extreme point.

We can suppose $\|x\| = 1$. It suffices to show there exists $g \in E^*$ such that $g(x) = 1$ and $\|g\| = 1$, $g(e) = 1$. Such a g must be positive, since if $y \geq 0$, $\|y\| = 1$, then $0 \leq y \leq e$, so $e-y \geq 0$ and $\|e-y\| \leq \|e\| \leq 1$. Thus

$$1 - g(y) = g(e) - g(y) \leq 1,$$

i.e. $g(y) \geq 0$.

The case that $x = e$ is trivial. Now suppose x and e are linearly independent and define g on the subspace they span by

$$g(\lambda x + \mu e) = \lambda + \mu .$$

Certainly $g(x) = g(e) = 1$. To show $\|g\| = 1$ we can suppose $\lambda \geq 0$.

Then if $\mu \geq 0$

$$\begin{aligned} g(\lambda x + \mu e) &= \lambda + \mu = \|(\lambda + \mu) x\| \\ &\leq \|\mu(e-x) + (\lambda+\mu)x\| \\ &= \|\lambda x + \mu e\| . \end{aligned}$$

If $\mu < 0$,

$$|g(\lambda x + \mu e)| = |\lambda + \mu| = |\lambda - |\mu|| = ||\lambda| - |\mu||$$
$$\leq \|\lambda x - |\mu| e\| = \|\lambda x + \mu e\| .$$

Thus $\|g\| = 1$ in the subspace. A norm preserving extension to E will now be an element of B.

Lemma 2.20. An extreme point h of Σ has the property:

If $0 \leq g \leq h$, then $g = g(e)h$.

Proof. It is true if $g = 0$ or $g = h$. If $0 \neq g \neq h$, then

$$\frac{g}{\|g\|} = \frac{g}{g(e)} \quad \text{and} \quad \frac{h-g}{\|h-g\|} = \frac{h-g}{(h-g)(e)}$$

are in Σ and since $1 \geq h(e)$, also

$$\frac{h-g}{1-g(e)} \in \Sigma$$

and

$$h = g(e) \frac{g}{g(e)} + (1-g(e)) \frac{h-g}{1-g(e)}$$

$$\Longrightarrow \quad h = \frac{g}{g(e)} .$$

Lemma 2.21. Let E be a vector lattice, $u \geq 0$ and f a positive functional. Then there exists g such that

(a) $0 \leq g \leq f$.

(b) $g(u) = f(u)$.

(c) $g(x) = 0$ for each $x \geq 0$ such that $x \wedge u = 0$.

Proof. For each $x \geq 0$ define

$$g(x) = \sup\{f(y) : \ 0 \leq y \leq x, \ \ y \leq tu \ \text{ for some } \ t \geq 0\}.$$

We show g is additive on the positive cone P. Let $x, y \geq 0$. Let $0 \leq z \leq x+y$, $z \leq tu$ for some $t \geq 0$. Then there exists a, b such that

$$z = a + b, \qquad 0 \leq a \leq x, \quad 0 \leq b \leq y.$$

Then

$$a \leq z \leq tu, \qquad b \leq z \leq tu.$$

So

$$f(z) \leq f(a) + f(b) \leq g(x) + g(y).$$

Therefore

$$g(x+y) \leq g(x) + g(y).$$

On the other hand if $0 \leq a \leq x$, $0 \leq b \leq y$, $a \leq tu$, $b \leq su$ for some $t, s \geq 0$, then $0 \leq a+b \leq x+y$, $a+b \leq (t+s)u$. So

$$f(a) + f(b) = f(a+b) \leq g(x+y),$$

i.e.

$$g(x) + g(y) \leq g(x+y).$$

Since g is additive on P we may extend it to a linear functional on all of E. Clearly $g(x) \leq f(x)$, $x \geq 0$, so $0 \leq g \leq f$. Also

$$g(u) = \sup\{f(y):\ 0 \leq y \leq u\} = f(u).$$

Thus (a) and (b) hold. To prove c let $x \geq 0$ and $x \wedge u = 0$. Suppose $0 \leq y \leq x$, $y \leq tu$ for some $t \geq 0$. Then $0 \leq y \leq x$, $y \leq (t+1)u \implies 0 \leq y \leq x \wedge (t+1)u \leq (t+1)(x \wedge u) = 0 \implies y = 0$.

Lemma 2.22. If $h \in \Sigma$ and $0 \leq g \leq h \implies g = g(e)h$, then h is a real lattice homomorphism.

Proof. First note if $x, y \geq 0$ and $x \wedge y = 0$, then $h(x) \wedge h(y) = 0$. To see this, suppose $h(x) \neq 0$. By Lemma 2.21, there exists $g \in E^*$, $0 \leq g \leq h$ with

$$g(x) = h(x), \quad g(y) = 0.$$

Then

$$g = g(e)h \implies h(y) = 0,$$

i.e.

$$h(x) \wedge h(y) = 0.$$

If $x, y \in E$, $x - x\wedge y$ and $y - x\wedge y$ are non-negative. Setting $a = x$, $c = y$, $h = -(x\wedge y)$ in the identity

$$(a+h)\wedge(c+h) = a\wedge c + h$$

we get

$$[x - x\wedge y]\wedge[y - x\wedge y] = x\wedge y - x\wedge y = 0.$$

Therefore

$$\begin{aligned} 0 &= h[x - x\wedge y]\wedge h[y - x\wedge y] \\ &= [h(x) - h(x\wedge y)]\wedge[h(y) - h(x\wedge y)] \\ &= h(x)\wedge h(y) - h(x\wedge y) \end{aligned}$$

(by the identity again), i.e. h is a real lattice homomorphism.

Corollary 2.23. If $x \geq 0$, there exists $h \in \Omega$ with $h(x) = \|x\|$.

Proof. The h we constructed in Lemma 2.19 satisfies $h(e) = 1$. [Indeed by Lemma 2.20 with $g = h$ gives $h = h(e)h$, i.e. $h(e) = 1$.]

We now know that T is isometric. Thus $T(E)$ is a closed sublattice of $C(\Omega)$ containing $1 = T(e)$. Moreover, $T(E)$ separates points of Ω ($h_1 \neq h_2 \Rightarrow \exists$ x with $h_1(x) \neq h_2(x)$). The standard proof of Stone-Weierstrass theorem gives

$$T(E) = C(\Omega).$$

APPENDIX

by M.Rao

We can give another proof of Aren's theorem as follows: This proof is more measure theoretic. Let $\mathfrak{A}$ be a commutative real Banach algebra with unit e with

$$\|x^2\| = \|x\|^2, \qquad x \in \mathfrak{A}$$

$$\|x^2+y^2\| \geq \|x^2\|, \qquad x,y \in \mathfrak{A}.$$

Let

$$P^* = \{\xi : \xi \in \mathfrak{A}^*, \ \|\xi\| = \xi(e) = 1\}$$

and let Ω denote the extreme points of P^*. We shall show that Ω is precisely the set of multiplicative linear functionals on $\mathfrak{A}$.

Proposition 1. $x \in \mathfrak{A}$, $\|x\| \leq 1 \implies e-x = y^2$ for some $y \in \mathfrak{A}$.

The proof is in Lemma 2.3.

Proposition 2. $x \in \mathfrak{A}$, $\xi \in P^* \implies \xi(x^2) \geq 0$.

Proof. If $\|x^2\| \leq 1$, then

$$e-x^2 = y^2 \implies e = x^2+y^2 \implies \|e-x^2\| = \|y^2\| \leq \|e\| = 1.$$

Hence

$$|\xi(e-x^2)| \leq 1, \quad \text{i.e.} \quad \xi(x^2) \geq 0.$$

Proposition 3. $x \in \mathfrak{A}$, $\|y\| \leq 1$, $\xi \in P^* \implies$

$$|\xi(x^2y)| \leq \xi(x^2).$$

Proof. $x^2 \pm x^2y = x^2(e \pm y)$ and both $e \pm y$ are squares $\Longrightarrow$ $x^2(e \pm y)$ are both squares $\Longrightarrow$ $\xi(x^2(e \pm y)) \geq 0$ (by Proposition 2). Hence

$$|\xi(x^2y)| \leq \xi(x^2).$$

Proposition 4. $\Omega = \mathrm{ext}(P^*) =$ set of multiplicative functionals on $\mathfrak{A}$.

Proof. Let $\xi \in \mathrm{ext}(P^*)$ and $x \in \mathfrak{A}$, $\|x\| \leq 1$. Let us first show that $\xi(x^2y) = \xi(x^2)\xi(y)$. If $\xi(x^2) = 0$, nothing to prove (see Proposition 3). Let $0 < \xi(x^2) \leq 1$. Choose α so that $0 < \xi(\alpha^2 x^2) < 1$. Let ξ_1 and ξ_2 be defined by

$$\xi_1(y) = \frac{\xi(\alpha^2x^2y)}{\xi(x^2\alpha^2)}, \qquad \xi_2(y) = \frac{\xi((e-\alpha^2x^2)y)}{\xi(e-x^2\alpha^2)}.$$

Proposition 3 shows that $\xi_1, \xi_2 \in P^*$. Also

$$\xi = \xi(x^2\alpha^2)\xi_1 + (1-\xi(\alpha^2x^2))\xi_2.$$

Since ξ is extreme we should have $\xi = \xi_1$, i.e.

$$\xi(\alpha^2x^2)\xi(y) = \xi(\alpha^2x^2y)$$

i.e.

$$\xi(x^2)\xi(y) = \xi(x^2y).$$

Since

$$4x = (e+x)^2 - (e-x)^2$$

we deduce that

$$\xi(xy) = \xi(x)\xi(y).$$

Conversely suppose ξ is multiplicative and

$$\xi = t\xi_1 + (1-t)\xi_2.$$

Let

$$\xi(x) = 0 \Longrightarrow \xi(x^2) = 0 \Longrightarrow \xi_1(x^2) = 0$$

(because $\xi_1(x^2) \geq 0$, $\xi_2(x^2) \geq 0$ by Proposition 2). $\Longrightarrow$ by Propo sition 2, $\xi_1(x) = 0$. [Indeed $\xi_1((e \pm \lambda x)^2) \geq 0$, i.e. $1 + 2\lambda \xi(x) \geq 0$ for all λ $\Rightarrow \xi_1(x) = 0$. Thus the kernels of ξ and ξ_1 are the same $\Longrightarrow \xi \equiv \xi_1$, i.e. ξ is extreme.

Only one more proposition is necessary. Namely that given $x \in \mathfrak{A}$, there exists $\xi \in P^*$ with

$$\xi(x^2) = \|x\|^2.$$

<u>Proposition 5</u>. $x \in \mathfrak{A}$, $\|x\| = 1 \Longrightarrow$

$$|\lambda + \mu| \leq \|\lambda e + \mu x^2\| \quad \text{for all} \quad \lambda, \mu .$$

<u>Proof</u>. If $\lambda, \mu > 0$, then

$$\|\lambda e + \mu x^2\| = \|\lambda(e - x^2) + (\lambda + \mu)x^2\| \geq (\lambda + \mu)$$

because $e - x^2$ is a square. If $\lambda > 0$, but $\mu < 0$, we have

$$\|\lambda e + \mu x^2\| = \|\lambda e - |\mu| x^2\| \geq |\lambda \|e\| - |\mu| \|x^2\||$$
$$= |\lambda - |\mu|| = |\lambda + \mu| .$$

<u>Proposition 6</u>. If $x \in \mathfrak{A}$, then there exists $\xi \in P^*$ with

$$\xi(x^2) = \|x^2\| .$$

<u>Proof</u>. May assume x^2 is linearly independent of e, and $\|x^2\| = 1$. From Proposition 5, the functional

$$\lambda e + \mu x^2 \to \lambda + \mu$$

is of norm 1. So it can be extended to all of $\mathfrak{A}$ by Hahn-Banach theorem.

The rest of the proof is the same.

SECTION 3

Simultaneous linear extension.

Let S be a compact Hausdorff space and $C(S)$ the real or complex continuous functions on S. We consider $C(S)$ as a Banach algebra. The following characterizes closed ideals in $C(S)$

Theorem 3.1. I is a closed ideal iff there exists a closed subset $K \subset S$ with

$$I = I(K) = \{f : f \in C(S), \quad f(K) = 0\}.$$

Proof. The "if" part is clear. Let I be a closed ideal and

$$K = \{s \in S : \quad f(s) = 0 \text{ for all } f \in I\}.$$

Let $f_o \in C(S)$ and $f_o(K) = 0$. We shall show that $f_o \in I$. This will of course imply that $K \neq \emptyset$ because if K were empty, $f \equiv 1$ is such a function. Let $\varepsilon > 0$ and

$$F = \{s : \ |f_o(s)| \geq \varepsilon\}.$$

By definition of K, for each $x \in F$ there exists a function in I which does not vanish at x. By compactness we may select a finite number of functions $f_1, \ldots, f_n$ in I such that not all of them vanish at any point of F. Now $f \in I \Rightarrow \bar{f} \in C(S) \Rightarrow f\bar{f} \in I$. Thus

$$g = \sum_{i=1}^{n} |f_i|^2 \in I \qquad \text{and} \qquad g = \sum_{i=1}^{n} |f_i|^2 \geq \varepsilon$$

on F, $g = 0$ on K. The function $h_1 = \frac{1}{g}$ on F, $h_1 = 0$ on K can be extended continuously to all of S. Then the function $h = h_1 \wedge \frac{1}{g}$ is continuous, so $gh \in I$. It is clear that $gh = 1$ on F, $gh = 0$ on K and $0 \leq gh \leq 1$. Also $f_o gh \in I$ and $|f_o - f_o gh| \leq \varepsilon$ everywhere. Since I is closed this implies $f_o \in I$.

If I is a closed ideal, we give $C(S)|_I$ the quotient norm

$$\|f+I\| = \inf_{g \in I} \|f+g\|.$$

Theorem 3.2. Let I be a closed ideal with zero set K. Then the restriction map

$$R: f \longrightarrow f|_K$$

defines an isometric isomorphism of $C(S)|_I$ onto $C(K)$.

Proof. Clearly f_1 and f_2 lie in the same coset iff

$$R(f_1) = R(f_2).$$

Thus

$$R: C(S)|_I \longrightarrow C(K)$$

is well defined and $\|R\| \leq 1$. Also R is a homomorphism, indeed an isomorphism. If $h \in C(K)$, then h has an extension $g \in C(S)$ with $\|g\|_S = \|h\|_K$. [Take any extension say f and put

$$h = \begin{cases} f & \text{if } |f| \leq 1 \\ f/|f| & \text{if } |f| \geq 1. \end{cases}]$$

Thus

$$\|R(g)\| = \|h\| = \|g+I\|$$

so R is isometric onto.

Theorem 3.3. Let S and T be compact and

$$u: C(S) \longrightarrow C(T)$$

be continuous and linear. The following are equivalent.

(1) $u(1_S) = 1_T$ and $\|u\| = 1$.

(2) $u(1_S) = 1_T$ and $f \geq 0 \implies u(f) \geq 0$.

(3) $u^*P(T) \subset P(S)$, [$P(T), P(S)$ are the probability measures on T, S]. Or equivalently u^* maps the positive cone in $M(T)$ isometrically into the positive cone of $M(S)$

Proof. (1) $\implies$ (2). We show if $f \in C(S)$

$$u(f)(T) \subset \overline{co}\, f(S).$$

Let α be a scalar. Then

$$\max_{t\in T} |\alpha-(uf)(t)| = \max_{t\in T}|u(\alpha 1_S-f)(t)|$$

$$\leq \max_{s\in S}|(\alpha 1_S-f)(s)| = \max_{s\in S}|\alpha-f(s)|.$$

Therefore the range of $u(f)$ lies in every circle in the plane containing the range of f. This implies that the closed convex hull of the range of f contains the range of uf. In particular $f\geq 0 \implies uf\geq 0$.

[Another way of seeing this is as follows: Let x be a complex number. Then $1\geq x\geq 0$ is equivalent to $|1-x+xe^{i\alpha}|\leq 1$ for all real α. In fact if $1\geq x\geq 0$, then

$$|1-x+xe^{i\alpha}| \leq |1-x| + |x| = 1-x+x = 1.$$

Conversely if x satisfies the inequality we may choose α so that $xe^{i\alpha} = |x|$. Then we get $|1-x+|x|| \leq 1 \implies x$ is real and positive. Since x can be replaced by $1-x$ we see that also $1\geq x$. Now if

$$u1_S = 1_T, \quad \|u\| = 1, \quad \text{and} \quad 1\geq f\geq 0,$$

we have

$$1 \geq \|u(1_S-f+fe^{i\alpha})\| \quad \text{for all } \alpha,$$

i.e.

$$|1_T-uf+e^{i\alpha}uf| \leq 1 \quad \text{for all } \alpha \implies uf\geq 0 \text{ and } 1\geq uf.]$$

(2) $\implies$ (3). Let $\mu\in P(T)$, i.e. $\|\mu\| = 1$, $\mu\geq 0$. Then

$$(u^*\mu)(f) = \mu(u(f)) \geq 0 \quad \text{if} \quad f\geq 0$$

$$u^*\mu(1_S) = \mu(1_T) = \|\mu\| = 1.$$

So

$$u^*: \quad P(T) \longrightarrow P(S)$$

is an isometry.

(3) $\Longrightarrow$ (1). (3) implies

$$(u^*\mu)(1_S) = \mu(u(1_S)) = \mu(1_T)$$

for all $\mu \geq 0$, hence for all μ, so

$$u(1_S) = 1_T.$$

Now let $f \in C(S)$, $t \in T$. Then

$$|(uf)(t)| = |(u^*\delta_t)(f)| \leq \|u^*\delta_t\| \|f\| = \|f\|,$$

so

$$\|u\| \leq 1.$$

Since $u1_S = 1_T$, $\|u\| = 1$.

Definition 3.4. A map $u: C(S) \longrightarrow C(T)$ is called regular if it satisfies any of the above equivalent conditions of Theorem 3.3.

Definition 3.5. Let X and Y be Banach spaces and $T: X \longrightarrow Y$ be continuous linear and onto. Let

$$N(T) = \{x: \ Tx = 0\}.$$

A continuous linear map $U: Y \longrightarrow X$ is a linear inverse for T if

$$TU(y) = y, \qquad y \in Y.$$

Lemma 3.6. (a) If U is a linear inverse for $T: X \longrightarrow Y$ then $U(Y)$ is closed in X and

$$X = U(Y) \oplus N(T).$$

If P is the projection of X onto $U(Y)$, then $P = UT$.

(b) Suppose W is a closed subspace such that

$$X = W \oplus N(T)$$

and P is the projection of X onto W. Then $U: Y \longrightarrow X$ defined by

$$Uy = Px, \quad \text{where} \quad y = Tx$$

is a linear inverse for T.

Proof. (a) Let Uy_n converge to x_o. Then

$$TUy_n = y_n \longrightarrow Tx_o = y_o$$

and

$$UTUy_n = Uy_n \longrightarrow Uy_o,$$

so

$$Uy_o = x_o.$$

Thus $U(Y)$ is closed in X. Also

$$N(T) \cap U(Y) = \{0\}$$

because

$$TU(y) = 0 \implies y = TU(y) = 0.$$

Let $x \in X$. Then

$$x = (x-UTx) + UTx$$

and

$$T(x-UTx) = Tx - TUTx = Tx - Tx = 0,$$

and

$$(UT)^2 = UTUT = U(TU)T = UT.$$

(b) Note

$$y = Tx_1 = Tx_2 \implies x_1-x_2 \in N(T)$$
$$\implies P(x_1-x_2) = 0 \implies Px_1 = Px_2.$$

Thus U is well defined. Also TU = identity because

$$y = Tx \implies TUy = TPx = Tx.$$

Since

$$x = Px + (I-P)x \quad \text{and} \quad (I-P)x \in N(T)$$

we see that $U(Y) = W$. It remains to show that U is continuous. This is easy because T restricted to W is 1-1 continuous onto Y. Its inverse (which is U) is therefore continuous by Banach's theorem.

<u>Definition 3.7</u>. Let T be compact and $S \subset T$ be closed. A continuous linear operator $e: C(S) \to C(T)$ is an <u>operator</u> of <u>simultaneous linear extension</u> (S.L.E.), if

(1) $\quad e(g)|_S = g, \qquad g \in C(S)$

(2) $\quad e(1_S) = 1_T.$

<u>Remarks</u>. (1). Let $R: C(T) \to C(S)$ be the restriction operator

$$Rf = f|_S, \qquad f \in C(T).$$

Then

$$(Re)(g) = g, \qquad g \in C(S),$$

so e is a linear inverse for R which maps 1_S into 1_T. Conversely, any linear inverse for R which carries 1_S into 1_T is an operator of S.L.E.

(2) Given a S.L.E. e, $e(C(S))$ is closed, and

$$C(T) = e(C(S)) \oplus I(S).$$

$I(S) = \{f \in C(T): \ Rf = 0\}$ is the ideal of functions vanishing on S.

(3) The existence of a S.L.E. is equivalent to the existence of a complementary closed subspace W to $I(S)$ containing 1_T:

$$C(T) = W \oplus I(S), \qquad 1_T \in W.$$

(4) If P is the projection of $C(T)$ onto W, then

$$\|e\| = \|P\|.$$

Indeed:

$$P = eR \implies \|P\| \leq \|e\|\,\|R\| = \|e\|.$$

But $g \in C(S)$ implies that there exists $f \in C(T)$ with $Rf = g$, $\|f\| = \|g\|$. Then

$$f - eg \in I(S),$$

so

$$eg = Peg = Pf$$

$$\implies \|eg\| \leq \|P\|\,\|f\| = \|P\|\,\|g\|.$$

(5) If $e : C(S) \to C(T)$ is a regular S.L.E. (this simply means $\|e\| = 1$) we have $g \geq 0 \Rightarrow eg \geq 0$ and e is isometric.

We can now state

Theorem 3.8. Let $S \subseteq T$ be closed. The following are equivalent:

(a) There exists a regular operator of S.L.E. for $C(S)$.

(b) We have $C(T) = W \oplus I(S)$,

where W is a closed subspace containing 1_T and the projection $P: C(T) \to W$ has norm 1.

(c) We have $C(T) = W \oplus I(S)$, where $1_T \in W$ and $P: C(T) \to W$ is a positive operator.

It is convenient to have a somewhat more general formulation.

Definition 3.9. Let S, T be compact and $\varphi : S \to T$ be a homeomorphic imbedding. The operator $\varphi^\circ : C(T) \to C(S)$ is defined by $(\varphi^\circ f)(s) = f(\varphi(s))$, $f \in C(T)$, $s \in S$. An operator $u: C(S) \to C(T)$ is a (regular) linear inverse of extension for φ° if

(i) $u(1_S) = 1_T$.

(ii) $\varphi^\circ u = I_{C(S)}$ = identity on $C(S)$.

((iii) u is regular.)

Note that the null space $N(\varphi^\circ) = I(\varphi(S))$ and φ = identity $\Rightarrow \varphi^\circ = R$ = restriction. We know that under these conditions $u(C(S))$ is closed and

$$C(T) = u(C(S)) \oplus I(\varphi(S)).$$

We can state:

Theorem 3.10. Let $\varphi : S \to T$ be a homeomorphic imbedding The following are equivalent:

(1) There exists a (regular) linear inverse of extension for φ° .

(2) There exists a (regular) operator of S.L.E. for $C(\varphi(S))$.

(3) $C(T) = W \oplus I(\varphi(S))$ with $1_T \in W$.

Our next goal is to prove:

Theorem 3.11. (Borsuk-Dugundji.) Let S be compact metric and T be any compact space. If $\varphi : S \to T$ is any homeomorphic imbedding, there exists a regular linear inverse of extension for φ^{o}.

Definition 3.12. Let X be a topological space. An open cover $\mathcal{U}$ of X is a family of open sets whose union is X. An open cover $\mathcal{V}$ is a refinement of an open cover $\mathcal{U}$ if for each $V \in \mathcal{V}$ there is a $U \in \mathcal{U}$ such that $V \subseteq U$. An open cover $\mathcal{U}$ is locally finite if for each $x \in X$ there exists a neighbourhood of x which intersects only finitely many members of $\mathcal{U}$. If every open cover has a locally finite refinement we say that X is paracompact. A reference for paracompact spaces is [21], pp. 156-161.

What we need is

Theorem 3.13. (A.H.Stone.) Every metric space is paracompact.

For a proof, see [21], pp. 156-160.

Lemma 3.14. Let X be a metric space and S a closed subset of X. Put $G = X-S$ the complement of S. There exists an open cover $\mathcal{U}$ of G such that

(1) $\mathcal{U}$ is locally finite.

(2) For each point $a \in S$ and each neighbourhood V_a of a, there exists a neighbourhood W_a such that $U \in \mathcal{U}$, $U \cap W_a \neq \emptyset \implies U \subset V_a$.

Proof. For $x \in G$ let

$$K_x = \{ y : \rho(x,y) < \tfrac{1}{2} \rho(x,S) \}$$

where ρ is a metric in X. Clearly each $K_x \subset G$ and the family $\{K_x ; x \in G\}$ is an open cover of G. G is a metric space. By Theorem 3.13 there exists a locally finite open refinement of this cover. Call this $\mathcal{U}$. We shall show that $\mathcal{U}$ has the required properties.

$\mathcal{U}$ is locally finite by choice. Since each member of $\mathcal{U}$ is contained in K_x for some $x \in G$, it is enough to prove (2) for the family $\{K_x; x \in G\}$. Since V_a is a neighbourhood of a, there exists $\varepsilon > 0$ such that the ball $\{y: \rho(a,y) < \varepsilon\} \subset V_a$. Let $W_a = \{y: \rho(a,y) < \frac{\varepsilon}{3}\}$. Suppose for some $x \in G$, $K_x \cap W_a \neq \emptyset$. Let us show that $K_x \subset V_a$. Let $z_0 \in K_x \cap W_a$. Then

$$\rho(z_0,a) < \frac{\varepsilon}{3}, \qquad \rho(z_0,x) < \tfrac{1}{2}\rho(x,S)$$

$$\Longrightarrow \rho(x,S) \leq \rho(x,a) \leq \rho(x,z_0) + \rho(z_0,a) < \frac{\varepsilon}{3} + \tfrac{1}{2}\rho(x,S)$$

$$\Longrightarrow \rho(x,S) \leq \frac{2\varepsilon}{3}.$$

Thus

$$\rho(x,a) < \frac{\varepsilon}{3} + \frac{1}{2}\rho(x,S) \leq \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \frac{2\varepsilon}{3}.$$

Now

$$z \in K_x \Longrightarrow \rho(x,z) < \frac{1}{2}\rho(x,S) \leq \frac{\varepsilon}{3}.$$

So

$$\rho(z,a) \leq \rho(x,z) + \rho(a,x) < \frac{\varepsilon}{3} + \frac{2\varepsilon}{3} = \varepsilon \Longrightarrow z \in V_a.$$

Lemma 3.15. Let $\mathcal{U} = \{U_\alpha : \alpha \in A\}$ be an open locally finite cover of a metric space X. Then there exists a family $\{\lambda_\alpha : \alpha \in A\}$ of continuous functions such that

(1) $0 \leq \lambda_\alpha \leq 1$.

(2) For each α, $\lambda_\alpha(X-U_\alpha) = 0$.

(3) $\sum \lambda_\alpha(x) \equiv 1, \quad x \in X$.

Proof. Define

$$\tau_\alpha(x) = \rho(x, X-U_\alpha).$$

The functions τ_α are continuous; $\tau_\alpha(x) = 0$, iff $x \in X-U_\alpha$. Each $x \in X$ has a neighbourhood which intersects only finitely many U_α, so

$$\tau = \sum_{\alpha \in A} \tau_\alpha(x)$$

is a well defined continuous strictly positive function. Let

$$\lambda_\alpha(x) = \frac{\tau_\alpha(x)}{\tau(x)} .$$

We call $\{\lambda_\alpha \; ; \alpha \in A\}$ a partition of unity subordinate to $\mathcal{U}$.

Let X be a metric space, S a closed subset of X and $G = X - S$. Let $\mathcal{U}$ be a locally finite open cover of G determined as in Lemma 3.14. Let $\mathcal{U} = \{U_\alpha : \alpha \in A\}$ and $\{\lambda_\alpha ; \alpha \in A\}$ be a partition of unity subordinate to $\mathcal{U}$. For each $\alpha \in A$, let x_α be a point in U_α and choose a point $a_\alpha \in S$ such that $\rho(x_\alpha, a_\alpha) < 2\rho(x_\alpha, S)$.

Let g be continuous on S and define

$$\begin{aligned} \tilde{g}(x) &= g(x), && x \in S \\ &= \sum_{\alpha \in A} \lambda_\alpha(x) g(a_\alpha), && x \in G \end{aligned}$$

The sum is meaningful because $\lambda_\alpha(x) = 0$ except for a finite number of α . Clearly we have $g \geq 0 \implies \tilde{g} \geq 0$, $g \equiv 1 \implies \tilde{g} \equiv 1$. Also $0 \leq \lambda_\alpha \leq 1$, $\sum \lambda_\alpha \equiv 1 \implies$ that the range of $\tilde{g}$ is contained in the convex hull of $g(S)$.

Lemma 3.16. $\tilde{g}$ is continuous on X.

Proof. Clearly the continuity of $\tilde{g}$ need be verified only at the boundary points of S.

Let $p \in \bar{G} \cap S$ and let V be a convex neighbourhood of $g(p)$ (note that $\tilde{g}(p) = g(p)$). By continuity of g on S, there exists $\varepsilon > 0$ such that $a \in S$, $\rho(a,p) < 3\varepsilon \implies g(a) \in V$. Let

$b(p,\varepsilon) = \{y \in X: \; \rho(a,y) < \varepsilon\}$. Then as we showed in Lemma 3.14

$$U \in \mathcal{U}, \quad U \cap b(p, \tfrac{\varepsilon}{3}) \neq \emptyset \implies U \subset b(p,\varepsilon).$$

So let $x \in b(p, \frac{\varepsilon}{3})$. If $\lambda_\alpha(x) \neq 0$, then

$$\begin{aligned} & x \in U_\alpha \implies U_\alpha \cap b(p,\tfrac{\varepsilon}{3}) \neq \emptyset \implies U_\alpha \subset b(p,\varepsilon) \\ \implies & x_\alpha \in b(p,\varepsilon) \implies \rho(x_\alpha, S) < \varepsilon \implies \\ & \rho(a_\alpha, x_\alpha) < 2\rho(x_\alpha, S) < 2\varepsilon . \end{aligned}$$

Since $x_\alpha \in b(p,\varepsilon)$ we deduce $\rho(a_\alpha,p) < 3\varepsilon \Longrightarrow g(a_\alpha) \in V$. What we have said is: $x \in b(p,\frac{\varepsilon}{3})$, $\lambda_\alpha(x) \neq 0$ imply that $g(a_\alpha) \in V$. Since

$$\tilde{g}(x) = \sum_\alpha \lambda_\alpha(x) g(a_\alpha) = \sum_{\lambda_\alpha(x)\neq 0} \lambda_\alpha(x) g(a_\alpha)$$

we deduce that $\tilde{g}(x) \in V$, since V was assumed convex. This proves the continuity of $\tilde{g}$ at p.

We have thus proved (since $g \to \tilde{g}$ is clearly linear)

Theorem 3.17. Let X be a metric space and S a closed subset of X. Then there exists a map $e: C(S) \to C(X)$ such that $e(1_S) = 1_X$, $e(f) \geq 0$ if $f \geq 0$, and e is linear, $|e(f)| \leq \sup_{s\in S}|f(s)|$.

Corollary 3.18. Let $S \subset T$, S and T compact metric. Then there exists a (regular) operator of S.L.E. for $C(S)$.

We have also the following general Tietze theorem. [Only let V of Lemma 3.16 be a convex neighbourhood in a linear space.]

Theorem 3.19. (Dugundji.) Let S be a closed subset of a metric space X and let Y be a locally convex linear space. Every continuous map $f: S \to Y$ can be extended continuously to all of X in such a way that the range of the extension is contained in the convex hull of $f(S)$.

A proof of Theorem 3.19 can also be found in Borsuk [5], p.77.

Definition 3.20. A compact Hausdorff space is called a Dugundji space if for every compact space T and every homeomorphic imbedding $\varphi : S \to T$, there exists a regular linear inverse of extension $u: C(S) \to C(T)$ for φ°. [Equivalently if there exists a regular operator of S.L.E. for $C(\varphi(S))$.]

We will prove that every compact metric space is a Dugundji space.

Remark. Let S and T be compact and $\varphi : S \to T$ be any continuous map. We may define $\varphi^\circ : C(T) \to C(S)$ by

$$(\varphi^\circ g)(s) = g(\varphi(s)), \qquad s \in S, \quad g \in C(T).$$

Then φ° is an algebraic homomorphism and $\varphi^\circ(1_T) = 1_S$, $\|\varphi^\circ\| = 1$. More about this later. Now we only need $\varphi : S \to T$, $\psi : T \to U$ and $\eta = \psi\varphi \Rightarrow \eta^\circ = \varphi^\circ\psi^\circ$.

Lemma 3.21. For a compact space S the following are equivalent:

(a) S is a Dugundji space.

(b) There exists a Dugundji space T and a homeomorphic imbedding $\varphi : S \to T$ which admits a regular linear inverse of extension (R.L.I.E.) for φ°.

(c) For some cardinal number m, there exists a homeomorphic imbedding $\psi : S \to I^m$ $(I = [0,1])$ which admits a R.L.I.E. for ψ°.

Proof.

(a) $\Rightarrow$ (b). Take $T = S$, φ = identity.

(b) $\Rightarrow$ (c). Let T be Dugundji, $\varphi : S \to T$ and $u: C(S) \to C(T)$ a R.L.I.E. for φ°. There exists a homeomorphic imbedding $\theta : T \to I^m$. Since T is Dugundji there exists $v: C(T) \to C(I^m)$ which is a R.L.I.E. for θ°. Now $\psi = \theta\varphi : S \to I^m$. Define $\omega = vu$. Then $\omega : C(S) \to C(I^m)$; $\omega(1_S) = 1_{I^m}$. Since $\psi^\circ = \varphi^\circ\theta^\circ$ we get $\psi^\circ\omega = \varphi^\circ\theta^\circ\omega = \varphi^\circ\theta^\circ v u = \varphi^\circ \mathrm{id}_{C(T)} u = \varphi^\circ u = \mathrm{id}_{C(S)}$. Since $1 \leq \|\omega\| \leq \|u\|\|v\| = 1$, ω is a R.L.I.E. for ψ°.

(c) $\Rightarrow$ (a). Let $\psi : S \to I^m$ with R.L.I.E. $\omega : C(S) \to C(I^m)$. Let T be arbitrary and $\varphi : S \to T$ be a homeomorphism. We will define a map $\theta : T \to I^m$ as follows: Let A be an index set of cardinal m. Then

$$\psi(s) = \{f_\alpha(s) : \alpha \in A\}, \quad f_\alpha : S \to I.$$

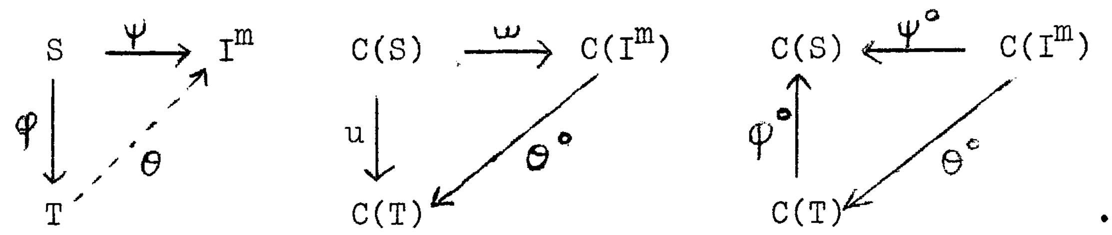
Define g_α on $\varphi(S)$ by

$$g_\alpha(\varphi(s)) = f_\alpha(s).$$

Extend each g_α to all of T such that $g_\alpha(T) \subset I$. Define Θ by

$$\Theta(t) = (g_\alpha(t) : \alpha \in A).$$

It should be clear that $\theta \circ \varphi = \psi$. Now define $u: C(S) \to C(T)$ by $u = \Theta^\circ \omega$. It is simple to show that it is a R.L.I.E. for φ°.



Theorem 3.22. Every compact metric space S is Dugundji.

Proof. Imbed S into a Hilbert cube and use Lemma 3.14.

SECTION 4

Projections onto self-adjoint subalgebras of $C(S)$.

We begin by discussing some basic material which might have been discussed earlier.

If S, T compact and $\varphi : S \to T$ continuous, then $\varphi^\circ : C(T) \to C(S)$ is a homomorphism, $\varphi^\circ(1_T) = 1_S$, $\|\varphi^\circ\| = 1$.

Theorem 4.1.

(a) $\varphi^\circ[C(T)]$ is a closed self-adjoint subalgebra of $C(S)$.

(b) φ° is onto $C(S)$ iff φ is one to one.

(c) φ° is one to one $\iff$ φ is onto $\iff$ φ° is an isometric isomorphism into $C(S)$.

Proof.

(a) We only need to show that $\varphi^\circ[C(T)]$ is closed. Let $\varphi^\circ g_n \to h \in C(S)$. Then $\lim g_n(\varphi(s)) = h(s)$ uniformly on S. $\Longrightarrow$ $g_n(t)$ converges uniformly on $\varphi(S)$ to a continuous function $g_o(t)$. Extend g_o to T. Then $h = \varphi^\circ g_o$.

(b) $\varphi^\circ[C(T)] = C(S) \iff \varphi^\circ(C(T))$ separates points of S (by Stone-Weierstrass) $\iff s_1 \neq s_2$, $s_1, s_2 \in S$ implies that there exists $g \in C(T)$ such that $\varphi^\circ(g)(s_1) \neq \varphi^\circ(g)(s_2) \iff g(\varphi(s_1)) \neq g(\varphi(s_2)) \iff \varphi(s_1) \neq \varphi(s_2)$.

(c) φ° is one to one iff $\varphi^\circ g = 0 \iff g = 0$. Thus if $\varphi(S) \neq T$, there exists $g \neq 0$ with $g(\varphi(S)) = 0$, i.e. $\varphi^\circ g = 0$. The other direction is clear. If φ is onto, isometry of φ° is clear.

The above arguments are more general than they seem.

Theorem 4.2. Suppose $\upsilon : C(T) \to C(S)$ is any homomorphism such that $\upsilon(1_T) = 1_S$. Then there exists a continuous map $\varphi : S \to T$ such that $\upsilon = \varphi^\circ$. Thus $\|\upsilon\| = 1$ and $\upsilon(C(T))$ is closed.

Proof. We have for each $s \in S$

$$\mathcal{U}(g_1 g_2)(s) = \mathcal{U}(g_1)(s)\mathcal{U}(g_2)(s), \qquad g_1, g_2 \in C(T).$$

Thus the function

$$F_s(g) = \mathcal{U}(g)(s), \qquad g \in C(T),$$

is non-zero and multiplicative, hence continuous, and there exists $t \in T$ such that

$$F_s(g) = g(t), \qquad g \in C(T).$$

The map $\varphi : S \to T$ defined by $\varphi(s) = t$ is thus continuous and

$$\mathcal{U}(g)(s) = g(\varphi(s)) = (\varphi^\circ g)(s), \quad s \in S, \quad g \in C(T).$$

Thus $\mathcal{U} = \varphi^\circ$.

Now let $\varphi : S \to T$ be any continuous onto map. We know $\varphi^\circ : C(T) \to C(S)$ is an isometric isomorphism of $C(T)$ onto a closed self-adjoint subalgebra $\varphi^\circ [C(T)] \subseteq C(S)$ containing 1_S. Define

$$\mathcal{R} = \{\varphi^{-1}(t) : \ t \in T\} ;$$

$\mathcal{R}$ is a family of disjoint closed subsets of S, whose union is S. We call the set $\varphi^{-1}(t)$ the fiber over $t \in T$. We can identify the subalgebra $\varphi^\circ C(T)$ in terms of $\mathcal{R}$.

Lemma 4.3. If $f \in C(S)$, then $f \in \varphi^\circ C(T)$ iff f is constant on each set of $\mathcal{R}$.

Proof. Let $f \in C(S)$ be constant on each fiber. Define $g \in C(T)$ by

$$g(t) = f(\varphi^{-1}(t)).$$

If $t_\alpha \to t_0$, choose $s_\alpha \in \varphi^{-1}(t_\alpha)$; we may suppose $s_\alpha \to s_0$. Then

$$\varphi(s_0) = \lim \varphi(s_\alpha) = \lim t_\alpha = t_0.$$

Thus

$$g(t_\alpha) = f(\varphi^{-1}(t_\alpha)) = f(s_\alpha) \to f(s_o) = f(\varphi^{-1}(t_o)) = g(t_o).$$

Thus g is indeed in $C(T)$.

As

$$g(\varphi(s)) = f(s)$$

we have $f \in \varphi^\circ[C(T)]$. The other direction is easy.

We note an important property of $\mathcal{R}$.

Lemma 4.4. If V is an open set in S, then

$$\bigcup\{\varphi^{-1}(t): \varphi^{-1}(t) \subset V\}$$

is also open in S. [This is the property characterizing upper semi continuous decompositions. See Kelley [21], p. 99.]

Proof. Since $\varphi(V^c)$ is closed, $\varphi^{-1}[[\varphi(V)^c]^c]$ is open and

$$\bigcup\{\varphi^{-1}(t): \varphi^{-1}(t) \subset V\} = \varphi^{-1}\{\varphi(V^c)^c\} \subset V.$$

Theorem 4.5. Let $\mathfrak{A} \subset C(S)$ be a closed self-adjoint subalgebra containing 1_S. Define

$$s_1 \approx s_2 \text{ iff } f(s_1) = f(s_2) \quad \text{for all } f \in \mathfrak{A}.$$

Then the equivalence relation partitions S into a collection $\mathcal{R}$ of disjoint closed sets. There exists a compact space T and a continuous onto map $\varphi : S \to T$ such that

$$\mathcal{R} = \{\varphi^{-1}(t): t \in T\}.$$

Proof. By the Gelfand-Naimark theorem (Theorem 2.13) $\mathfrak{A}$ is isometrically isomorphic to a $C(T)$. This isometry gives an isometry $\upsilon : C(T) \to C(S)$ with range $\mathfrak{A}$. By Theorem 4.1 and Theorem 4.2 we get a continuous map $\varphi : S \to$ onto T such that $\varphi^\circ = \upsilon$. Clearly

$$\mathcal{R} = \{\varphi^{-1}(t): t \in T\}.$$

Definition 4.6. A continuous linear operator $u: C(S) \to C(T)$ is an averaging operator for φ° if $u\varphi^\circ = id_{C(T)}$.

If u is regular, we call u a regular averaging operator.

Remarks.

(1) An averaging operator for φ° is onto $C(T)$.

(2) $u: C(S) \to C(T)$ is an averaging operator iff

$$u = \varphi^{\circ -1} \text{ on } \varphi^\circ(C(T)).$$

(3) necessarily $u(1_S) = 1_T$.

(4) necessarily $\|u\| \geq 1$. Thus for an averaging operator the following are equivalent:

(a) u is regular

(b) $\|u\| = 1$

(c) $u \geq 0$.

(5) φ° is a regular linear inverse for u. Thus

$$C(S) = \varphi^\circ C(T) \oplus N(u).$$

We can now state the connections between linear averaging operators, and projections of $C(S)$ onto $\varphi^\circ(C(T))$.

Lemma 4.7.

(a) Let $u: C(S) \to C(T)$ be an averaging operator for φ°. Then $P = \varphi^\circ u$ is a projection of $C(S)$ onto $\varphi^\circ(C(T))$. Moreover $\|u\| = \|P\|$, $N(u) = N(P)$.

(b) Let P be a projection of $C(S)$ onto $\varphi^\circ(C(T))$ and define $u = \varphi^{\circ -1}P$. Then u is an averaging operator for φ°. The projection P is regular $\iff P \geq 0 \iff \|P\| = 1 \iff u$ is regular.

Proof.

(a) From Lemma 3.6 we see that $P = \varphi^\circ u$ is a projection of $C(S)$ onto $C(T)$ with null space $N(P) = N(u)$. Since φ° is isometric $\|P\| = \|u\|$.

(b) If P is a projection and $u = \varphi^{\circ -1}P$, then $u = \varphi^{\circ -1}$ on $\varphi^\circ(C(T))$, so by Remark (2) above, u is an averaging operator. The rest is easy.

We now discuss multiplicative linear operators of simultaneous extension. The connection with averaging operators will become clear as we proceed.

Definition 4.8. Let T be compact, $S \subset T$ be closed. A continuous onto map $\rho : T \rightarrow S$ is called a retraction if $\rho(s) = s$, $s \in S$. We say S admits a retraction from T or that S is a retract of T.

Theorem 4.9. There exists a multiplicative linear operator e of simultaneous extension from S to T, i.e.

$$e(f_1 f_2) = e(f_1) \cdot e(f_2), \qquad f_1, f_2 \in C(S)$$

iff S admits a retraction from T. Moreover e is regular.

Proof. Suppose $\rho : T \rightarrow S$ is a retraction. Then $e = \rho^{\circ}$ is the desired multiplicative extension operator. Conversely, let $e: C(S) \rightarrow C(T)$ be a multiplicative operator of S.L.E. From Theorem 4.2 (it is clear that $e 1_S = 1_T$), there exists a map $\rho: T \rightarrow S$ with $\rho^{\circ} = e$. We have $eg = g \circ \rho$, i.e. $eg(t) = g(\rho(t))$ for all $t \in T$. If $s \in S$, $eg(s) = g(s)$, i.e. $g(\rho(s)) = g(s)$ for all $g \in C(S)$. Thus $\rho(s) = s$, i.e. ρ is a retraction.

Corollary 4.10. Let $\mathfrak{A}$ be a closed self-adjoint subalgebra of $C(T)$ containing 1_T and $\mathcal{R}$ be the collection of sets of constancy for $\mathfrak{A}$. Then $\mathfrak{A}$ is the range of a multiplicative projection $P: C(T) \rightarrow \mathfrak{A}$, i.e.

$$P(fg) = P(f)P(g), \qquad f, g \in C(T)$$

iff there exists a closed set $S \subset T$ which meets each set in $\mathcal{R}$ in exactly one point. Necessarily S is a retract of T and the restriction map

$$g \rightarrow g|_S, \qquad g \in C(T)$$

is an isometric isomorphism of $\mathfrak{A}$ onto $C(S)$.

Proof. Let $P: C(T) \longrightarrow \mathfrak{A}$ be a multiplicative projection and $I = N(P)$. Since I is a closed ideal there exists a closed subset S of T such that

$$I = \{ g: \ g(S) = 0 \} .$$

Also $C(T) = \mathfrak{A} \oplus I$, so $\mathfrak{A}|S = C(S)$. The map $e: C(S) \longrightarrow C(T)$ defined by

$$e(f) = Pg \qquad \text{if } g|S = f$$

is an operator of S.L.E. which is multiplicative since if $g_1|S = f_1$, $g_2|S = f_2$, $(g_1 \cdot g_2)|S = (g_1|S)(g_2|S)$, so

$$e(f_1 f_2) = P(g_1 g_2) = P(g_1)P(g_2) = e(f_1)e(f_2).$$

By the proof of Theorem 4.9 there is s retraction $\rho: T \longrightarrow S$ such that $\rho^{\circ} = e$. The range of e is $\mathfrak{A}$. If $g \in \mathfrak{A}$, $g = e(f)$, where $f = g|S$. Thus

$$g(t) = e(f)(t) = f(\rho(t)) = g(\rho(t)), \qquad t \in T, \quad g \in \mathfrak{A} .$$

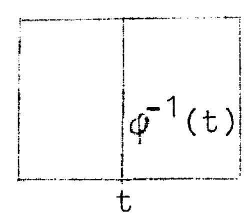
It follows that each set of constancy for $\mathfrak{A}$ meets S. Since $\mathfrak{A}|S = C(S)$, it meets S in exactly one point. Clearly the sets of constancy for $\mathfrak{A}$ are precisely the sets of constancy for ρ.

Conversely, suppose there exists a closed set S meeting each set of $\mathcal{R}$ in exactly one point. Define $\rho(t) = s$ if the set of $\mathcal{R}$ which contains t meets S at s. Suppose $t_\alpha \longrightarrow t_0$. By compactness of S we can suppose $\rho(t_\alpha) \longrightarrow s_0$. Then $\rho(t_0) = s_0$ by uniqueness and the continuity of the functions in $\mathfrak{A}$. Thus ρ is continuous and a retraction of T onto S. The map $e = \rho^{\circ}$ is a multiplicative S.L.E. from $C(S)$ whose range is $\mathfrak{A}$. The map $Pg = e(g|S)$, $g \in C(T)$ is a multiplicative projection.

Example 1. Let (Ω, Σ, μ) be a complete σ-finite measure space. Let M be the algebra of all bounded measurable functions with supremum norm. Then M is isometrically isomorphic to a $C(T)$, and the closed ideal N of null functions consists of those functions in $C(T)$ which vanish on a certain closed subset S of T. The algebra L_∞ of equivalence classes of essentially bounded measurable functions is thus isometrically isomorphic to $C(S)$. A deep theorem of the Tulceas states that there exists a positive, linear, multiplicativ map $T: L_\infty \longrightarrow M$ such that for each class $\bar{f} \in L_\infty$, the function $T\bar{f}$ is a member of the class $\bar{f}$. Viewed differently, T is a multiplicative linear inverse for the natural homomorphism of M onto L_∞ (i.e. of $C(T)$ onto $C(S)$). It follows from the preceding theorems that S must be a retract of T.

Example 2. Let $S = I \times I$, $I = [0,1] = T$.
Let φ be the natural projection map: $(x,y) \longrightarrow x$.
Let μ_t = Lebesgue measure on $\varphi^{-1}(t)$. Define

$$u: C(S) \longrightarrow C(T)$$



by

$$(uf)(t) = \int_{\varphi^{-1}(t)} f(s)\mu_t(ds)$$

(μ_t is regarded as a measure on the square S.)
u averages f over the fiber $\varphi^{-1}(t)$ and

$$u\varphi^\circ = id_{C(T)}.$$

Thus u is an averaging operator for φ°.
We need not take Lebesgue measure on $\varphi^{-1}(t)$. Other choices v_t will work as long as the map

$$(vf)(t) = \int_{\varphi^{-1}(t)} f(s)v_t(ds)$$

maps into $C(T)$. The condition is that $t \rightarrow v_t$ is weak star continuous from T to $M(S)$. Let us see how general this situation is.

Theorem 4.11. Let $\varphi : S \to T$ be a continuous onto map and let $u: C(S) \to C(T)$ be an averaging operator for φ°, i.e. $u\varphi^\circ = id_{C(T)}$. Then u^* maps $M(T) \to M(S)$. For each $t \in T$ let $\mu_t = u^*(\delta_t)$. Then $t \to \mu_t$ is weak star continuous from T into $M(S)$ and

$$uf(t) = \int_S f(s)\mu_t(ds), \qquad f \in C(S).$$

Moreover,

$$\|\mu_t\| \geq 1 \quad \text{and} \quad \|u\| = \sup_{t\in T}\|\mu_t\|.$$

Proof. That $t \to \mu_t$ is weak star continuous is clear because T is homeomorphic to $\{\delta_t : t \in T\}$ and u^* is continuous in the respective weak star topologies. Also

$$(uf)(t) = \delta_t(u(f)) = (u^*\delta_t)(f)$$

$$= \int_S f(s)\mu_t(ds).$$

$$\|u\| = \max_{t\in T}\max_{\|f\|\leq 1}|(uf)(t)| = \max_{\substack{t\in T\\ \|f\|\leq 1}}\left|\int_S f(s)\mu_t(ds)\right|$$

$$= \max_{t\in T}\|\mu_t\|.$$

Since

$$\int 1_S\mu_t(ds) = u(1_S) = 1_T, \quad \|\mu_t\| \geq 1.$$

Now we will investigate the question of existence of maps from S onto T which admit averaging operators. Given spaces S and T it may happen that certain maps from S onto T admit averaging operators and some others do not.

It is a theorem of Milutin that given a compact metric space T there is always a map from the cantor set onto T which admits a linear averaging operator. Our next task is to prove this.

Theorem 4.12. (Localization Principle.) Let $\varphi : S \to T$ be a continuous onto map. Let

$$T = \bigcup_{i=1}^{N} T_i$$

where T_i are compact and

$$T = \bigcup_{i=1}^{N} \mathring{T}_i ,$$

$\mathring{T}_i$ = interior of T_i. Let

$$S_i = \varphi^{-1}(T_i,) \qquad \varphi_i = \varphi | S_i, \qquad i = 1,2,\dots,N.$$

Suppose that for each i, $u_i : C(S_i) \to C(T_i)$ is a regular averaging operator for φ_i. Then there exists a regular averaging operator $u: C(S) \to C(T)$ for φ .

Proof. We have $(u_i \mathring{\varphi}_i)g = g$ for each $g \in C(T_i)$, and for $g \in C(T)$, $(\mathring{\varphi}g)|S_i = \mathring{\varphi}_i(g|T_i)$. Let $\lambda_1, \dots, \lambda_N$ be a partition of unity subordinate to the cover $\{\mathring{T}_i\}$. Since $\lambda_i = 0$ outside $\mathring{T}_i$ the function $\lambda_i(t)u_i(f|S_i)(t)$ is defined on all of T and continuous. For $f \in C(S)$ define

$$(uf)(t) = \sum_{i=1}^{N} \lambda_i(t)\, u_i(f|S_i)(t).$$

Then $uf \in C(T)$. Let $g \in C(T)$. Then $\mathring{\varphi}g \in C(S)$ and

$$\begin{aligned}(u\mathring{\varphi}g)(t) &= \sum \lambda_i(t)\, u_i(\mathring{\varphi}g\,|\,S_i)(t) \\ &= \sum \lambda_i(t)\, u_i(\mathring{\varphi}_i(g|T_i))(t) \\ &= \sum \lambda_i(t)(g|T_i)(t) = g(t), \qquad t \in T\end{aligned}$$

and clearly

$$\|uf\| \leq \|f\|, \quad u\,1_S = 1_T.$$

Thus u is a regular averaging operator for φ .

Theorem 4.13. (Milutin.) Let T be a compact metric space. There exists a continuous map from the cantor set C onto T which admits a regular averaging operator.

The proof we shall give is due to Seymoor Ditor. The proof proceeds by steps. Let X be a compact space and

$$X = \bigcup_{i=1}^{N} X_i,$$

where X_i are compact. We shall say that $\{X_i\}$ form a strong cover if

$$X = \bigcup_{i=1}^{N} \mathring{X}_i ,$$

$\mathring{X}_i$ = interior of X_i.

Let T be compact metric. We construct inductively a sequence $\{S_n\}$ of compact metric spaces as follows: Let $S_o = T$. Strongly cover S_o by compact sets $T_1, \ldots, T_N$ such that diameter $(T_i) \leq 1$ and let S_1 = disjoint union of $T_1, \ldots, T_N$. We shall say that $T_1, \ldots, T_N$ are constituents of S_1. Now strongly cover each constituent of S_1 by compact sets of diameter $\leq \frac{1}{2}$ and let S_2 = disjoint union of all the compact sets thus obtained. Continue inductively. At the n'th stage all the constituents of S_n have diameter $\leq \frac{1}{n}$. The natural injections define continuous onto maps:

$$\varphi_o^1 : S_1 \to S_o = T, \quad \varphi_1^2 : S_2 \to S_1, \ldots, \varphi_n^{n+1} : S_{n+1} \to S_n.$$

Define $\varphi_m^n : S_n \to S_m$, $n > m$ by composition

$$\varphi_m^n = \varphi_m^{m+1} \circ \cdots \circ \varphi_{n-1}^n, \qquad m < n$$

$$\varphi_n^n = \text{id} .$$

Define $\varphi_n = \varphi_o^n$, $\varphi_n : S_n \to T$. By the Localization Principle there exists a regular averaging operator u_n^{n+1} for φ_n^{n+1}, i.e.

$$u_n^{n+1}: \; C(S_{n+1}) \longrightarrow C(S_n)$$

$$u_n^{n+1}(\varphi_n^{n+1})^\rho = id_{C(S_n)} .$$

By composition we get a regular averaging operator u_m^n for φ_m^n :

$$u_m^n = u_m^{m+1} \; \cdots \; u_{n-1}^n .$$

In particular

$$v_n = u_o^n : \; C(S_n) \longrightarrow C(T)$$

is a regular averaging operator for $\varphi_n : \; S_n \longrightarrow T$.
We have

$$v_n = v_m \, u_m^n, \qquad m < n.$$

Let S_∞ be the "inverse limit" of the system $\{S_n, \varphi_m^n\}$ i.e.

$$S_\infty = \Big\{ s: \; s = \{s_n\} \in \prod_{i=0}^{\infty} S_i \; \text{ such that for all}$$

$$m \leq n, \; s_m = \varphi_m^n(s_n) \Big\} .$$

Let us show that S_∞ is a totally disconnected compact metric space. The map $s = \{s_n\} \longrightarrow s_n$ is continuous $\Longrightarrow$ $s \longrightarrow \varphi_m^n(s_n)$ is continuous $\Longrightarrow$ $A_n = \{ s: s = \{s_n\} ; \; s_m = \varphi_m^n(s_n)$ for all $m \leq n \}$ is closed. Clearly A_n are decreasing and $S_\infty = \bigcap A_n$. Define $\psi_n : \; S_\infty \rightarrow S_n$ by $\psi_n(s) = s_n$ if $s = \{s_n\}$ Then $\psi_m = \varphi_m^n \psi_n$, $m \leq n$. Let $\psi = \psi_o : S_\infty \rightarrow T$.

Now ψ_n is continuous and onto S_n for all $n \geq 0$. To see that S_∞ is totally disconnected let $x \neq y \in S_\infty$. Then there exists n with $x_n \neq y_n$ and x_n and y_n lie in distinct constituents X and Y of S_n. [Note that the diameters of the constituents $\longrightarrow 0$.] Then $\psi_n^{-1}(X)$, $\psi_n^{-1}(Y)$ are disjoint clopen sets containing x and y. We are now going to define a regular linear inverse $v: C(S_\infty) \longrightarrow C(T)$ for $\dot{,}$

$\psi^\circ: C(T) \to C(S_\infty)$. For each n, $\psi_n: S_\infty \overset{\text{onto}}{\Longrightarrow} S_n \Longrightarrow$ $\psi_n^\circ: C(S_n) \to C(S_\infty)$ is an isometric imbedding. Let

$$M = \bigcup_{n=0}^{\infty} \psi_n^\circ \quad [C(S_n)].$$

Remark that M is a self-adjoint subalgebra of $C(S_\infty)$ separating points of S_∞. [Since $\psi_m = \varphi_m^n \psi_n$, $m<n \Longrightarrow \psi_m^\circ = \psi_n^\circ(\varphi_m^n)^\circ$ $\Longrightarrow \psi_m^\circ[C(S_m)] \subset \psi_n^\circ[C(S_n)]$ and $x \neq y \Longrightarrow$ there exists n with $\psi_n(x) = x_n \neq y_n = \psi_n(y)$]. Hence M is dense in $C(S_\infty)$. It will suffice to define v on M. Let $f \in M$. Then for some m, $f \in \psi_m^\circ[C(S_m)]$. Define

$$vf = v_m(\psi_m^\circ)^{-1} f.$$

[Recall $\psi_m^\circ: C(S_m) \to \psi_m^\circ[C(S_m)]$ is an isometric isomorphism.] We check v is well defined on M. For $n > m$

$$\begin{aligned} v_n[\psi_n^\circ]^{-1} f &= v_m u_m^n (\varphi_m^n)^\circ [(\varphi_m^n)^\circ]^{-1} [\psi_n^\circ]^{-1} f \\ &= v_m[(\varphi_m^n)^\circ]^{-1} [\psi_n^\circ]^{-1} f \\ &= v_m[\psi_m^\circ]^{-1} f, \end{aligned}$$

since

$$v_n = v_m u_m^n$$

and

$$(\psi_m^\circ)^{-1} = [(\varphi_m^n)^\circ]^{-1} [\psi_n^\circ]^{-1}, \quad u_m^n(\varphi_m^n)^\circ = id_{C(S_m)}.$$

Finally, to show that v is an averaging operator for ψ we must show $v\psi^\circ = id_{C(T)}$. However,

$$\psi^\circ[C(T)] = \psi_o^\circ[C(S_o)] \subset M,$$

so

$$v\psi^\circ g = v_o \psi_o^\circ g = g$$

by our definition of v. Note that since $\|v_n\| = 1$ for all n, $\|v\| = 1$. So v is a regular averaging operator for ψ. Of course v is extended uniquely to all of $C(S_\infty)$.

Now let $Z = C \times S_\infty$. Then Z is compact metric, totally disconnected and perfect. [We take Z instead of S_∞ because S_∞ may have isolated points.] Let $\rho : Z \longrightarrow S_\infty$ be the canonical projection and define $\varphi : Z \longrightarrow T$ by $\varphi = \psi \circ \rho$. From the following lemma, it follows that φ has a regular averaging operator.

Lemma 4.14. Let X and Y be compact and $\pi : X \times Y \to X$ be the canonical projection. Let μ be a probability measure on Y and for each $x \in X$ let $\mu_x = \mu$ on the fiber $Y_x = \{x\} \times Y$. Then

$$(pf)(x) = \int_{Y_x} f(x,y) \mu_x(dy)$$

is regular and averaging.

Proof. Easy.

Now use Theorem 2-97, p. 99, Topology, Hocking and Young, which states that every compact totally disconnected perfect metric space is homeomorphic to the Cantor set.

SECTION 5

Milutin's isomorphism Theorem.

Definition 5.1. Let X and Y be Banach spaces. We write $X \sim Y$ if X and Y are linearly homeomorphic (i.e. there exists a 1-1 continuous onto linear map $T: X \longrightarrow Y$). If $X \sim Y$, we say X and Y are isomorphic Banach spaces. If $X \sim Y$ and the map T can be chosen to be isometric we write $X \approx Y$ and call X and Y equivalent Banach spaces. We remark that both $\sim$ and $\approx$ are equivalence relations.

If X and Y are Banach spaces, $X \times Y$ denotes the Banach space of all pairs (x,y) and norm

$$\|(x,y)\| = \max(\|x\|,\|y\|).$$

(This is an arbitrary choice of norms. We could just as well as use $(\|x\|^p + \|y\|^p)^{1/p}$, $1 \leq p < \infty$ or many others. The different choices yield isomorphic spaces.)

Definition 5.2. If X and Y are Banach spaces, we write X/Y and call X a factor of Y if there exists a Banach space Z such that $Y \sim X \times Z$.

Remark. Clearly X/Y iff X is isomorphic to a complemented subspace of Y.

Definition 5.3. Let S be compact and X a Banach space. Then $C(S,X)$ denotes the linear normed space of all continuous functions F of S into X with norm

$$\|F\| = \sup_{s \in S} \|F(s)\| .$$

It is not difficult to show that $C(S,X)$ is a Banach space.

Verification of the following is elementary:

(1) $X_1 \sim X_2 \implies C(S,X_1) \sim C(S,X_2)$.

(2) $C(S,X_1 \times X_2) \sim C(S,X_1) \times C(S,X_2)$.

(3) If $X \sim Z \times W$, then

$$C(S,X) \sim C(S,Z) \times C(S,W).$$

Our aim in this section is to prove:

Theorem 5.4. (Milutin.) If S and T are any uncountable compact metric spaces, then

$$C(S) \sim C(T).$$

We do this by proving that for any uncountable compact metric space S: $C(S) \sim C(C)$, $C =$ Cantor set. Hereafter we shall denote by K the Cantor set C.

Lemma 5.5. If S is any compact metric space, then $C(S) | C(K)$.

Proof. There exists $\varphi : K \to S$(onto), with a regular linear averaging operator. This is Theorem 4.13. Hence $C(S)$ is isometric (under φ°) to a self-adjoint subalgebra $\varphi^\circ(C(S))$ of $C(K)$ which is the range of a projection of norm 1.

We show next that if S is uncountable, $C(K) | C(S)$.

For this we need:

Lemma 5.6. Every uncountable compact metric space contains a homeomorph of K.

Proof. Let

$$P = \{ s : s \in S \text{ such that every neighbourhood of } s \text{ contains uncountably many points} \}.$$

Then P is non-empty and closed. Also no point in P is isolated in P, since such a point p has a neighbourhood V in

S with $V-\{p\} \subset S-P$. However, each point of $S-P$ has a neighbourhood which contains at most countably points of S. $\Longrightarrow$ $V-P$ is countable (S has a countable base). Thus P is perfect. We shall find a homeomorph of K in P so we might as well suppose S is perfect.

If S is perfect and U is open, then $\bar{U}$ is perfect. Thus we can find perfect subsets of S of small diameter. Choose perfect sets $S_0, S_1 \subset S$ such that $S_0 \cap S_1 = \emptyset$ and diameter $(S_i) \leq 1$, $i = 0,1$. Put $A_0 = S_0 \cup S_1$. Find disjoint perfect subsets of S_0 and S_1 of diameter $\leq \frac{1}{2}$ and let A_1 be the union of all these four disjoint perfect sets. Next find two disjoint perfect subsets of each of these four perfect sets of diameter $\leq \frac{1}{2^2}$, and let A_2 be the union of these eight disjoint perfect sets. Continue this way. In general A_n will consist of 2^{n+1} disjoint perfect sets each of which has diameter $\leq \frac{1}{2^n}$. Also $A_{n+1} \subset A_n$. Let $A = \bigcap A_n$. Clearly A is non-empty. Given $x,y \in A$, $x \neq y$, there exists an n such that x and y are contained in disjoint sets whose union is A_n; since each of these sets is clopen in A_n we see that x and y are contained in disjoint clopen subsets of $A \Longrightarrow A$ is totally disconnected. The same kind of argument shows that A has no isolated points. Hence A is homeomorphic to K. [Theorem 2-97, p.99, Topology by Hocking and Young.]

Corollary 5.7. If S is uncountable $C(K) | C(S)$.

Proof. Let A be a homeomorph of K in S. By the Borsuk-Dugundji Theorem $C(K)$ is isometric to a closed subspace of $C(S)$ which is the range of a projection of norm one.

Now let $N = \{1,2,\dots\}$ and $N^* = N \cup \{\infty\}$ be the Alexandroff compactification of N. If X is a Banach space $C(N^*,X) =$ all sequences $(x_0,x_1,\dots)$ such that $x_0 = \lim x_n$, $\|(x_0,x_1,\dots)\| = \sup\|x_n\|$. Therefore

$$C(N^*,X) \approx X \times C(N^*,X).$$

Now take $X = C(K)$. Now $N^* \times K$ is compact metric, totally disconnected and perfect. So $N^* \times K$ is homeomorphic to $K \Rightarrow C(K) \approx C(N^* \times K)$. But as is easily seen, $C(N^* \times K) \approx C(N^*, C(K))$. Thus

$$C(K) \approx C(N^*, C(K)) \approx C(K) \times C(N^*, C(K))$$

by what we said above.

Proof of Milutin's theorem. We have $C(S) | C(K)$ and $C(K) | C(S)$. Hence $C(S) \sim W \times C(K)$, $C(K) \sim C(S) \times Z$.

We have

$$C(S) \sim W \times C(K) \sim W \times C(N^*, C(K))$$

$$\sim W \times C(K) \times C(N^*, C(K)) \sim C(S) \times C(N^*, C(K))$$

$$(\text{since } C(S) \sim W \times C(K))$$

$$\sim C(S) \times C(N^*, C(S) \times Z) \sim C(S) \times C(N^*, C(S)) \times C(N^*, Z)$$

$$\sim C(N^*, C(S)) \times C(N^*, Z)$$

$$(\text{because } X \times C(N^*, X) \approx C(N^*, X))$$

$$\sim C(N^*, C(S) \times Z) \sim C(N^*, C(K)) \sim C(K).$$

Example. Let l_∞ denote the Banach space of all bounded sequences $\xi = \{\xi_n\}$ of scalars with $\|\xi\| = \sup|\xi_n|$. Let C be the closed subspace of convergent sequences and C_o be the closed subspace of sequences which converge to zero. Note that we have the identifications

$$l_\infty = C(\beta(N))$$

$$C = C(N^*)$$

C_o = those functions in $C(N^*)$ vanishing at ∞.

We shall prove that there exists no bounded projections of l_∞ onto C or C_o. Thus we will have contructed examples of uncomplemented ideals and subalgebras of $C(\beta(N))$.

Lemma 5.8. There exists a family $\{U_\alpha\}$ of subsets of N of cardinality of the continuum such that

(1) each U_α is an infinite set

(2) $U_\alpha \cap U_\beta$ is finite for $\alpha \neq \beta$.

Proof. Let R be the set of rationals in $[0,1]$. Then R is countable. Let S = set of irrationals and for each $s \in S$ let U_s be a sequence of distinct rationals converging to S. Now map R onto N.

Theorem 5.9. There exists no bounded projection of l_∞ onto C_o.

Proof. Let $\{U_\alpha\}$ be the family of sets of Lemma 5.8 and let $\xi_\alpha = (\xi_\alpha(1), \xi_\alpha(2), \ldots)$ denote the indicator function of U_α, i.e. $\xi_\alpha(n) = 1$, if $n \in U_\alpha$, $\xi_\alpha(n) = 0$ if $n \notin U_\alpha$. Consider a finite subset $\xi_{\alpha_1}, \ldots, \xi_{\alpha_k}$ of the set $\{\xi_\alpha\}$ and let $b_1, \ldots, b_k$ be scalars. Let us show that there exists an element $\xi_o \in C_o$ such that

$$\left\| \sum_{i=1}^{k} b_i \xi_{\alpha_i} - \xi_o \right\| \leq \max\{|b_1|, \ldots, |b_k|\}.$$

Indeed define the vector

$$\xi_o = (\xi_o(1), \xi_o(2), \ldots)$$

as follows:

$$\xi_o(n) = \begin{cases} 0 & \text{if } n \text{ belongs to at most one } U_{\alpha_i}, \\ & \text{otherwise} \\ \sum b_i & \text{the summation being over those indices} \\ & \alpha_i \text{ such that } n \in U_{\alpha_i}. \end{cases}$$

Since $U_{\alpha_i} \cap U_{\alpha_j}$ is finite for $\alpha_i \neq \alpha_j$, it is clear that $\xi_o(n) = 0$ except for a finite number of n. Thus $\xi_o \in C_o$. Also $\sum_{i=1}^{k} b_i \xi_{\alpha_i}(n) - \xi_o(n)$ is zero if n belongs to at least two of U_{α_i}, and equals b_i if n belongs only to U_{α_i}. We have proved the assertion. Now if $l \in C_o^{\perp}$ and $\varepsilon_\alpha = 1$, if $l(\xi_\alpha) \geq 0$, $\varepsilon_\alpha = -1$ if $l(\xi_\alpha) \leq 0$, we have

$$\sum_{i=1}^{k} |l(\xi_{\alpha_i})| = \sum_{i=1}^{k} \varepsilon_{\alpha_i} l(\xi_{\alpha_i}) = l(\sum_{i=1}^{n} \varepsilon_{\alpha_i} \xi_{\alpha_i}) = l(\sum_{i=1}^{n} \varepsilon_{\alpha_i} \xi_{\alpha_i} \xi_0)$$

$$\leq \|l\| \max|\varepsilon_{\alpha_i}| = \|l\|,$$

where ξ_0 is the element of C_0 determined above.

Thus $l \in C_0^{\perp} \Rightarrow$ the set of α s.t. $l(\xi_\alpha) \neq 0$ is at most countable. Now let $l_n \in l_\infty^*$ be defined by $l_n(\xi) = \xi_n$, if $\xi = (\xi_1, \xi_2, \ldots)$. Then l_n is total, i.e. $l_n(\xi) = 0$ for all $n \Rightarrow \xi = 0$. Assume that P is a projection of l_∞ onto C_0. Then $l_n - l_n P \in C_0$. Hence the set of α s.t.

$$(l_n - l_n P)(\xi_\alpha) \neq 0 \text{ is at most countable.}$$

Thus there exists α such that $(l_n - l_n P)\xi_\alpha = 0$ for all n.

But

$$l_n[(I-P)\xi_\alpha] = 0 \text{ for all } n \Rightarrow \xi_\alpha - P\xi_\alpha = 0,$$

i.e. $\xi_\alpha \in C_0$, which is a contradiction because U_α is infinite.

Remark. What we have proved above implies the following: Let S be a σ-compact, locally compact non-compact space; $B = B(S)$ = set of bounded continuous functions on S; B_0 = set of continuous functions vanishing at ∞. Then there exists no bounded projection of B onto B_0.

SECTION 6

Lower bounds for averaging operators.

We have seen in Theorem 4.11 that if $\varphi : S \rightarrow T$ is a continuous onto map and $u: C(S) \rightarrow C(T)$ is an averaging operator (A.O.) for φ, then $u^*: C(T)^* \rightarrow C(S)^*$ induces a weak star continuous map of T into $M(S)$ via $\mu_t^* = u^*(\delta_t)$ where δ_t is the unit point mass at $t \in T$. We have

$$(uf)(t) = \int_S f(s)\mu_t(ds), \qquad f \in C(S).$$

Moreover

$$\|\mu_t\| \geq 1 \quad \text{and} \quad \|u\| = \sup_{t\in T} \|\mu_t\|.$$

Our purpose in this section is to obtain lower bounds for $\|u\|$ in terms of the topological structure of the closed set decomposition

$$\mathcal{R} = \{ \varphi^{-1}(t) \mid t \in T \} \quad \text{of } S.$$

Lemma 6.1. If u is an A.O. for φ, then

$$\mu_t[\varphi^{-1}(A)] = \delta_t(A) \quad \text{for all Borel sets } A \subset T.$$

Proof. We have for $g \in C(T)$

$$g(t) = u(\varphi^\circ g)(t) = \int_S (\varphi^\circ g)(s)\mu_t(ds)$$
$$= \int_S g(\varphi(S))\mu_t(ds).$$

The same equality holds for all Borel functions on T. Hence

$$\delta_t(A) = \mu_t(\varphi^{-1}(A)).$$

Corollary 6.2. For each $t \in T$, $\mu_t(\varphi^{-1}(t)) = 1$.

Corollary 6.3. If u is an A.O. for φ, then u is regular iff for each $t \in T$, μ_t is a probability measure concentrated on $\varphi^{-1}(t)$.

The proof is immediate because u is regular iff

$$\|u\| = 1 \quad \text{and we have} \quad \|u\| = \sup_{t \in T} \|\mu_t\|.$$

Definition 6.4. For $\varphi: S \to T$ onto we define the extended real number $p(\varphi)$ by

$$p(\varphi) = \inf\{\|u\| : u \text{ is an A.O. for } \varphi\} .$$

Thus φ admits no A.O. iff $p(\varphi) = \infty$. We shall estimate $p(\varphi)$ via the nature of the decomposition

$$\mathcal{R} = \{\varphi^{-1}(t) : t \in T\}.$$

Definition 6.5. Let u be an A.O. for $\varphi : S \to T$ and $\mu_t = u^*(\delta_t)$. The residue of μ_t is

$$R(\mu_t) = \|\mu_t\| - |\mu_t|(\varphi^{-1}(t)) = |\mu_t|(S - \varphi^{-1}(t)).$$

Definition 6.6. If $\{A_\alpha\}$ is a net of sets in S

$$\begin{aligned}\limsup A_\alpha = \{s : & \text{ for each } \alpha_0 \text{ and a neighbourhood } U \text{ of } s \text{ there exists } \alpha \geq \alpha_0 \text{ such that } U \cap A_\alpha \neq \emptyset\} \\ = & \text{ all cluster points of nets } \{s_\alpha\} \text{ where } s_\alpha \in A_\alpha.\end{aligned}$$

Thus $\limsup \varphi^{-1}(t_\alpha)$ is a non-empty compact subset of $\varphi^{-1}(t_0)$, if $\{t_\alpha\}$ is a net in T converging to $t_0 \in T$.

The next result is crucial.

Lemma 6.7. (Ditor.) If $\{t_\alpha\}$ is a net in T with $t_\alpha \to t_o$, then

$$\liminf R(\mu_{t_\alpha}) \geq 1 + \|\mu_{t_o}\| - 2|\mu_{t_o}|(\limsup \varphi^{-1}(t_\alpha)).$$

We remark that if $t_\alpha \to t_o$ and U is any open neighbourhood of $S_o = \limsup \varphi^{-1}(t_\alpha)$, then eventually $\varphi^{-1}(t_\alpha) \subset U$. Indeed if for some U and each α_o there exists $\beta \geq \alpha_o$ with $\varphi^{-1}(t_\beta) \cap U^c \neq \emptyset$, then there exists a cluster point s_o of a net $\{s_\alpha\}$ with $s_\alpha \in \varphi^{-1}(t_\alpha)$ for all α and $s_o \in S_o^c$.

Proof of Lemma 6.7. Let $\varepsilon > 0$ and choose a compact $K \subset S - S_o$ with $|\mu_{t_o}|(K) > |\mu_{t_o}|(S - S_o) - \varepsilon$. Let V be a neighbourhood of S_o with $\bar{V} \cap K = \emptyset$ and let W be a closed neighbourhood of S_o with $W \subset V$. Let $h_1 \in C(S)$ with $\|h_1\| \leq 1$, $h_1(\bar{V}) = 0$ and

$$|\mu_{t_o}(h_1)| > |\mu_{t_o}|(K) - \varepsilon .$$

Choose $h_2 \in C(S)$ with $\|h_2\| = 1$, $h_2(W) = 1$, $h_2(V^c) = 0$.

We may assume that

$$|\mu_{t_o}|(V) < |\mu_{t_o}|(S_o) + \varepsilon .$$

Then also

$$|\mu_{t_o}|(h_2) \leq |\mu_{t_o}|(V) < |\mu_{t_o}|(S_o) + \varepsilon .$$

By weak star continuity eventually

$$|\mu_{t_\alpha}(h_1)| > |\mu_{t_o}|(K) - \varepsilon ,$$

$$|\mu_{t_\alpha}(h_2)| < |\mu_{t_o}|(S_o) + \varepsilon .$$

Since $h_2 = 1$ on W and $\varphi^{-1}(t_\alpha) \subset W$, $\mu_{t_\alpha}(\varphi^{-1}(t_\alpha)) = 1$, we see

$$\mu_{t_\alpha}(h_2) = \int_V h_2 \, d\mu_{t_\alpha} = 1 + \int_{V - \varphi^{-1}(t_\alpha)} h_2 \, d\mu_{t_\alpha} .$$

Thus

$$\left|\int_{V-\varphi^{-1}(t_\alpha)} h_2 d\mu_{t_\alpha}\right| > 1 - |\mu_{t_o}|(S_o) - \varepsilon ,$$

so

$$|\mu_{t_\alpha}|(V \sim \varphi^{-1}(t_\alpha)) \geq 1 - |\mu_{t_o}|(S_o) - \varepsilon .$$

Also

$$|\mu_{t_\alpha}|(V^c) \geq |\mu_{t_\alpha}(h_1)| > |\mu_{t_o}|(K) - \varepsilon .$$

Thus

$$\begin{aligned} |\mu_{t_\alpha}|(S - \varphi^{-1}(t_\alpha)) &\geq |\mu_{t_\alpha}|(V^c) + |\mu_{t_\alpha}|(V - \varphi^{-1}(t_\alpha)) \\ &\geq |\mu_{t_o}|(S - S_o) - 2\varepsilon + 1 - |\mu_{t_o}|(S_o) - \varepsilon \\ &= 1 + \|\mu_{t_o}\| - 2|\mu_{t_o}|(S_o) - 3\varepsilon . \end{aligned}$$

Definition 6.8. We call a closed subset $B \subset \varphi^{-1}(t_o)$ a cluster set for φ at t_o, if $B = \limsup \varphi^{-1}(t_\alpha)$ for some net $t_\alpha \rightarrow t_o$.

Theorem 6.9. (Arens.) Suppose $\varphi^{-1}(t_o)$ contains n disjoint cluster sets. Then

$$p(\varphi) \geq 3 - \frac{2}{n} .$$

Proof. Suppose u is an A.O. for φ. There exists a net $t_\alpha \rightarrow t_o$ with

$$\begin{aligned} |\mu_{t_o}|(\limsup \varphi^{-1}(t_\alpha)) &\leq \frac{1}{n}|\mu_{t_o}|(\varphi^{-1}(t_o)) \\ &\leq \frac{1}{n}\|\mu_{t_o}\| . \end{aligned}$$

From Ditor's Lemma we have for some α

$$\begin{aligned} \|\mu_{t_\alpha}\| &= |\mu_{t_\alpha}|(\varphi^{-1}(t_\alpha)) + R(\mu_{t_\alpha}) \\ &\geq 1 + 1 + \|\mu_{t_o}\| - 2|\mu_{t_o}|(\limsup \varphi^{-1}(t_\alpha)) \\ &\geq 2 + \|\mu_{t_o}\| - \frac{2}{n}\|\mu_{t_o}\| - \varepsilon \geq 3 - \frac{2}{n} - \varepsilon , \end{aligned}$$

since

$$|\mu_{t_\alpha}|(\varphi^{-1}(t_\alpha)) \geq 1 .$$

The above holds for all $\varepsilon > 0$.

Lemma 6.10. Suppose $\{t_\alpha\}$ and $\{t_\beta\}$ are nets in T with $t_\alpha \to t_o$ and $t_\beta \to t_o$, $[\limsup \varphi^{-1}(t_\alpha)] \cap [\limsup \varphi^{-1}(t_\beta)] = \emptyset$. Then for one of the nets, say $\{t_\alpha\}$, we have

$$\liminf R(\mu_{t_\alpha}) \geq 1 + R(\mu_{t_o}).$$

Proof. Since $\limsup \varphi^{-1}(t_\alpha)$ and $\limsup \varphi^{-1}(t_\beta)$ are disjoint, we may suppose

$$|\mu_{t_o}|(\limsup \varphi^{-1}(t_\alpha)) \leq \tfrac{1}{2}|\mu_{t_o}|(\varphi^{-1}(t_o)).$$

So

$$\begin{aligned}\liminf R(\mu_{t_\alpha}) &\geq 1 + \|\mu_{t_o}\| - 2|\mu_{t_o}|(\limsup \varphi^{-1}(t_\alpha)) \\ &\geq 1 + \|\mu_{t_o}\| - |\mu_{t_o}|(\varphi^{-1}(t_o)) \\ &= 1 + R(\mu_{t_o}).\end{aligned}$$

We define the sets $M^{(n)} \subset T$ as follows:

$$M^{(1)} = \{t : t \in T \text{ such that there exists nets } \{t_\alpha\}, \{t_\beta\} \text{ with } t_\alpha \to t,\ t_\beta \to t \text{ and } [\limsup \varphi^{-1}(t_\alpha)] \cap [\limsup \varphi^{-1}(t_\beta)] = \emptyset\}$$

and in general

$$M^{(n+1)} = \{t : \text{such that there exists nets } \{t_\alpha\}, \{t_\beta\} \text{ with } t_\alpha \to t,\ t_\beta \to t,\ t_\alpha, t_\beta \in M^{(n)} \text{ and } [\limsup \varphi^{-1}(t_\alpha)] \cap [\limsup \varphi^{-1}(t_\beta)] = \emptyset\}.$$

Theorem 6.11. If $M^{(n)} \neq \emptyset$, then $p(\varphi) \geq n$.

Proof. Let $\varepsilon > 0$. Then there exists $t_o \in M^{(n)}$ and a net $\{t_\alpha\} \subset M^{(n-1)}$, $t_\alpha \to t_o$, such that by Lemma 6.10

$$\liminf R(\mu_{t_\alpha}) \geq 1 + R(\mu_{t_o}).$$

Thus there exists $t_1 = t_{\alpha_1} \in M^{(n-1)}$ with

$$R(\mu_{t_1}) > 1 + R(\mu_{t_o}) - \frac{\varepsilon}{n} .$$

Now $t_1 = \lim t_\beta$, $\{t_\beta\} \subset M^{(n-2)}$ with

$$\lim\inf R(\mu_{t_\beta}) \geq 1 + R(\mu_{t_1}).$$

Thus there exists $t_2 \in M^{(n-2)}$ with

$$R(\mu_{t_2}) \geq 1 + R(\mu_{t_1}) - \frac{\varepsilon}{n}$$

$$\geq 2 + R(\mu_{t_o}) - \frac{2\varepsilon}{n} .$$

Continuing this way we see that there exists $t \in T$ with

$$R(\mu_t) \geq (n-1) + R(\mu_{t_o}) - \varepsilon \geq (n-1) - \varepsilon$$

$$\Longrightarrow \|\mu_t\| \geq n - \varepsilon \quad \Longrightarrow \|u\| \geq n - \varepsilon \quad \text{for all } \varepsilon > 0.$$

Corollary 6.12. If $M^{(n)} \neq \emptyset$ for all n then $p(\varphi) = +\infty$. Hence $\varphi^\circ(C(T))$ is uncomplemented in $C(S)$.

Example. Let C = Cantor set, and $T = [0,1]$. C is homeomorphic to $S = \{0,1\}^{\aleph_0}$ = set of sequences

$$x = (x_1, x_2, \dots) \text{ where } x_i = 0 \text{ or } 1.$$

Let φ be the map

$$\varphi(x) = \sum \frac{x_i}{2^i} .$$

Then φ is continuous onto $T = [0,1]$. $\varphi^{-1}\{t\}$ consists of one point if t is not a diadic rational and consists of exactly two points if t is a diadic rational. Each diadic rational is a point in $M^{(1)}$. Since diadics are dense in themselves each diadic is in $M^{(n)}$ for every n. $\Longrightarrow M^{(n)} \neq \emptyset$ for all n $\Longrightarrow \varphi^\circ(C(T))$ is uncomplemented in $C(S)$. Thus there exists no A.O. for φ. However, Milutin's theorem guarantees the existence of a map ψ: $S \to [0,1]$ which admits an A.O.

SECTION 7

Stonian spaces.

In this section we introduce a remarkable class of compact spaces which play a central role in several theories.

Definition 7.1. A compact Hausdorff space S is called stonian (or extremely disconnected) if disjoint open sets in S have disjoint closures.

Remark. S is stonian iff U open implies $\bar{U}$ is also open.

Proof. Let S be stonian and U be open. Then the disjoint open sets U and $\bar{U}^c$ have disjoint closures. But the equalities

$$\bar{U} \cap \bar{U}^c = \emptyset \quad \text{and} \quad \bar{U} \cap \overline{[\bar{U}^c]} = \emptyset$$

implies $\bar{U}^c = \overline{[\bar{U}^c]}$. Hence $\bar{U}$ is open. Conversely assume that open sets have open closures. Let U_i, $i = 1,2$ be open and disjoint. Since U_1 is open $U_1 \cap \bar{U}_2 = \emptyset$. But by assumption $\bar{U}_2$ is open, hence $\bar{U}_1 \cap \bar{U}_2 = \emptyset$.

We note first the simplest examples of stonian spaces. If D is any completely regular space, then $\beta(D)$ denotes the Stone-Čech compactification of D (cf. section 2, Theorem 2.14).

Theorem 7.2. Let D be any discrete set. Then $\beta(D)$ is stonian.

Proof. Let U_i, $i = 1,2$ be disjoint open sets in $\beta(D)$ and put $A_i = D \cap U_i$. (Note that $U_i \neq \emptyset$ and open implies $A_i \neq \emptyset$.) Since D is discrete, the functions 1_{A_i} are continuous, so let p_i denote the corresponding unique continuous extensions to $\beta(D)$. By continuity, p_i takes only the values 0 and 1 and $p_1 p_2 = 0$. As A_i is dense in U_i, $p_i(\bar{U}_i) = 1$ for $i = 1,2$. Hence $\bar{U}_1 \cap \bar{U}_2 = \emptyset$ and $\beta(D)$ is stonian.

We show next that the stonian spaces coincide with those spaces having another remarkable property, and characterize stonian spaces as retracts of spaces $\beta(D)$, where D is discrete.

Definition 7.3. A compact Hausdorff space S is projective, if for any compact Hausdorff spaces T and W and continuous maps $\theta: T \to W$ onto $\emptyset: S \to W$, there exists a continuous map $\psi: S \to T$ such that $\emptyset = \theta \circ \psi$.

$$\begin{array}{ccc} S & \overset{\psi}{\dashrightarrow} & T \\ & \searrow^{\emptyset} & \downarrow \theta \text{ onto} \\ & & W \end{array}$$

Theorem 7.4. For a compact Hausdorff space S, the following are equivalent

(1) S is stonian.

(2) S is a retract of a space $\beta(D)$, where D is discrete.

(3) S is projective.

The theorem will be proved by establishing (1) $\Longrightarrow$(2), (2) $\Longrightarrow$(3) and (3) $\Longrightarrow$(1). To prove (1) $\Longrightarrow$(2) we need some definitions and lemmas.

Definition 7.5. Let A and B be compact Hausdorff and $\pi: A \to B$ be continuous and onto. Then π is called irreducible if $\pi(A_o) \neq B$ for any proper closed subset A_o of A.

We note the following characterizations of irreducible maps.

Proposition 7.6. Let A and B be compact Hausdorff. For a continuous map π of A onto B the following are equivalent

(1) π is irreducible

(2) For each open set $U \subseteq A$, the set

$$V = \{ b \in B \mid \pi^{-1}(b) \subseteq U \}$$

is a non empty open set in B.

(3) For each open set $U \subseteq A$, the set

$$W = \bigcup \{ \pi^{-1}(b) \mid \pi^{-1}(b) \subseteq U \}$$

is an open set dense in U.

Proof. Clearly π is irreducible iff every open set U in A contains a hole fiber. Thus (1) $\Longleftrightarrow$ (2). (2) $\Longleftrightarrow$ (3) is elementary. That V is open follows from the equality $V = [\pi(U^c)]^c$ and W is open by Lemma 4.4 in Section 4.

Lemma 7.7. Let A and B be compact Hausdorff and π a continuous map of A onto B. Then there exists a compact set $A_o \subseteq A$ such that $\pi(A_o) = B$ and $\pi|_{A_o}$ is irreducible.

Proof. Let $\mathcal{A} = \{E \subseteq A \mid E$ is closed and $\pi(E) = B\}$, and order $\mathcal{A}$ by inclusion. Clearly $A \in \mathcal{A}$. If $\{E_\alpha\}$ is any chain in $\mathcal{A}$, then $\pi(E_o) = B$, where $E_o = \bigcap_\alpha E_\alpha$. For if $y \in B$ and $x_\alpha \in E_\alpha$ with $\pi(x_\alpha) = y$, then by compactness the net $\{x_\alpha\}$ has a cluster point $x_o \in A$. Also $x_o \in E_o$, and by continuity $\pi(x_o) = y$. Hence $E_o \in \mathcal{A}$ and is a lower bound for the chain $\{E_\alpha\}$. By Zorn's Lemma $\mathcal{A}$ has a minimal element, which is the set we require.

Lemma 7.8. Let A and B be compact Hausdorff and $\pi : A \to B$ be irreducible. If B is stonian, then π is a homeomorphism.

Proof. It suffices to show that π is one to one. Suppose $x_1 \neq x_2$, $x_1, x_2 \in A$ and $y = \pi(x_1) = \pi(x_2)$. Let U_i, $i = 1,2$, be disjoint open neighbourhoods of x_i, $i = 1,2$, and define

$$V_i = \{b \in B \mid \pi^{-1}(b) \subseteq U_i\}.$$

The sets V_i are open and disjoint, and since B is stonian $\bar{V}_1 \cap \bar{V}_2 = \emptyset$. But $y \in \bar{V}_i$, $i = 1,2$ by Proposition 7.6 (3). This contradiction proves the lemma.

Proof of (1) $\Rightarrow$ (2). Let S be stonian and S_d be S with the discrete topology. Since $C(S) \subseteq C(\beta(S_d))$ there is a continuous map, π, of $\beta(S_d)$ onto S (Section 4, Theorem 4.2 and 4.1) By Lemma 7.7 we can find a closed subset P of $\beta(S_d)$ such that $\pi|_P$ is irreducible, and hence by Lemma 7.8 $\pi|_P$ is a homeomorphism. The map

$$(\pi|_P)^{-1} \circ \pi : \beta(S_d) \to P$$

is clearly a retraction of $\beta(S_d)$ onto P. Hence S is homeomorphic to a retract of $\beta(S_d)$.

To prove (2) $\Longrightarrow$ (3) we need the following strengthened form of the Stone-Cěch extension theorem (Section 2, Theorem 2.14).

Lemma 7.9. Let R be a completely regular space, T compact Hausdorff and τ a continuous map of R into T. Then τ has a unique continuous extension $\tilde{\tau}$, mapping $\beta(R)$ into T.

Proof. For each bounded continuous function f on R, let f be its unique continuous extension to $\beta(R)$. Define the homomorphism $\vartheta : C(T) \to C(\beta(R))$ by

$$\vartheta(g) = \widetilde{g \circ \tau} \qquad \forall\, g \in C(T).$$

Clearly $\vartheta(1_T) = 1_{\beta(R)}$. Now $\vartheta^* : M(\beta(R)) \to M(T)$, and from the proof of Theorem 4.2, Section 4, it follows that ϑ^* maps unit point masses on $\beta(R)$ into unit point masses on T. Hence restricting ϑ^* to the unit point masses we get a continuous map from $\beta(R)$ into T, and since

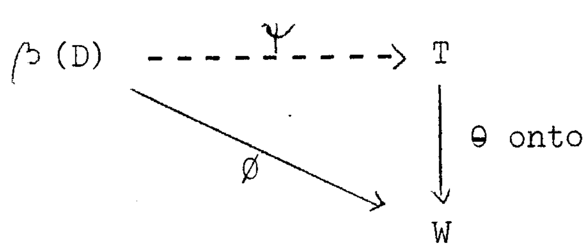
$$\begin{aligned}(\vartheta^* \delta_r)(g) &= \delta_r(\vartheta(g)) = \vartheta(g)(r) = (\widetilde{g \circ \tau})(r) \\ &= g(\tau(r)) \quad = \delta_{\tau(r)}(g)\end{aligned}$$

$$\forall\, g \in C(T) \text{ and } \forall\, r \in R,$$

ϑ^* extends τ.

Lemma 7.10. If D is discrete, then $\beta(D)$ is projective.

Proof. Let T and W be compact Hausdorff, $\emptyset : \beta(D) \to W$ continuous, and $\Theta : T \to W$ continuous and onto.

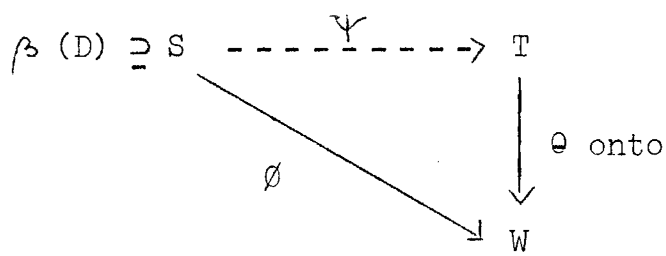


For each $d \in D$ choose a point $\psi(d) \in \theta^{-1}(\emptyset(d))$. Then we have defined a map $\psi : D \longrightarrow T$, which is continuous since D is discrete. By Lemma 7.9, ψ has a continuous extension (also denoted by ψ) to a map $\psi : \beta(D) \longrightarrow T$. Since

$$\theta(\psi(d)) = \emptyset(d) \qquad \forall\, d \in D,$$

we get $\emptyset = \theta \circ \psi$ on $\beta(D)$, because D is dense in $\beta(D)$.

Proof of (2) $\Longrightarrow$ (3). Let S be a retract of $\beta(D)$, where D is discrete. Then there exists a continuous map $\vartheta : \beta(D) \longrightarrow S$ such that $\vartheta|_S = id_S$. Now suppose we have the diagram



where T and W are compact Hausdorff, θ and $\emptyset$ continuous and θ onto. Extend $\emptyset$ to $\tilde{\emptyset}$ on $\beta(D)$ by $\tilde{\emptyset} = \emptyset \circ \vartheta$. Since $\beta(D)$ is projective (Lemma 7.10), we can find $\tilde{\psi} : \beta(D) \longrightarrow T$ such that $\tilde{\emptyset} = \theta \circ \tilde{\psi}$. Define $\psi = \tilde{\psi}|_S$. Then for $s \in S$

$$\emptyset(s) = \emptyset\,\vartheta(s) = \tilde{\emptyset}(s) = \theta(\tilde{\psi}(s)) = \theta(\psi(s))$$

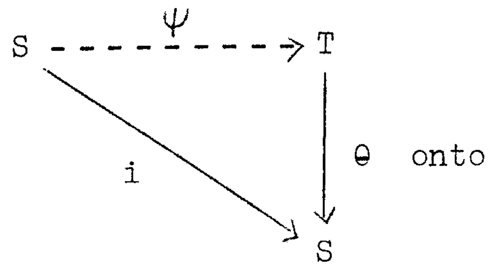
so $\emptyset = \theta \circ \psi$. Thus S is projective.

We note, that in fact we have proved that any retract of a projective space is projective.

Proof of (3) $\Longrightarrow$ (1). Let S be projective and G be open in S. We show $\bar{G}$ is open. Let $\{p,q\}$ be the two-point space, and let T be the closed set

$$T = [(S \setminus G) \times \{p\}] \cup [\bar{G} \times \{q\}]$$

in $S \times \{p,q\}$, with the product topology. Let $W = S$, π be the natural projection of $S \times \{p,q\}$ onto S and θ the restriction of π to T. Then we have the diagram



where i is the identity map. By the assumption on S, we can find a continuous map $\psi : S \to T$ so that $i = \theta \circ \psi$. Now θ is one to one from $G \times \{q\} \subseteq T$ to G, so

$$\psi(x) = (x,q) \qquad\qquad \forall x \in G.$$

Hence by continuity $\psi(x) = (x,q)$ for $x \in \bar{G}$. Similarly for $x \notin \bar{G}$, $\psi(x) = (x,p)$ and hence

$$\bar{G} = \psi^{-1}(\bar{G} \times \{q\}).$$

Since $\bar{G} \times \{q\}$ is open in T, the continuity of ψ implies that $\bar{G}$ is open in S as required. This completes the Proof of Theorem 7.2.

We show next that any compact Hausdorff space is the image under an irreducible map of an essentially unique stonian space. For this we need the following lemma:

Lemma 7.11. Let P be a compact Hausdorff space and $\emptyset : P \to P$ be continuous. If $\emptyset$ is not the identity map, then there exists a proper closed subset Q of P such that

$$P = Q \cup \emptyset^{-1}(Q).$$

Proof. Let $p \in P$ with $\emptyset(p) \neq p$, and choose open disjoint neighbourhoods U and V of p and $\emptyset(p)$ respectively. Let

$$Q = [U \cap \emptyset^{-1}(V)]^c = U^c \cup [\emptyset^{-1}(V)]^c.$$

Since $p \in U \cap \emptyset^{-1}(V)$, Q is a proper closed subset of P. Also

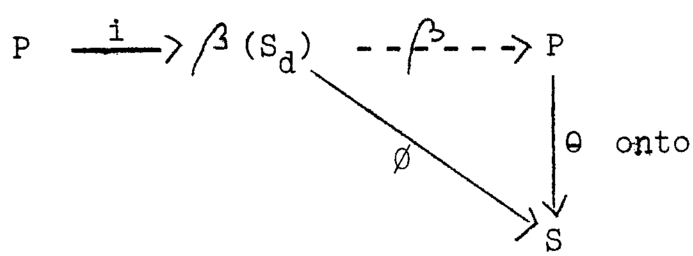
$$Q^c = U \cap \emptyset^{-1}(V) \subseteq \emptyset^{-1}(V) \subseteq \emptyset^{-1}(U^c) \subseteq \emptyset^{-1}(Q).$$

Thus $P = Q \cup \emptyset^{-1}(Q)$.

<u>Theorem 7.12</u>. (Gleason.) For every compact Hausdorff space S, there exists a pair (P,θ), where P is a stonian space[*] and θ a continuous irreducible map of P onto S. Moreover, if (P',θ') is another such pair, then there exists a homeomorphism $\beta : P \to P'$ such that $\theta = \theta' \circ \beta$.

<u>Proof</u>. Let S_d be S with the discrete topology and $\emptyset : \beta(S_d) \to S$ be continuous and onto (cf. (1) $\Rightarrow$ (2), Theorem 7.4

Let P be a closed subset of $\beta(S_d)$ such that $\emptyset|_P$ is irreducible and onto S, and consider the diagram



where $\theta = \emptyset|_P$ and i is the imbedding map. Since $\beta(S_d)$ is projective, there exists a continuous map $\beta : \beta(S_d) \to P$ such that $\emptyset = \theta \circ \beta$. We show that $\beta \circ i$ equals the identity on P. Hence β is a retraction of $\beta(D)$ onto P, and P is stonian by Theorem 7.4.

That $\theta = \emptyset|_P$ implies $\emptyset \circ i = \theta$ and hence

$$\theta \circ \beta \circ i = \emptyset \circ i = \theta.$$

If $\beta \circ i$ is not the identity, then by Lemma 7.11 there exists a proper closed subset Q of P such that

$$P = Q \cup (\beta \circ i)^{-1} Q.$$

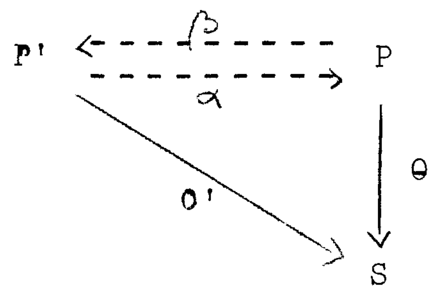
But then

$$\begin{aligned} \theta(P) &= \theta(Q) \cup \theta((\beta \circ i)^{-1} Q) \\ &= O(Q) \cup (\theta \circ \beta \circ i)(\beta \circ i)^{-1}(Q) \\ &= \theta(Q) \end{aligned}$$

contradicting the irreducibility of θ.

*) Called the <u>Gleason space</u> of S.

To prove the uniqueness let (P',Q') be another pair satisfying the requirements. Consider the diagram



By the projectivity of P' and P there exists continuous maps $\alpha : P' \to P$ and $\beta : P \to P'$ such that

$$\theta' = \theta \circ \alpha \qquad \text{and} \qquad \theta = \theta' \circ \beta \quad .$$

If $\beta \circ \alpha$ is not the identity on P', then there exists a proper closed set Q of P' such that

$$P' = Q \cup (\beta \circ \alpha)^{-1}(Q).$$

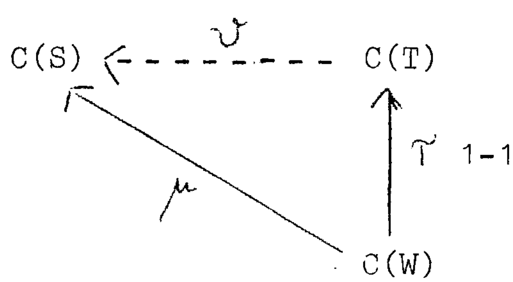
But as before $\theta' \circ \beta \circ \alpha = \theta \circ \alpha = \theta'$ so that

$$\theta'(P') = \theta'(Q) \cup \theta'(\beta \circ \alpha)^{-1}(Q) = \theta'(Q)$$

now contradicting the irreducibility of θ'. Thus $\beta \circ \alpha$ and by the same argument, $\alpha \circ \beta$, equals the identity on P' and P respectively. Hence β is a homeomorphism.

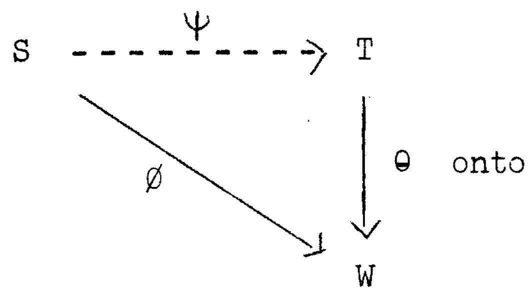
We now want to characterize $C(S)$ where S is stonian.

Definition 7.13. Let S be a compact Hausdorff space. Then $C(S)$ is called injective if for all compact Hausdorff spaces T and W and (algebra-) homomorphisms $\mu : C(W) \to C(S)$ and $\tau : C(W) \to C(T)$, where τ is one to one, there exists an (algebra-) homomorphism $\vartheta : C(T) \to C(S)$ such that $\mu = \vartheta \circ \tau$.

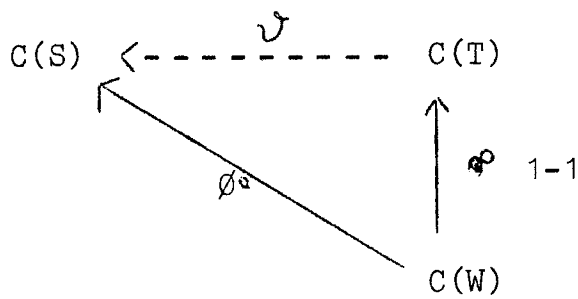


<u>Theorem 7.14</u>. Let S be compact Hausdorff. Then $C(S)$ is injective iff S is stonian.

<u>Proof</u>. Let $C(S)$ be injective and suppose we have the diagram



Then we have the dual diagram

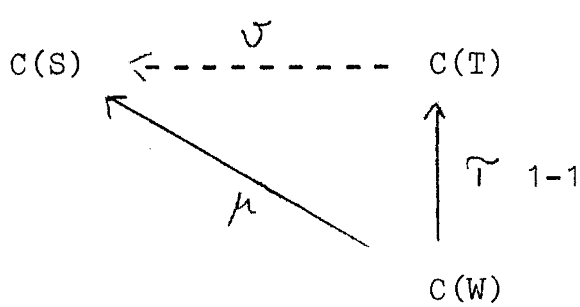


where $\emptyset^0$ and θ^0 are homomorphisms and θ^0 is one to one as θ is onto. By assumption there exist a homomorphism $\vartheta : C(T) \to C(S)$ such that $\emptyset^0 = \vartheta \circ \theta^0$. Thus by Theorem 4.2 in section 4, $\vartheta = \psi^0$ for some continuous map $\psi : S \to T$, and the equalities

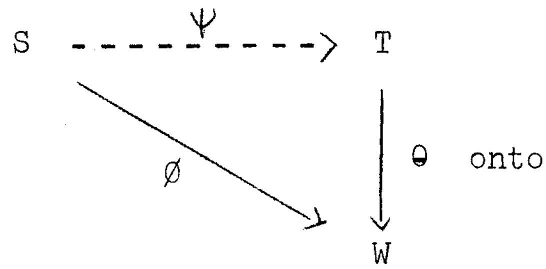
$$\emptyset^0 = \vartheta \circ \theta^0 = \psi^0 \circ \theta^0 = (\theta \circ \psi)^0$$

implies $\emptyset = \theta \circ \psi$. Hence S is projective.

Conversely assume S is projective and suppose we have the diagram



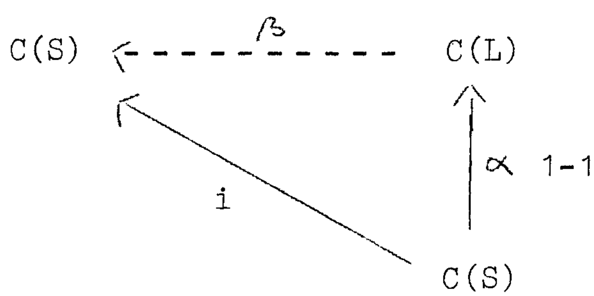
Again by Theorem 4.2, section 4 we get $\mu = \emptyset^{\circ}$ and $\Upsilon = \Theta^{\circ}$ where $\emptyset : S \longrightarrow W$ and $\Theta : T \longrightarrow W$ are continuous and Θ is onto as Υ is one to one. Now we have the diagram



and since S is projective, there exists a continuous map $\psi : S \longrightarrow T$ such that $\emptyset = \Theta \circ \psi$. Hence $\emptyset^{\circ} = \psi^{\circ} \circ \Theta^{\circ}$, and taking $\vartheta = \psi^{\circ}$ we see that $C(S)$ is injective.

Corollary 7.15. (Grothendieck.) Let S be stonian and L a compact Hausdorff space. Suppose that $C(S)$ is imbedded as a subalgebra of $C(L)$ with the same unit. Then $C(S)$ is the range of a projection in $C(L)$ which is a homomorphism and hence of norm one.

Proof. Let $\alpha : C(S) \longrightarrow C(L)$ be the imbedding map. Then we have the diagram



where i is the identity on $C(S)$. Since $C(S)$ is injective there exists a homomorphism $\beta : C(L) \longrightarrow C(S)$ such that $\beta \circ \alpha = i$. Hence β is a projection.

Corollary 7.16. Let S be stonian and L a compact Hausdorff space. If there exists a continuous map $\emptyset$ of L onto S, then S is a retract of L.

Proof. $\emptyset^{\circ}$ embeds $C(S)$ as a subalgebra in $C(L)$, so Corollary 7.15 implies that $C(S)$ is the range of multiplicative projection on $C(L)$. But by the corollary of Theorem 4.9 in section 4 this is possible iff S is a retract of L.

To state our next theorem we need some notation. Recall that for any compact space S the space $C(S)$ of all real continuous functions is a lattice under the operations

$$(f \vee g)(s) = \max[f(s), g(s)] \qquad \forall\, s \in S$$

$$(f \wedge g)(s) = \min[f(s), g(s)] \qquad \forall\, s \in S.$$

A family $\{f_\alpha\}$ of functions from $C(S)$ is said to be bounded above if there exists $f_o \in C(S)$ such that $f_\alpha \leq f_o$ for all α. We call f_o an upper bound for the family. If $f_o \leq g_o$ whenever g_o is an upper bound we call f_o the least upper bound and write $f_o = \bigvee f$. One defines bounded below and greatest lower bound similarly. The lattice $C(S)$ is said to be complete if every family of functions which is bounded above has a least upper bound. An equivalent definition could of course be given in terms of lower bounds.

Theorem 7.17. Let S be a compact Hausdorff space. Then S is stonian iff the space $C(S)$ of all real valued continuous functions is a complete lattice.

Proof. Suppose $C(S)$ is a complete lattice and V is open in S. By Urysohn's Lemma we can find a family $\{f_\alpha\}$ of continuous functions on S with $0 \leq f_\alpha \leq 1$, $f_\alpha(V^c) = 0$ and such that

$$1_V(s) = \sup f_\alpha(s), \qquad \forall\, s \in S.$$

Let $f_o = \bigvee f_\alpha$. Then $f_o(V) = 1$ so $f_o(\bar{V}) = 1$. However, if $s_o \notin \bar{V}$, we can construct a continuous function g such that $0 \leq g \leq 1$, $g(s_o) = 0$ and $g(\bar{V}) = 1$. Hence g is an upper bound for $\{f_\alpha\}$, so that $f_o \leq g$. Thus $f_o(\bar{V}^c) = 0$. As f_o is continuous, $\bar{V}$ is open and closed. This completes the easy half of the proof.

Conversely assume S is stonian and let $\{f_\alpha\}$ be a family of functions in C(S) which is bounded above. Define

$$h_o(s) = \sup_\alpha f_\alpha(s), \qquad \forall s \in S,$$

then h_o is clearly bounded, and for each real λ the set

$$\{s \in S \mid h_o(s) > \lambda\} = \bigcup_\alpha \{s \in S \mid f_\alpha(s) > \lambda\}$$

is open. Suppose $-M < h_o(s) < M$ for all $s \in S$, and let

$$\pi : -M = \lambda_o < \lambda_1 < \dots < \lambda_n = M$$

be a partition of $[-M,M]$. Then the sets

$$C_k = \{s \in S \mid h_o(s) > \lambda_k\}, \qquad k = 0,1,2,\dots,n,$$

are open and $C_o = S$, $C_n = \emptyset$ and $C_i \supseteq C_{i+1}$. Moreover, as the sets $\overline{C_k}$ are open and closed, we may write S as a union of disjoint open and closed sets

$$S = \bigcup_0^{n-1} (\bar{C}_i \setminus \bar{C}_{i+1}).$$

Define

$$g_\pi(s) = \lambda_i \quad \text{for} \quad s \in \bar{C}_i \setminus \bar{C}_{i+1}, \quad i = 0,1,2,\dots,n$$

and

$$G_\pi = \bigcup_{i=1}^{n} (\bar{C}_i \setminus C_i)$$

Then g_π is continuous, G_π nowhere dense, and the symmetric difference

$$(\bar{C}_i \setminus \bar{C}_{i+1}) \triangle (C_i \setminus C_{i+1}) \subseteq (\bar{C}_i \setminus C_i) \cup (\bar{C}_{i+1} \setminus C_{i+1})$$
$$\subseteq G_\pi, \qquad i = 0,1,\dots,n-1.$$

Since

$$\lambda_i < h_o(s) \leq \lambda_{i+1} \qquad \text{for} \quad s \in C_i \setminus C_{i+1}$$

we get

$$(*) \qquad |h_o(s) - g_\pi(s)| \leq \text{mesh}(\pi) \quad \text{for} \quad s \in S \setminus G_\pi ,$$

where $\text{mesh}\,\pi = \max\{\lambda_i - \lambda_{i-1} \mid i = 1,\ldots,n\}$. We now choose a sequence $\{\pi_n\}$ of partitions of $[-M,M]$ such that each is a refinement of the proceeding and $\text{mesh}\,\pi_n \to 0$. Then, if $m \geq n$

$$0 \leq g_{\pi_m}(s) - g_{\pi_n}(s) \leq \text{mesh}(\pi_n)$$

for all s in S, so that $\{g_{\pi_n}\}$ is a Cauchy sequence and thus converges to a continuous function g_o on S. By (*), p. 78 $g_o(s) = h_o(s)$ for $s \in S \setminus N$, where N is the first category set $N = \bigcup_1^\infty G_{\pi_n}$. Also $S \setminus N$ must be dense in S. Because, if not, then $\overline{S \setminus N}^c$ would be a non empty open subset of the compact Hausdorff space S. But also

$$\overline{S \setminus N}^c = \bigcup_1^\infty (\overline{(S \setminus N)}^c \cap G_{\pi_n}),$$

hence $\overline{S \setminus N}^c$ would be of first category, contradicting Baires Theorem. Thus $g_o(s) \geq f_\alpha(s)$ for s in dense set $S \setminus N$ and all α, so by continuity $g_o \geq f_\alpha$ for all α.

If $h \in C(S)$ and $h \geq f_\alpha$ for all α then

$$h(s) \geq h_o(s) = g_o(s)$$

for all $s \in S \setminus N$ and hence for all $s \in S$. Thus $g_o = \bigvee f_\alpha$, and $C(S)$ is a complete lattice.

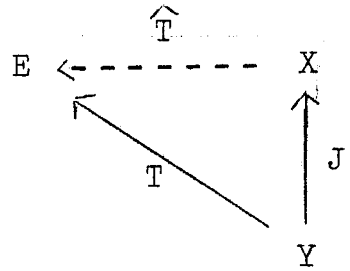
We have characterized the algebras $C(S)$, where S is stonian, among algebras $C(T)$, where T is compact Hausdorff, as being those which were injective. Our next object is to obtain a similar characterization of $C(S)$, where S is stonian, among Banach spaces. The associated maps here, will be continuous linear maps.

We start by considering two properties of Banach-spaces. Scalars may be real or complex.

Definition 7.18. Let E be a Banach space.
(a) E has the (Hahn-Banach) extension property if for every Banach space X and closed linear subspace $Y \subseteq X$ and continuous linear map $T: Y \to E$, there exist a linear extension $\hat{T}: X \to E$ of T such that $\|\hat{T}\| = \|T\|$.

(b) E has the projection property if whenever E is embedded as a subspace of a Banach space F, there exists a projection of norm one of F onto E.

Remark. The extension property can be stated in the following equivalent form, where we see the relation with injectivity. Consider the diagram



where X and Y are Banach spaces, $T: Y \to E$ is a continuous linear map and J is a linear isometry of Y into X. E has the extension property iff there always exists $\hat{T}: X \to E$ with

$$T = \hat{T} \circ J \quad \text{and} \quad \|\hat{T}\| = \|T\|$$

The projection property should be compared with Corollary 7.15.

Let E be a Banach space. Then a subset B of E of the form

$$B = B(x_o, x) = \{ x \in E \mid \|x - x_o\| \leq r \}$$

is called a ball with center x_o and radius r.

Definition 7.19. Let E be a Banach space. We say E has the binary intersection property for balls if given any family $\{B_\alpha\}$ of balls in E such that for each pair α_1, α_2, we have $B_{\alpha_1} \cap B_{\alpha_2} \neq \emptyset$, then necessarily

$$\bigcap_\alpha B_\alpha \neq \emptyset.$$

Theorem 7.20. For a real or complex Banach space E, the following (1)-(3) are equivalent

(1) E has the extension property.

(2) E has the projection property.

(3) There exists a linear isometric map af E onto a space C(S), where S is stonian.

In case of real scalars the following (4) is equivalent to the other

(4) E has the binary intersection property for balls.

The proof of Theorem 7.20 is contained in our next results. Specificly (1) $\Longleftrightarrow$ (2) is 7.21 and 7.25, (1) $\Longrightarrow$ (3) is corollary 7.30 and (3) $\Longrightarrow$ (1) follows from Proposition 7.23. The equivalence of (4) and (1)-(3) in the real case follows from Theorem 7.26 ((3) $\Longrightarrow$ (4)) and Theorem 7.27 ((4) $\Longrightarrow$ (1)).

Lemma 7.21. If E has the extension property, then E has the projection property.

Proof. Let E be imbedded as a closed subspace of F and let $T: E \rightarrow E$ be the identity map. Then T has an extension P of norm one, which is a projection of F onto E.

To prove (3) $\Longrightarrow$ (1) we note first the following:

Lemma 7.22. Let D be a descrete set. Then the Banach space $l_\infty(D)$ has the extension property.

Proof. Let $Y \subseteq X$ and $T: Y \rightarrow l_\infty(D)$. For each $d \in D$ define the functional y_d^* in Y^* by

$$y_d^*(y) = T(y)(d) \qquad \forall\, y \in Y.$$

and let x_d^* be a norm-preserving extension of y_d^* to all of X. Define $\hat{T}: X \rightarrow l_\infty(D)$ by

$$\hat{T}(x)(d) = x_d^*(x) \qquad x \in X.$$

Then $\hat{T}$ is a norm-preserving extension of T to all of X.

Proposition 7.23. If S is stonian then $C(S)$ has the extension property.

Proof. By the Stone-Cech theorem, section 2, Theorem 2.14 we may identify $l_\infty(S_d)$ with $C(\beta(S_d))$, where S_d is S with the discrete topology. From Corollary 7.15 we get that there exists a projection $P: C(\beta(S_d)) \rightarrow C(S)$ of norm one. Now assume $Y \subseteq X$ and $T: Y \rightarrow C(S)$. Let T_1 be T considered as a map of Y into $C(\beta(S_d))$. By Lemma 7.22 there exists an extension $\hat{T}_1: X \rightarrow C(\beta(S_d))$ with $\|\hat{T}_1\| = \|T_1\|$. Define

$$\hat{T} = P\hat{T}_1.$$

Then

$$\hat{T}(y) = P\hat{T}_1(y) = PT_1(y) = T(y) \qquad \forall y \in Y$$

and

$$\|\hat{T}\| \leq \|P\| \, \|\hat{T}_1\| = \|T_1\| = \|T\|.$$

Thus $\|\hat{T}\| = \|T\|$, since $\hat{T}$ extends T.

We note that the proof above shows more.

Corollary 7.24. If a Banach space E has the extension property and a subspace F of E is the range of a projection of norm one, then F also has the extension property.

The precedent corollary can be used to give a direct proof that (2) $\Longrightarrow$ (1).

Proposition 7.25. If a Banach space has the projection property then it has the extension property.

Proof. Let E have the projection property. By Corollary 7.24 it suffices to show that we can imbed E as a subspace of a space with the extension property. Let Q be the unit ball in the dual E^* endowed with the w*-topology. The map $V: E \longrightarrow C(Q)$ defined by

$$(Vx)(x^*) = x^*(x) \qquad \forall x^* \in Q, \ \forall x \in E$$

is an isometric isomorphism since

$$\|Vx\| = \sup_{\|x^*\| \leq 1} |x^*(x)| = \|x\| \qquad \forall x \in E.$$

Now $C(Q)$ is a subspace of $l_{\infty}(Q_d)$, so we are done by Lemma 7.22.

In the next two theorems we only consider real spaces and we prove that (3) $\Longrightarrow$ (4) and (4) $\Longrightarrow$ (1).

Theorem 7.26. If S is stonian, and $C(S)$ the real continuous functions on S, then $C(S)$ has the binary intersection property for balls.

Proof. Let $\{B_\alpha\}$ be a family of balls in $C(S)$, such that any two intersect. Now $f \in B_\alpha = B(f_\alpha, r_\alpha)$ iff

$$f_\alpha(s) - r_\alpha \leq f(s) \leq f_\alpha(s) + r_\alpha \qquad \forall\, s \in S$$

Since any two balls intersect, for any α_1, α_2 we can find an $f \in C(S)$ such that

$$f_{\alpha_i}(s) - r_{\alpha_i} \leq f(s) \leq f_{\alpha_i}(s) + r_{\alpha_i}$$

for all $s \in S$ and $i = 1,2$. Thus

$$\max_\alpha [f_\alpha(s) - r_\alpha] \leq \min_\beta [f_\beta(s) + r_\beta] \qquad \forall\, s \in S.$$

Let $g_0 = \bigvee_\alpha [f_\alpha - r_\alpha]$. This last upper bound exists, since $C(S)$ is a complete lattice by Theorem 7.17. Since each function $f_\beta + r_\beta$ is an upper bound for the family $\{f_\alpha - r_\alpha\}$ we have

$$f_\alpha - r_\alpha \leq g_0 \leq f_\beta + r_\beta$$

for all α and β. Thus for all α

$$f_\alpha - r_\alpha \leq g_0 \leq f_\alpha + r_\alpha ,$$

which means $g_0 \in B_\alpha$ for all α, hence $g_0 \in \bigcap_\alpha B_\alpha$.

Theorem 7.27. Let E be a real Banach space. If the family of balls in E has the binary intersection property, then E has the extension property.

Proof. Let X be a real Banach space, Y a subspace of X, and let $T: Y \to E$ be a continuous linear transformation. Denote by $\bar{\cdot}$ the collection of all continuous linear transformations $U: Z \to E$, where $\|U\| = \|T\|$ and Z is a closed subspace of X containing Y. We partially order $\bar{\cdot}$ by $U_1 \leq U_2$ if $Y \subseteq Z_1 \subseteq Z_2 \subseteq X$ and U_2 is an extension of U_1. Suppose $\{U_\alpha\}$ is any chain in $\bar{\cdot}$. Let

$$Z_0 = \bigcup_\alpha Z_\alpha$$

where Z_α is the domain of U_α. If $x \in Z_0$, then for some α we have $x \in Z_\alpha$. If also $x \in Z_\beta$, then either $U_\alpha \subseteq U_\beta$ or $U_\beta \subseteq U_\alpha$, but in both cases $U_\alpha(x) = U_\beta(x)$. Thus $U_0: Z_0 \to E$:

$$U_o(x) = U_\alpha(x) \qquad \text{if} \quad x \in Z_\alpha$$

is well-defined and $U_o(x) = T(x)$ for $x \in Y$. Clearly U_o is linear, and

$$\|U_o(x)\| = \|U_\alpha(x)\| \leq \|T\| \, \|x\| \qquad \forall \; x \in Z_o .$$

The unique continuous extension of U_o to the closure $\bar{Z}_o$ of Z_o is in $\underline{\cdot}$ and is an upper bound of $\{U_\alpha\}$. Thus by Zorns Lemma, there exists a maximal extension of T in $\underline{\cdot}$.

To complete the proof it is sufficient to show that if $U \in \underline{\cdot}$ is an extension of T whose domain Z is not all of X, then U cannot be maximal. Suppose $x_o \in X \setminus Z$. Every element x in the subspace Z_1 generated by Z and x_o has a unique representation

$$x = Z + \lambda x_o, \qquad \lambda \in \mathring{R}.$$

We wish to select a vector $e_o \in E$ and define a map $U_1 : Z_1 \to E$ by setting

$$U_1(x) = U_1(z + \lambda x_o) = U(z) + \lambda e_o, \qquad \forall z \in Z, \; \forall \lambda \in \mathring{R}$$

while maintaning the relation

$$\|U(z) + \lambda e_o\| \leq \|T\| \cdot \|z + \lambda x_o\|, \qquad \forall z \in Z, \; \forall \lambda \in \mathring{R}.$$

For $\lambda = 0$ this relation will be true, no matter how e_o is chosen. For $\lambda \neq 0$ we can rephrase this inequality in the form

$$\|U(-z/\lambda) - e_o\| \leq \|T\| \cdot \|(-z/\lambda) - x_o\|.$$

Thus if we can find a vector $e_o \in E$ such that

$$\|U(z) - e_o\| \leq \|T\| \cdot \|z - x_o\|, \qquad \forall z \in Z ,$$

the proof will be complete.

For each $e \in U(Z)$ consider the ball

$$B_e = \{e' \mid \|e' - e\| \leq \beta(e)\}$$

where

$$\beta(e) = \|T\| \inf\{\|z - x_o\| \mid z \in U^{-1}(e)\} .$$

We assert that any pair B_{e_1}, B_{e_2} of these balls intersect To prove this, let $z_i \in U^{-1}(e_i)$, $i = 1,2$. Then

$$\|e_1 - e_2\| = \|U(z_1-z_2)\| \leq \|T\| \; \|z_1-z_2\|$$
$$\leq \|T\| \; \|z_1-x_o\| + \|T\| \; \|z_2-x_o\| .$$

The left side of this inequality is independent of $z_i \in U^{-1}(e_i)$, so

$$\|e_1-e_2\| \leq \beta(e_1) + \beta(e_2).$$

Let e be the vector

$$e = (\beta(e_2)e_1 + \beta(e_1)e_2) / (\beta(e_1) + \beta(e_2)),$$

then

$$\|e-e_1\| = \|(-\beta(e_1))(e_1-e_2) / (\beta(e_1) + \beta(e_2))\|$$
$$\leq \beta(e_1).$$

Similarly $\|e-e_2\| \leq \beta(e_2)$ so $e \in B_{e_1} \cap B_{e_2}$.

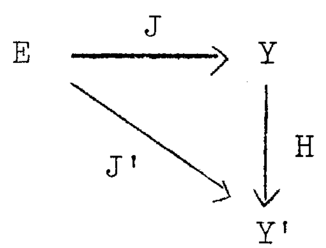
Now by the binary intersection property there is a vector $e_o \in E$ common to all the balls B_e, so

$$\|e-e_o\| = \|U(z) - e_o\|$$
$$\leq \beta(e) \leq \|T\| \; \|z-x_o\|$$

for all $z \in Z$, and the proof is complete.

For convenience let us call a (real or complex) Banach space E <u>injective</u> if it has the extension property (cf. Definiton 7.18). Our next theorem shows that to each Banach space there corresponds an essentially unique smallest injective Banach space into which E may be imbedded.

<u>Theorem 7.28</u>. Let E be e real or complex Banach space. There exists a pair (J,Y), where Y is an injective Banach space and $J: E \to Y$ is a linear isometry, such that if Z is any injective subspace of Y with $J(E) \subseteq Z \subseteq Y$, then $Z = Y$. Specificly Y is of the form $Y = C(S)$, where S is a stonian space. Moreover, if (J',Y') is another such pair, then there exists an isometry H of Y onto Y' such that $H \circ J = J'$.



<u>Definition 7.29</u>. The pair (J,Y) constructed in Theorem 7.28 is called the <u>injective envelope</u> of E.

Before we prove Theorem 7.28 let us note the following immediate corollary, which in fact is $(1) \Rightarrow (3)$.

<u>Corollary 7.30</u>. An injective Banach space E is linearly isomorphic to a space $C(S)$, where S is stonian.

<u>Proof</u>. Let $(J,C(S))$ be the injective envelope of E. Since $J(E)$ is injective, we must have $J(E) = C(S)$.

To prove Theorem 7.28 we need some lemmas. Let E be a Banach space and let Q denote the unit ball in E^*. Since Q is weak star compact the Krein-Milman theorem shows that $Q = \overline{co}(\text{ext}(Q))$. This is also valid in the complex case, since Q is also compact in E^* considered as a real topological vector space. We construct certain subsets of $\text{ext}(Q)$.

Lemma 7.31. Let E be a real Banach space. There exists a subset U of ext Q such that

(1) $\overline{U \cup (-U)} = \overline{\text{ext}(Q)}$

(2) $(-U) \cap (\bar{U}) = \emptyset$.

Proof. We note

(i) Every non-empty symmetric open subset W of $\overline{\text{ext}(Q)}$ contains a non-empty open subset V with $V \cap (-V) = \emptyset$.

To see this, choose $x_0^* \in W$ and $x_0 \in E$ such that $x_0^*(x_0) = 1$ Then

$$V = \{ x^* \in W \mid x^*(x_0) > \tfrac{1}{2} \}$$

satisfies $V \cap (-V) = \emptyset$. Now consider all open subsets V of $\overline{\text{ext}(Q)}$ which satisfy $V \cap (-V) = \emptyset$. If $\{V_\alpha\}$ is any chain of such subsets, $V = \bigcup_\alpha V_\alpha$ is an upper bound of $\{V_\alpha\}$ with the same properties. Thus by Zorn's Lemma:

(ii) There exists a subset W of $\overline{\text{ext}(Q)}$ which is maximal with respect to being open in $\overline{\text{ext}(Q)}$ and satisfying $W \cap (-W) = \emptyset$.

For such a W we have

(iii)

$$(W \cap \text{ext}(Q)) \cup ((-W) \cap \text{ext}(Q))$$

is W^*-dense in $\overline{\text{ext}(Q)}$.

It suffices to show that $W \cup (-W)$ is dense in $\overline{\text{ext}(Q)}$. If not, the set

$$\overline{\text{ext } Q} \setminus \overline{W \cup (-W)}$$

is open symmetric and non-empty in $\overline{\text{ext } Q}$, and it follows from (i) that W is not maximal.

Now define $U = W \cap \text{ext}(Q)$. Then (1) is just statement (iii). To prove (2) suppose $x^* \in (-U) \cap \bar{U}$. Since $\bar{U} = \bar{W}$, the open set $-W$ intersects $\bar{W}$, and hence $(-W) \cap (W) \neq \emptyset$ contradicting (ii).

We now prove a similar lemma in the case of complex scalars. First we need a definition.

Definition 7.32. Let E be a complex Banach space and A be a subset of E. We define the circled hull of A by

$$ci(A) = \{\lambda x \mid x \in A, \quad |\lambda| = 1\} .$$

A is circled if $A = ci(A)$. A is called deleted if $x \in A$ implies $\lambda x \in A$ for all but one λ on the unit circle.

Lemma 7.33. Let E be a complex Banach spaxe. There exists a subset U of $ext(Q)$ such that

(1) $\overline{ci(U)} = \overline{ext(Q)}$

(2) $ci(\{u^*\}) \cap \bar{U} = \{u^*\} \qquad \forall\, u^* \in U.$

Proof. As before we have

(i) every non-empty circled open subset W of $\overline{ext\ Q}$ contains a non-empty open deleted subset.

To prove (i), let D be the open unit dise with the interval $[0,1)$ removed. Choose $x_0^* \in W$ and $x_0 \in E$ such that $x_0^*(x_0) \in D$. Then we may take

$$V = \{x^* \in W \mid x^*(x_0) \in D\} .$$

Using Zorn's Lemma, we again get

(ii) There exists a subset W of $\overline{ext\ Q}$ which is maximal with respect to the properties of being open in $\overline{ext\ Q}$ and deleted.

For such a W we have

(iii)

$$ext(Q) \cap ci(W) \text{ is } W^*\text{-dense in } \overline{ext(Q)}.$$

This follows since $ci(W)$ is dense in $\overline{ext(Q)}$. For otherwise $\overline{ext(Q)} \setminus \overline{ci(W)}$ is non-void, circled and open and we can invoke (i) as before to see that W is not maximal.

Now define

$$U = \{u^* \in ext(Q) \mid u^* \notin W \text{ but } \lambda u^* \in W \text{ if } |\lambda| = 1, \ \lambda \neq 1\}.$$

Then (1) follows from (iii), since

$$ci(U) = ext(Q) \cap ci(W).$$

To prove (2) note that W is open in $\overline{\text{ext}(Q)}$ and $U \subseteq W^c$, so $\bar{U} \cap W = \emptyset$. If $u^* \in U$ and $\lambda \neq 1$, $|\lambda| = 1$, then $\lambda u^* \in W$ so $\lambda u^* \notin \bar{U}$.

We now proceed with both the real and complex cases together. Let K be the w*-closure of U. Then K is a compact subset of Q. Let S be the (stonian) <u>Gleason space</u> of K and $\beta : S \to K$ be the irreducible <u>Gleason map</u> ((S, β) is the space and map constructed in Theorem 7.12). Define the map

$$J(x)(s) = \beta(s)(x) \qquad \forall\, s \in S, \ \forall\, x \in E.$$

This make sense since $\beta(s) \in K \subseteq E^*$. We note that

$$J^*(\delta_s) = \beta(s).$$

<u>Lemma 7.34</u>. J is a linear isometry.

<u>Proof</u>. Clearly

$$\|J(x)\| = \sup_{s \in S} |\beta(s)(x)| \leq \|x\|,$$

since $\beta(s) \in Q$. However, given $x \in E$ and $\varepsilon > 0$, we can find an extreme point x^* of Q such that $x^*(x) = \|x\|$. By (1) in either Lemma 7.31 or Lemma 7.33 we can find $u^* \in U$ such that

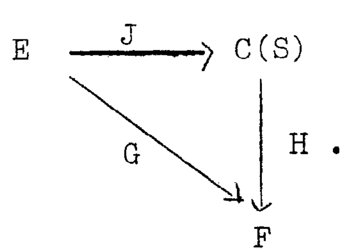
$$|u^*(x)| > \|x\| - \varepsilon .$$

Now $u^* = \beta(s_0)$ for some $s_0 \in S$, so

$$|u^*(x)| = |\beta(s_0)(x)| = |J(x)(s_0)| \geq \|x\| - \varepsilon .$$

Hence J is an isometry.

<u>Proposition 7.35</u>. The pair $(J, C(S))$ has the property that for each Banach space F and linear isometry $G: E \to F$ and linear map $H: C(S) \to F$ with $\|H\| \leq 1$ satisfying $H \circ J = G$, we have that H is an isometry.



Proof. Let $f \in C(S)$ and $\varepsilon > 0$ be given. Since we are interested in the norm of $H(f)$ we can suppose without loss of generality that $\|f\| = f(s_0)$ for some $s_0 \in S$. Define

$$V_\varepsilon = \{ s \in S \mid |f(s) - \|f\|| < \varepsilon \} .$$

Then V_ε is open in S and since $\beta : S \to K$ is irreducible, there is a non-void open set $A \subseteq K$ such that $\beta^{-1}(A)$ is dense in V_ε (Proposition 7.6). Since U is dense in K, there exists an extreme point u^* of Q in A such that the fiber $\beta^{-1}(u^*) \subseteq V_\varepsilon$. Hence, writing $\hat{S}$ for S considered as measures in $C(S)^*$, we have

$$(*) \qquad |\mu(f)| \geq \|f\| - 2\varepsilon \quad , \qquad \forall \mu \in \overline{co}[\widehat{\beta^{-1}(u^*)}].$$

Now

$$\widehat{\beta^{-1}(u^*)} = J^{*-1}(u^*) \cap \hat{S}.$$

However, it follows from (2) in the Lemmas 7.31 and 7.33 that

$$J^{*-1}(u^*) \cap \hat{S} = J^{*-1}(u^*) \cap ci(\hat{S}).$$

To see this, suppose we have $|\lambda| = 1$ and

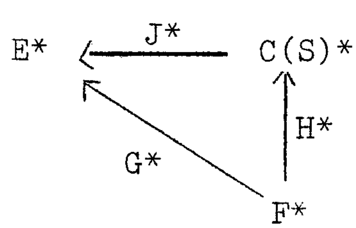
$$J^*(\lambda \delta_s) = \lambda \beta(s) = u^*;$$

then $u^* \in U$ and $\lambda^{-1} u^* = \beta(s) \in K = \bar{U}$. Thus $\lambda^{-1} = 1 = \lambda$.

Since $u^* \in ext(Q)$ and J^* is continuous and linear, the set $J^{*-1}(u^*) \cap Q_{C(S)}$ is a closed support of the unit ball $Q_{C(S)}$ in $C(S)^*$. Thus we have

$$\begin{aligned} \overline{co}(\widehat{\beta^{-1}(u^*)}) &= \overline{co}(J^{*-1}(u^*) \cap ci(\hat{S})) \\ &= \overline{co}(J^{*-1}(u^*) \cap ext(Q_{C(S)})) \\ (**) \qquad &= \overline{co}\ ext(J^{*-1}(u^*) \cap Q_{C(S)}) \\ &= J^{*-1}(u^*) \cap Q_{C(S)}. \end{aligned}$$

We now show that H is an isometry. Consider the diagram



By the Hahn-Banach theorem there exists $y^* \in F^*$ such that $\|y^*\| = 1$ and $G^*y^* = u^*$. Thus

$$J^*(H^*y^*) = (J^* \circ H^*)y^* = (J \circ H)^*y^* = G^*y^* = u^*,$$

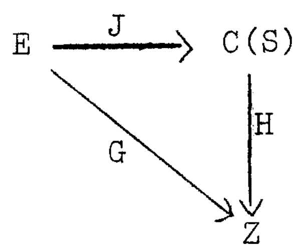
so H^*y^* is a measure in $J^{*-1}(u^*) \cap Q_{C(S)}$. Consequently by (*) and (**), p. 90.

$$|(H^*y^*)f| = |y^*(Hf)| \geq \|f\| - \varepsilon ,$$

so $\|Hf\| \geq \|f\| - \varepsilon$. Since ε was arbitrary, H is an isometry.

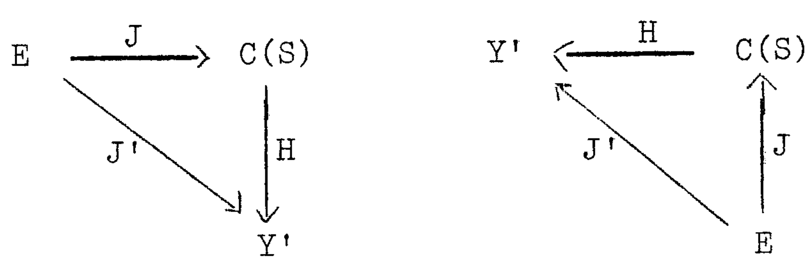
<u>Proof of Theorem 7.28</u>. Let $(J, C(S))$ be the pair constructed in the precedent. Then J is a linear isometry (Lemma 7.37) and $C(S)$ is injective since S is stonian. (Proposition 7.23.) Suppose now Z is an injective subspace of $C(S)$ containing $J(E)$. Since by Lemma 7.21 Z also has the projection property, there exists a norm one projection $H: C(S) \to Z$.

Now consider the diagram



where $G = H \circ J$. Since J is isometric and $J(E) \subseteq Z$, G is isometric. Hence by Propositiin 7.35, H is an isometry, so $Z = C(S)$.

To see the uniqueness, suppose (J', Y') is another injective envelope of E. Consider the following diagram written in two ways:



Since Y' is injective, we get, considering the second diagram, that there exists an $H: C(S) \to Y'$ such that $\|H\| = \|J'\| = 1$ and $J' = H \circ J$. However, considering the first diagram, H must be an isometry. Thus $H(C(S))$ is an injective subspace of Y' containing $J'(E)$ so $H(C(S)) = Y'$. This completes the Proof of Theorem 7.28.

Corollary 7.36. Let T be a compact Hausdorff space. The injective hull of $C(T)$ is $(J, C(S))$ where S is the Gleason space of T and $J = \beta^{\circ}$, where $\beta : S \to T$ is the irreducible Gleason map.

Proof. The set $\hat{T}$ in $Q_{C(T)}$ satisfies the lemmas. Since T is compact, we may take $U = \hat{T} = K$.

Remark. If we examine the proofs of Lemma 7.21 and Proposition 7.25, we see that the following theorem is true.

Theorem 7.37. Let E be a real or complex Banach space and $\lambda \geq 1$. Then the following are equivalent

(1) For every Banach space X, closed linear subspace $Y \subseteq X$ and continuous linear map $T: Y \to E$, there exists a continuous linear extension $\hat{T}: X \to E$ of T (such that $\|\hat{T}\| \leq \lambda \|T\|$).

(2) Whenever E is imbedded as a subspace of a Banach space F, there exists a projection (of norm less than λ) of F onto E.

The Banach spaces E for which (1) and (2) hold are called P spaces (P_λ spaces). It is known that every P space is a P_λ space for some λ. The injective Banach spaces are just the P_1 spaces, and we know they are just the spaces for which $E = C(S)$, where S is stonian. Any space isomorphic to a P_1 space is a P_λ space for some λ (determined by the norm of the isomorphism). The following conjecture has withstood much research effort.

Conjecture. Every P_λ space is isomorphic to a space $C(S)$, where S is stonian.

SECTION 8.

Hyperstonian spaces.

This section will be concerned with a subclass of the stonian spaces, which arise in several settings. They are

(a) The stone-representation spaces of measure algebras.

(b) The maximal ideal spaces of commutative W*-algebras (B*-algebras, which admit a faithful *-representation as an algebra of bounded operators on a Hilbert space which is closed for the weak operator topology).

(c) The compact Hausdorff spaces S, for which $C(S)$ is a conjugate space.

(d) The maximal ideal spaces of algebras $L_\infty(R,\sum,\mu)$.

Before introducing the hyperstonian spaces, we must discuss a class of measures. Throughout this discussion S is a compact stonian space and $C(S)$ represents either the real or complex continuous functions. A measure is an element of $C(S)^*$, represented as usual as a regular set function defined on the Borel sets of S. Recall that the real $C(S)$ forms a complete lattice (Theorem 7.17).

Definition 8.1. A real or complex measure μ is normal if for each bounded monotone increasing net $\{f_\alpha\}$ of real functions in $C(S)$ we have

$$\lim_\alpha \int_S f_\alpha \, d\mu = \int_S f_0 \, d\mu$$

where $f_0 = \bigvee_\alpha f$.

Clearly, we could equivalently consider bounded monotone decreasing nets in Definition 8.1. We recall also that a set $A \subseteq S$ is called nowhere dense, if $\operatorname{int}(\bar{A}) = \emptyset$.

Theorem 8.2. A positive measure μ is normel iff it vanishes on all nowhere dense Borel sets.

Proof. Let μ be a positive normal measure and A a nowhere dense Borel set. Let $\{f_\alpha\}$ be the net of characteristic functions of clopen sets containing A ordered by inclusion. Then clearly

$$0 \leq \bigwedge_\alpha f_\alpha \leq 1_{\bar{A}} ,$$

so $\bigwedge_\alpha f_\alpha = 0$, since $\bigwedge_\alpha f_\alpha$ is continuous and A nowhere dense. Thus

$$0 \leq \mu(A) = \lim_\alpha \int_S f_\alpha \, d\mu = 0.$$

Conversely suppose μ is a positive measure vanishing on nowhere dense Borel sets. Let $\{f_\alpha\}$ be a bounded monotone increasing net of real functions in $C(S)$ and $f_o = \bigvee_\alpha f_\alpha$. Define

$$h_o(s) = \sup_\alpha f_\alpha(s), \quad \forall \, s \in S.$$

Then h_o is lower semicontinuous and $h_o \leq f_o$. The set

$$N = \{s \in S \mid f_o(s) - h_o(s) > 0\}$$

is a first category Borel set; in fact it is an F_σ set. This follows from the proof of Theorem 7.17, (p. 79, 1.7) but can also be proved directly. By the properties of μ, $\mu(N) = 0$ hence $\int_S f_o \, d\mu = \int_S h_o \, d\mu$, so the proof will be complete if we show

$$\lim_\alpha \int_S f_\alpha \, d\mu = \int_S h_o \, d\mu .$$

To see this let M be a constant such that $\| f_\alpha \| \leq M$ for all α and let $\varepsilon > 0$. By Luzins Theorem there is a Borel set E such that

$$\mu(S \setminus E) \leq \varepsilon/8M,$$

and such that h_o is continuous on E. By regularity, we can find a compact set $K \subseteq E$ such that

$$\mu(S \setminus K) \leq \varepsilon/4M.$$

Now by Dinis Theorem $f_\alpha \longrightarrow h_o$ uniformly on K, so there exists an α_o such that

$$\int_K (h_o - f_{\alpha_o}) d\mu \leq \varepsilon/2.$$

Thus for $\alpha \geq \alpha_o$

$$0 \leq \int_S (h_o - f_\alpha) d\mu \leq \int_S (h_o - f_{\alpha_o}) d\mu$$

$$\leq \int_K (h_o - f_{\alpha_o}) d\mu + \int_{S \setminus K} (h_o - f_{\alpha_o}) d\mu$$

$$\leq \varepsilon/2 + 2M\varepsilon/4M = \varepsilon .$$

This completes the proof.

Lemma 8.3. A measure μ is normal iff its total variation $|\mu|$ is normal.

Proof. If $|\mu|$ is normal, $\{f_\alpha\}$ a monotone increasing net and $f_o = \bigvee_\alpha f_\alpha$, then

$$\left| \int_S (f_o - f_\alpha) d\mu \right| \leq \int_S (f_o - f_\alpha) d|\mu| \xrightarrow[\alpha]{} 0$$

Now let μ be normal. Clearly its real and imaginary parts must be normal, so we can suppose that μ is real. Let $S = E^+ \cup E^-$ be a Hahn decomposition of S relative to μ, and μ^+ and μ^- be the positive and negative parts of μ. Then

$$\mu^+(E) = \mu(E \cap E^+)$$

and

$$\mu^-(E) = \mu(E \cap E^-).$$

We show that μ^+ and μ^- are normal, hence $|\mu| = \mu^+ - \mu^-$ must be normal.

Let $\varepsilon > 0$ be given. By regularity of $|\mu|$ we can find an open set U and a compact set K such that $K \subseteq E^+ \subseteq U$ and $|\mu|(U \setminus K) \leq \varepsilon$. Since S is totally disconnected, there exists a clopen set V with $K \subseteq V \subseteq U$. Hence

$$|\mu|(E^+ \setminus V) + |\mu|(V \setminus E^+) =$$

$$= |\mu|((E^+ \setminus V) \cup (V \setminus E^+)) \leq |\mu|(U \setminus K) \leq \varepsilon .$$

Now if $\{f_\alpha\}$ is monotone increasing, $f_0 = \bigvee_\alpha f_\alpha$ and M a constant such that $\|f_\alpha\| \leq M$ for all α, then

$$|\int_S (f_0 - f_\alpha) d\mu^+| =$$

$$= |\int_{E^+ \setminus V} (f_0 - f_\alpha) d\mu + \int_V (f_0 - f_\alpha) d\mu - \int_{V \setminus E^+} (f_0 - f_\alpha) d\mu|$$

$$\leq 2M\varepsilon + |\int_V (f_0 - f_\alpha) d\mu| =$$

$$= 2M\varepsilon + |\int_S (f_0 - f_\alpha) 1_V d\mu| .$$

Since V is clopen 1_V is continuous. Clearly $f_\alpha 1_V$ is monotone increasing and $\bigvee_\alpha f_\alpha \cdot 1_V = f_0 1_V$, so by normality of μ the last integral converges to zero with α. Thus μ^+ is normal, so $\mu^- = \mu - \mu^+$ is also normal and the proof is complete.

Lemma 8.4. Let μ be a normal measure. Then for any Borel set $A \subseteq S$

$$|\mu|(\bar{A} \setminus \text{int}(A)) = 0$$

Proof. Since by Lemma 8.3 $|\mu|$ is normal, we can suppose $\mu \geq 0$. We show that for any Borel set A

$$\mu(\bar{A}) = \mu(A).$$

Since $\overline{A^c} = [\mathrm{int}(A)]^c$, we also have

$$\mu(\mathrm{int}(A)) = \mu((\overline{A^c})^c) = \mu((A^c)^c) = \mu(A).$$

Thus

$$\mu(\bar{A} \setminus \mathrm{int}(A)) = 0.$$

Let A be a Borel set in S. Using the regularity of μ, we can find a sequence $\{U_n\}$ of open sets containing A, such that

$$\mu(\cap U_n) = \mu(A).$$

For each n, $\overline{U_n} \setminus U_n$ is nowhere dense, so by Theorem 8.2 it has measure zero. Now

$$A \subseteq \bar{A} \subseteq \cap \overline{U_n}$$

and

$$(\cap \overline{U_n}) \setminus (\cap U_n) \subseteq \bigcup (\overline{U_n} \setminus U_n)$$

which has measure zero by the precedent.

Thus

$$\begin{aligned}\mu(A) &= \mu(\cap U_n) = \\ &= \mu((\cap U_n) \cup ((\cap \overline{U_n}) \setminus (\cap U_n)) \\ &= \mu(\cap \overline{U_n}) \geq \mu(\bar{A}),\end{aligned}$$

so

$$\mu(A) = \mu(\bar{A}),$$

and the lemma is proved.

Let μ be a positive measure. Using the regularity of μ it is easily seen that the union of the family of all open sets of zero measure has zero measure. Hence the following is non-ambiguous.

Definition 8.5. The support of a positive measure μ is the complement of the largest open set of μ-measure zero. The support of a measure μ is the support of $|\mu|$.

Theorem 8.6. The support of a normal measure is both open and closed.

Proof. We may suppose $\mu \geq 0$. Let F be the support of μ and $U = \operatorname{int} F$. Since F is closed $\bar{U} \subseteq F$ and since S is stonian $\bar{U}$ is open. Hence $\bar{U} \subseteq U$, so $U = \bar{U}$. Now $F \setminus U$ is nowhere dense, so by Theorem 8.2

$$\mu(S \setminus U) = \mu(S \setminus F) + \mu(F \setminus U) = 0.$$

Thus $F = U$ by the definition of the support.

Corollary 8.7. If μ is a positive normal measure and $\mu(A) = 0$, then $A \cap \operatorname{supp}(\mu)$ is nowhere dense.

Theorem 8.8. The normal measures form a closed subspace of $M(S)$.

Proof. Clearly the normal measures form a subspace in $M(S)$, so let $\{\mu_n\}$ be a sequence of normal measures and $\|\mu_n - \mu_o\| \longrightarrow 0$. Let $\{f_\alpha\}$ be a bounded monotone increasing net of real valued continuous functions and let $f_o = \bigvee_\alpha f_\alpha$. If M is a constant such that $\|f_\alpha\| \leq M$ for all α, then

$$\begin{aligned} &\left| \int_S (f_o - f_\alpha) d\mu_o \right| \leq \\ &\leq \left| \int_S (f_o - f_\alpha) d\mu_n \right| + \left| \int_S (f_o - f_\alpha) d(\mu_n - \mu_o) \right| \leq \\ &\leq \left| \int_S (f_o - f_\alpha) d\mu_n \right| + 2M\|\mu_o - \mu_n\| , \qquad n \in \mathbb{N}. \end{aligned}$$

Let $\varepsilon > 0$ be given and fix an n_o so large that $2M\|\mu_o - \mu_{n_o}\| \leq \varepsilon/2$. By the normality of μ_{n_o} we can find an α_o such that for $\alpha \geq \alpha_o$ we have

$$\left| \int_S (f_o - f_\alpha) d\mu_{n_o} \right| \leq \varepsilon/2.$$

Thus for $\alpha \geq \alpha_0$ we have

$$\left| \int_S (f_0 - f_\alpha) d\mu_0 \right| \leq \varepsilon/2 + \varepsilon/2 = \varepsilon ,$$

proving the normality of μ_0.

We are now able to define the concept of a hyperstonian space.

Definition 8.9. A stonian space S is called hyperstonian if the union of the supports of all normal measures is dense in S.

We next prove a decomposition theorem due to Dixmier, which shows that every stonian space is the disjoint union of a hyperstonian space and two stonian spaces of distrinct pathological type.

Theorem 8.10. Let S be a stonian space. There exists a unique decomposition of S into the union of three disjoint open and closed sets S_1, S_2, and S_3, which are stonian spaces with the properties:

(1) In S_1 there is a dense first category set, and no measure is normal except the zero measure.

(2) S_2 is hyperstonian.

(3) In S_3, every first category set is nowhere dense, and every measure has nowhere dense support (hence only the zero measure is normal).

Proof. Consider the family $\mathcal{F}$ of all open sets in S, each of which contains a dense first category set. By Zorn's Lemma, there exists a maximal collection $\{G_\alpha\}$ of open disjoint sets from $\mathcal{F}$. For each α let $\{A_\alpha^n\}$ be a sequence of nowhere dense sets such that $\bigcup_{n=1}^{\infty} A^n$ is dense in G_α. Let

$$A^n = \bigcup_\alpha A_\alpha^n , \qquad \forall n \in \mathbb{N}.$$

Then A^n is nowhere dense, for if $\overline{A^n}$ contained an open set U, then $U \cap G_{\alpha_0}$ is open and non-empty for some α_0, while

$$U \cap G_{\alpha_0} \subseteq G_{\alpha_0} \cap (\overline{A^n_{\alpha_0}} \cup (\overline{\bigcup_{\alpha \neq \alpha_0} A^n_\alpha}))$$
$$= G_{\alpha_0} \cap \overline{A^n_{\alpha_0}}$$

which is nowhere dense. Thus

$$A = \bigcup_{n=1}^{\infty} A^n$$

is of first category and

$$\overline{A} = \overline{\bigcup_\alpha G_\alpha}$$

is open and closed. Define

$$S_1 = \overline{\bigcup_\alpha G_\alpha}.$$

Then clearly S_1 is stonian in its relative topology. By maximality of the family $\{G_\alpha\}$, each first category set in $S \setminus S_1$ is nowhere dense. Now let μ be a positive normal measure with support in S_1. Then, by Theorem 8.2

$$\mu(A^n) = 0 \qquad \text{for all } n \in \mathbb{N}.$$

Thus

$$\mu(A) = \mu(\bigcup_1^\infty A_n) \leq \sum_1^\infty \mu(A_n) = 0,$$

so

$$\mu(S_1) = \mu(\overline{A}) = 0$$

by Lemma 8.4. Hence $\mu = 0$.

Now let $\{\mu_\beta\}$ be the net of all normal measures on S, and let F_β be the support of μ_β. By Theorem 8.6 each F_β is both open and closed. Define

$$F = \bigcup_\beta F_\beta$$

and

$$S_2 = \overline{F}.$$

Then $S_2 \subseteq S \setminus S_1$, and S_2 is open and closed and hyperstonian. Let

$$S_3 = S \setminus (S_1 \cup S_2).$$

By construction of S_1, every first category set in S_3 is nowhere dense, and by construction of S_2, the zero measure is the only normal measure on S_3.

We must show that any positive measure ϑ on S_3 has nowhere dense support. Let

$$a = \text{l.u.b.} \{ \vartheta(B) \mid B \subseteq S_3 \text{ nowhere dense Borel set} \}.$$

Choose $\{B_n\}$ with

$$\lim_n \vartheta(B_n) = a,$$

and let

$$B_o = \bigcup_n B_n.$$

Then B_o is of first category, hence nowhere dense, and clearly

$$\vartheta(B_o) = a.$$

We show that the support of ϑ is contained in $\overline{B_o}$. Let $G \subseteq S_3$ be open and closed. It suffices to prove that $\vartheta(G) = 0$. To see this, let C be an arbitrary nowhere dense subset of G. Then $\overline{B_o} \cup C$ is nowhere dense, so $\vartheta(\overline{B_o} \cup C) = a$. But then $\vartheta(C) = 0$. Thus the restriction ϑ_o of ϑ to G is a positive measure vanishing on nowhere dense sets, so by Theorem 8.2 ϑ_o is normal. But then, since ϑ_o is concentrated on S_3, $\vartheta_o = 0$.

It remains to prove the uniqueness. Suppose

$$S = S_1' \cup S_2' \cup S_3'$$

is another partition with the properties of the theorem. In $S_1' \cap (S_2 \cup S_3)$ there is a dense first category set, hence $S_1' \subseteq S_1$. As S_2' is hyperstonian, we must have $S_2' \subseteq S_2$, by the construction of S_2, and finally, since

$$S_3' \cap S_1 = \emptyset = S_3' \cap S_2$$

we get $S_3' \subseteq S_3$, completing the proof.

Corollary 8.11. In a hyperstonian space every first category set is nowhere dense.

We now proceed to alternate characterizations of $C(S)$ where S is hyperstonian. The next two theorems should rightly have been given earlier, as we have needed some of the ideas before.

Definition 8.12. A subset E in a topological space has the Baire property if there exists an open set U such that the symmetric difference $E \Delta U$ is of first category.

We restrict the discussion to Borel subsets of a compact Hausdorff space S.

Theorem 8.13. Let S be a compact Hausdorff space. Then every Borel set in S has the Baire property.

Proof. Let $\mathfrak{F}$ denote the class of first category Borel sets in S. We define A congruent with B

$$A \equiv B \pmod{\mathfrak{F}} \quad \text{if} \quad A \Delta B \in \mathfrak{F}.$$

Note that

$$A \equiv B, \quad B \equiv C \implies A \equiv C,$$

so $\equiv$ is an equivalence relation. Since

$$\begin{aligned} A \Delta B &= (A \cap B^c) \cup (B \cap A^c) \\ &= (A^{cc} \cap B^c) \cup (B^{cc} \cap A^c) \\ &= A^c \Delta B^c \end{aligned}$$

we have

$$A \equiv B \implies A^c \equiv B^c.$$

Let $\mathfrak{G}$ denote the class of Borel sets congruent to an open set. We note that if U is open and $E \equiv U$, then $U \equiv \bar{U}$, so

$$E^c \equiv \bar{U}^c,$$

which is open. Let $E_n \equiv U_n$, where U_n is open. Then

$$(\bigcup_n U_n)\setminus(\bigcup_n E_n) \subseteq \bigcup_n (U_n\setminus E_n) \in \mathfrak{J}$$

$$(\bigcup_n E_n)\setminus(\bigcup_n U_n) \subseteq \bigcup_n (E_n\setminus U_n) \in \mathfrak{J} ,$$

so

$$(\bigcup_n E_n) \equiv (\bigcup_n U_n).$$

Thus $\mathfrak{F}$ contains all open sets and is closed under forming complements and countable unions. Thus $\mathfrak{F} = \mathfrak{B}$.

Remark. An open set U is called regular if

$$U = \mathrm{int}(\bar{U}).$$

Since

$$U \equiv \bar{U} \equiv \mathrm{int}(\bar{U})$$

every Borel set in a compact Hausdorff space is congruent to a regular open set.

Corollary 8.14. Let S be a stonian space. If $E \in \mathfrak{B}$, then there exists a unique open and closed set U such that $E \Delta U$ is of first category.

Proof. If S is stonian, every regular open set is open and closed. Since uniqueness is obvious, the corollary follows from the precedent remark.

Remark. If S is a stonian space and E_i, $i = 1,2$ are Borel sets in S with $E_1 \cap E_2 \in \mathfrak{J}$ (in particular if $E_1 \cap E_2 = \emptyset$), then $U_1 \cap U_2 = \emptyset$, where U_i are the open and closed sets congruent with E_i mod $\mathfrak{J}$.

Theorem 8.15. Let S be a stonian space. If f is a bounded Borel measurable function on S, then there exists a unique continuous function g such that

$$\{ s \mid |f(s) - g(s)| > 0 \}$$

is of first category.

Proof. A simple function is a Borel function of the form

$$f = \sum_{i=1}^{n} \alpha_i \, 1_{E_i}$$

where the sets E_i are disjoint. Let U_i be the open and closed sets with $E_i \equiv U_i$. Then by the preceding remark, the sets U_i are disjoint, and if

$$g = \sum_{i=1}^{n} \alpha_i \, 1_{U_i}$$

we have

(1) $$\{ s \mid |f(s) - g(s)| > 0 \} \in \mathfrak{I} .$$

(2) $$\sup_{s \in S} |g(s)| \leq \sup_{s \in S} |f(s)|,$$

where the inequality results from the fact that some of the set U_i may be void. We note that every bounded Borel function is a uniform limit of simple Borel functions, and that the simple functions form a linear set.

Now let f be a bounded Borel function, and let $\{f_n\}$ be a sequence of simple Borel functions converging uniformly to f. Let $\{g_n\}$ be the corresponding sequence of continuous simple Borel functions. Then by (2)

$$\|g_n - g_m\| \leq \sup_{s \in S} |f_n(s) - f_m(s)|,$$

so $\{g_n\}$ converges uniformly, to a continuous function g. If

$$|f(s) - g(s)| > 0$$

then for n large enough we have

$$|f_n(s) - g_n(s)| > 0.$$

Thus

$$\{ s \mid |f(s) - g(s)| > 0 \} \subseteq \bigcup_{n=1}^{\infty} \{ s \mid |f_n(s) - g_n(s)| > 0 \}$$

which is a set of first category. Since uniqueness of g is clear, by Baires theorem, the proof is complete.

Remark. If S is hyperstonian, we may replace the words "first category" by nowhere dense in Theorem 8.15 (cf. Corollary 8.11).

We now suppose that S is hyperstonian and want to show that C(S) is a conjugate space. We first consider the case where there exists a positive normal measure μ whose support is all of S. By $L_\infty(S,\mathcal{B},\mu)$ we mean the class of all bounded Borel measurable functions on S, where we identify two functions which differ at most on a Borel set of μ -measures zero. Clearly this class of sets coincides with the nowhere dense Borel sets. For $f \in L_\infty(S,\mathcal{B},\mu)$

$$\|f\|_\infty = \text{ess.sup } |f(s)|$$
$$= \inf \{\alpha \mid \{s \in S \mid |f(s)| > \alpha\} \text{ is of } \mu\text{-measure zero}\} .$$

Lemma 8.16. Let S be hyperstonian, and let μ be a positive normal measure with $\text{supp}(\mu) = S$. Then every equivalence class in $L_\infty(S,\mathcal{B},\mu)$ contains a unique element of C(S), and C(S) is isometrically isomorphic to $L_\infty(S,\mathcal{B},\mu) = L_1(S,\mathcal{B},\mu)^*$.

Proof. If f is a bounded Borel function on S, then by Theorem 8.15 there exists a unique continuous function g which differs from f at most on a nowhere dense set E. Since $\mu(E) = 0$, f and g belongs to the same equivalence class in $L_\infty(S,\mathcal{B},\mu)$, and since $\text{supp}(\mu) = S$.

$$\|f\|_\infty = \text{ess.sup } |f(s)| = \text{ess.sup } |g(s)| = \|g\|_{C(S)}.$$

Now let S be an arbitrary hyperstonian space. Then the union of the supports of all normal measures on S is dense in S. Let $\{\mu_\alpha\}$ be a maximal family of positive normal measures, with $\|\mu_\alpha\| = 1$, whose supports $\{G_\alpha\}$ are disjoint. Let

$$R = \bigcup_\alpha G_\alpha .$$

Then R in its relative topology is locally compact. The next lemma shows that $S = \beta(R)$.

Lemma 8.17. The restriction map $f \to f|_R$ is an isometric isomorphism of $C(S)$ onto $BC(R)$.

Proof. Clearly if $f \in C(S)$, then $f|_R \in BC(R)$, and since $S \setminus R$ is nowhere dense

$$\| f|_R \|_{BC(R)} = \| f \|_{C(S)}.$$

Now let $g \in BC(R)$. Without loss of generality we can suppose that g is real and positive. Let $\{H_\beta\}$ be the net of finite unions of the open and closed sets G_α, ordered by inclusion. For each β

$$g \cdot 1_{H_\beta} = f_\beta \in C(S)$$

and $\{f_\beta\}$ is a bounded monotone net. Let $f = \bigvee_\beta f_\beta$. Then $f \in C(S)$ and $f = f_\beta$ on H_β, so $f|_R = g$.

Before we do the general case, some remarks. Suppose that A is an index set and that X_α is a Banach space for each $\alpha \in A$. We define $(\sum_{\alpha \in A} X_\alpha)_{l_1}$ to be the class of all functions F on A such that for each α, $F(\alpha) \in X_\alpha$ and

$$\| F \|_1 = \sum_{\alpha \in A} \| F(\alpha) \|_{X_\alpha}$$

$$= \sup \Big\{ \sum_{\alpha \in H} \| F(\alpha) \|_{X_\alpha} \;\Big|\; H \in \bar{A} \Big\} < \infty ,$$

where $\bar{A}$ is the collection of all finite subsets of A. Similarly we define $(\sum_{\alpha \in A} X_\alpha)_{l_\infty}$ to be the class of all functions G on A such that for each α, $G(\alpha) \in X_\alpha$ and

$$\| G \|_\infty = \sup_{\alpha \in A} \| G(\alpha) \|_{X_\alpha} < \infty .$$

The standard proofs show

Lemma 8.18. $(\sum X_\alpha)_{l_1}$ and $(\sum X_\alpha)_{l_\infty}$ are Banach spaces and

$$(\sum_{\alpha \in A} X_\alpha)^*_{l_1} = (\sum_{\alpha \in A} X^*_\alpha)_{l_\infty} .$$

Returning to R let us write

$$Y = \left(\sum_{\alpha} L_{\infty}(G_{\alpha}, \mathcal{B}_{\alpha}, \mu_{\alpha})\right)_{l_{\infty}}$$

$$Z = \left(\sum_{\alpha} L_{1}(G_{\alpha}, \mathcal{B}_{\alpha}, \mu_{\alpha})\right)_{l_{1}}.$$

Then $Y = Z^{*}$.

<u>Theorem 8.19</u>. Let S be hyperstonian. Then $C(S) = N(S)^{*}$, where $N(S)$ is the subspace of $M(S)$ consisting of all normal measures.

<u>Proof</u>. It follows from Lemma 8.16 that that map $g \to \{g_{\alpha}\}$, where $g_{\alpha} = g|_{G_{\alpha}}$, is an isometric isomorphism of $BC(R)$ onto Y. Thus composing with the restriction map $f \to f|_{R}$ we obtain an isometric isomorphism J of $C(S)$ onto Y (Lemma 8.17). Again we write

$$J(f) = \{f_{\alpha}\}, \quad \text{where } f_{\alpha} = f|_{G_{\alpha}}, \ \forall f \in C(S).$$

Now

$$J^{*} : \quad Y^{*} = Z^{**} \longrightarrow M(S)$$

is isometric and onto. We must show that J^{*} maps the natural image $\hat{Z}$ of Z in Z^{**} onto $N(S)$.

Let $\{h_{\alpha}\} \in Z$ and $\nu = J^{*}(\widehat{\{h_{\alpha}\}})$; then

$$\begin{aligned} \int_{S} f(s)\,\nu(ds) &= \langle J^{*}(\widehat{\{h_{\alpha}\}}), f\rangle \\ &= \langle \widehat{\{h_{\alpha}\}}, J(f)\rangle \\ &= \sum \int_{G_{\alpha}} f_{\alpha}\, h_{\alpha}\, d\mu_{\alpha} \qquad \forall f \in C(S). \end{aligned}$$

(Note that since $\{h_{\alpha}\} \in Z$, only countable many of the h_{α} are different from zero in $L_{1}(G_{\alpha}, \mathcal{B}_{\alpha}, d\mu_{\alpha})$.) Now let $\{f^{\beta}\}$ be a bounded monotone increasing net of real functions in $C(S)$ and let $f^{0} = \bigvee_{\beta} f^{\beta}$. Then for each α

$$(f^{\beta})_{\alpha} \nearrow (f^{0})_{\alpha} .$$

Let $\xi > 0$ be given. Then there exists a finite set H of indices such that

$$\sum_{\alpha \notin H} \int_{G_\alpha} |h_\alpha| \, d\mu_\alpha < \xi .$$

Thus

$$| \int_S (f^0 - f^\beta) d\vartheta | \leq$$

$$\leq \sum_{\alpha \in H} \int_{G_\alpha} |f^\beta - f^0| \, |h_\alpha| d\mu_\alpha + 2M\xi ,$$

where M is a constant such that $\|f^\beta\| \leq M$ for all β .

From the proof of Theorem 7.17 (p. 79 , 1.7) it follows that $\sup_\beta f^\beta = f^0$ except on a first category Borel set. Hence it follows from Corollary 8.11 and Theorem 8.2 that for all α $\lim_\beta (f^\beta)_\alpha = (f^0)_\alpha$ almost everywhere with respect to μ_α. Thus the integral on the right go to zero with β , and ϑ is normal.

To see that J* is onto, suppose that ϑ is a normal measure on S. Then $|\vartheta|(S \setminus R) = 0$, and for each α the restriction of ϑ_α of ϑ to G_α is absolutely continuous with respect to μ_α by Theorem 8.2. For each α let h_α be the Radon-Nikodym derivative of ϑ_α ; then $h_\alpha \in L_1(G_\alpha, \mathcal{B}_\alpha, \mu_\alpha)$ for every α . Thus $\{h_\alpha\} \in Z$ and for all $f \in C(S)$

$$\int_S f(s) d\vartheta = \sum_\alpha \int_{G_\alpha} f_\alpha \, h_\alpha \, d\mu_\alpha ,$$

so

$$\vartheta = J^*(\widehat{\{h_\alpha\}}),$$

and we are done.

SECTION 9

Weak compactness in M(S).

While one has no really useful characterization of M(S)*, it is still possible to obtain much useful information concerning weak convergence and weak compactness in M(S). These results form an interesting chapter in measure theory, and are useful in the study of C(S). In this section we develope these ideas. It is convenient to start by investigating weak compactness in L_1 with respect to a fixed measure.

For the present, S is a compact Hausdorff space and $\mathcal{B}$ denotes the Borel sets for S. We begin by recalling a classical fact about absolute continuity of measures.

Definition 9.1. Let $\mu, \lambda \in M(S)$ with $\mu \geq 0$. Then λ is μ-continuous if

$$\forall \varepsilon > 0 \; \exists \; \delta > 0 : \mu(E) < \delta \implies |\lambda(E)| < \varepsilon .$$

Lemma 9.2. Let $\mu, \lambda \in M(S)$ with $\mu \geq 0$. Then λ is μ-continuous iff $\lambda(E) = 0$ whenever $\mu(E) = 0$.

Proof. Clearly μ-continuity implies the second condition. Now suppose λ satisfies

(*) $\lambda(E) = 0$ whenever $\mu(E) = 0$.

To show that λ is μ-continuous it is enough to show that the positive and negative parts of the real and imaginary parts of λ are μ-continuous. Clearly, the real and imaginary parts satisfy (*), and taking there Hahn-decompositions and using the positivity of μ we see that there positive and negative parts satisfy (*). Hence we may assume $\lambda \geq 0$. Suppose λ is not μ-continuous. Then there exists $\varepsilon > 0$ and Borel sets E_n with $\lambda(E_n) \geq \varepsilon$ and $\mu(E_n) < \frac{1}{2^n}$. Let

$$E_o = \limsup E_n = \bigcap_{n=1}^{\infty} \bigcup_{m=n}^{\infty} E_m .$$

For each n

$$\mu(E_o) \leq \mu\left(\bigcup_{m=n}^{\infty} E_m\right) \leq \sum_{m=n}^{\infty} \frac{1}{2^m},$$

so $\mu(E_o) = 0$. Thus $\lambda(E_o) = 0$, but also

$$\lambda(E_o) = \lim_{n\to\infty} \lambda\left(\bigcup_{m=n}^{\infty} E_m\right) \geq \varepsilon .$$

This contradiction completes the proof.

<u>Remark</u>. If $\mu \in M(S)$, $\mu \geq 0$ and $f \in L_1(S,\mathcal{B},\mu)$, then the measure λ defined by

$$\lambda(E) = \int_E f\, d\mu \qquad \forall E \in \mathcal{B}$$

is absolutely continuous with respect to .

The first theorem is a result on equi-continuity:

<u>Theorem 9.3</u>. Let $\{f_n\} \subseteq L_1(S,\mathcal{B},\mu)$ and suppose

$$\lim_{n\to\infty} \int_A f_n\, d\mu = 0 \qquad \forall A \in \mathcal{B}.$$

Then for each $\varepsilon > 0$, there exists $\delta > 0$ such that if $\mu(A) < \delta$, then

$$\int_A |f_n|\, d\mu \leq \varepsilon \qquad \forall n \in \mathbb{N}.$$

<u>Proof</u>. Let $H = \{1_A \mid A \in \mathcal{B}\}$. Then H is a closed subset of $L_1(S,\mathcal{B},\mu)$, since if $g = \lim 1_{A_n}$ in L_1, then there exists a subsequence which converges to g a.e., hence g takes only the values 0 and 1 a.e. We consider H as a complete metric space. For $f \in L_1$ the map from H into the scalars defined by

$$1_E \to \int_S f \cdot 1_E\, d\mu = \int_E f\, d\mu$$

is continuous, since if

$$\int_S |1_{E_n} - 1_{E_o}|\, d\mu = \mu(E_n \Delta E_o) \to 0$$

then

$$\left| \int_S f(1_{E_n} - 1_{E_o})\,d\mu \right| \leq \int_{E_n \Delta E_o} |f|\,d\mu \longrightarrow 0$$

by Lemma 9.2 and the remark.

Let $\varepsilon > 0$, and for each n let G_n be the set

$$G_n = \left\{ 1_E \;\middle|\; \left|\int_E f_m\,d\mu\right| \leq \varepsilon, \quad m \geq n \right\}.$$

Then each G_n is closed and

$$H = \bigcup_{n=1}^{\infty} G_n.$$

By Baire's theorem there is an n_o such that G_{n_o} has non-empty interior. Thus there exists $A_o \in \mathcal{B}$ and $\delta_o > 0$ such that

$$\mu(A \Delta A_o) \leq \delta \Longrightarrow \left|\int_A f_n\,d\mu\right| \leq \varepsilon \qquad \text{for } n \geq n_o.$$

Now if $\mu(B) \leq \delta_o$ and $B \cap A_o = \emptyset$, then

$$\left|\int_B f_n d\mu\right| = \left|\int_{A_o \cup B} f\,d\mu - \int_{A_o} f_n d\mu\right| \leq 2\varepsilon, \text{ for } n \geq n_o.$$

Similarly, if $B \subseteq A_o$. If B is an arbitrary Borel set with $\mu(B) \leq \delta_o$, then $B = (B \cap A_o) \cup (B \setminus A_o)$, so

$$\left|\int_B f_n\,d\mu\right| \leq 4\varepsilon \qquad \text{for } n \geq n_o.$$

Using the inequality

$$\int_B |f|\,d\mu \leq 4 \sup_{C \subseteq B} \left|\int_C f\,d\mu\right|$$

we get

$$\mu(B) \leq \delta_o \Longrightarrow \int_B |f_n|\,d\mu \leq 16\,\varepsilon \qquad \text{for } n \geq n_o.$$

It remains to deal with the functions $f_1, \ldots, f_{n_0-1}$, but since each of the measures $\int_E f_n d\mu$ is μ-continuous, by Lemma 9.2, we can find a $\delta \leq \delta_0$ such that

$$\mu(B) \leq \delta \quad \text{implies} \quad \int_B |f_n| d\mu \leq 16\varepsilon \qquad \forall n \in \mathbb{N},$$

and the proof is complete.

Corollary 9.4. If

$$\lim_{n\to\infty} \int_A f_n \, d\mu = 0$$

for each Borel set A, then

$$\sup_n \|f_n\|_1 < \infty .$$

Proof. Choose any $\varepsilon > 0$ and take by Theorem 9.3, $\delta > 0$ corresponding to ε. There exists at most a finite set F

$$F = \{ s_1, \ldots, s_N \}$$

such that $\mu(\{s_i\}) > \delta/2$. If $s \notin F$, then by regularity there exists an open neighbourhood V_s of s such that $\mu(V_s) \leq \delta$. Again by regularity each $s \in F$ has an open neighbourhood U_s such that $\mu(U_s \setminus \{s\}) \leq \delta$. By compactness, we get a decomposition

$$S = F \cup \left(\bigcup_{i=1}^{K} A_i \right)$$

where $\mu(A_j) \leq \delta$. Hence

$$\|f_n\| = \int_S |f_n| d\mu \leq$$

$$\leq \sum_{i=1}^{K} \int_{A_i} |f_n| d\mu + \sum_{i=1}^{N} |f_n(s_i)| \mu(\{s_i\}) \leq$$

$$\leq K\varepsilon + \sum_{i=1}^{N} \left| \int_{\{s_i\}} f_n \, d\mu \right|$$

which is bounded as a function of n.

<u>Theorem 9.5</u>. Let $\mu \in M(S)$ with $\mu \geq 0$, $\{f_n\} \subseteq L_1(S,\mathcal{B},\mu)$ and suppose

$$\lim_{n\to\infty} \int_A f_n \, d\mu = \theta(A)$$

exists for every $A \in \mathcal{B}$. Then

(1) $\sup_n \|f_n\|_1 < \infty$.

(2) $\lim_{n\to\infty} \int_S f_n g \, d\mu$ exists for every $g \in L_\infty(S,\mathcal{B},\mu)$.

(3) There exists $f_o \in L_1(S,\mathcal{B},\mu)$ such that

$$\lim_{n\to\infty} \int_S f_n g \, d\mu = \int_S f_o g \, d\mu \qquad \forall \, g \in L_\infty(S,\mathcal{B},\mu),$$

so

$$\theta(A) = \int_A f_o \, d\mu \qquad A \in \mathcal{B}.$$

(4) Given $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\int_A |f_n| d\mu \leq \varepsilon \qquad \forall \, n \in \mathbb{N}$$

if $\mu(A) \leq \delta$.

<u>Proof</u>. (1). To show $\sup_n \|f_n\| \leq \infty$ it suffices to prove $\sup_n \|t_n f_n\| < \infty$ for each sequence t_n with $t_n \searrow 0$. But

$$\lim_n \int_A t_n f_n \, d\mu = 0 \qquad \forall \, A \in \mathcal{B}$$

so we can apply Corollary 9.4.

(2). If $g \in L_\infty(S,\mathcal{B},\mu)$ is a simple function. (2) follows immediately from the assumptions. If $g \in L_\infty(S,\mathcal{B},\mu)$ is arbitrary, we may without loss of generality suppose that g is bounded, and hence there exists a sequence $\{g_n\}$ of simple functions, such that g_n converges uniformly to g. Using this and (1) we easily get that $\int_S f_n g \, d\mu$ is a Cauchy-sequence, hence convergent.

(3). The set function

$$\Theta(A) = \lim_{n\to\infty} \int_A f_n d\mu$$

is additive. We show it is countable additive and μ-continuous. We assert

$$(\#) \quad \begin{cases} \text{Given } \varepsilon > 0 \text{ there exists } \delta > 0 \text{ and} \\ \text{an integer } n_o \text{ such that if } \mu(A) < \delta, \text{ then} \\ \int_A |f_m - f_n| d\mu \leq \varepsilon \qquad \forall\, n,m \geq n_o. \end{cases}$$

Suppose not. Then there exists an $\varepsilon_o > 0$, strictly increasing sequences $\{m_k\}$ and $\{n_k\}$ and sets $\{A_k\}$ with $\mu(A_k) \leq \frac{1}{k}$ such that

$$\int_{A_k} |f_{m_k} - f_{n_k}| d\mu \geq \varepsilon, \qquad \forall\, k \in \mathbb{N}.$$

Since

$$\lim_{k\to\infty} \int_A (f_{m_k} - f_{n_k}) d\mu = 0 \qquad \forall\, A \in \mathcal{B}.$$

we contradict Theorem 9.3. From (#) it follows that if $\mu(A) \leq \delta$ then

$$|\Theta(A) - \int_A f_{n_o} d\mu| \leq \varepsilon.$$

Now as before we choose $\delta_o \leq \delta$ such that

$$\int_A |f_{n_o}| d\mu < \varepsilon.$$

if $\mu(A) \leq \delta_o$. Then if $\mu(A) \leq \delta_o$ we have

$$|\Theta(A)| \leq |\Theta(A) - \int_A f_{n_o} d\mu| + |\int_A f_{n_o} d\mu|$$

$$\leq 2\varepsilon.$$

Hence if $\{E_n\}$ are Borel sets such that $E_n \searrow \emptyset$, then $\mu(E_n) \downarrow 0$ since μ is countable additive. Thus θ is countable additive and μ-continuous. Now by the Radon-Nikodym theorem there exists an $f_o \in L_1(S,\mathcal{B},\mu)$ such that

$$\theta(A) = \int_A f_o \, d\mu \qquad \forall A \in \mathcal{B} .$$

If $g \in L_\infty(S,\mathcal{B},\mu)$ is simple, then clearly

$$\lim_{n\to\infty} \int_S f_n g \, d\mu = \int_S f_o g \, d\mu ,$$

and hence again using the fact that the simple functions are uniformly dense in $L_\infty(S,\mathcal{B},\mu)$ and (1), we get that $f_n \to f_o$ weakly and (3) is proved.

(4). Let $\varepsilon > 0$ be given. Then using (#) we can find $\delta > 0$ and $n_o \in N$ such that if $\mu(A) < \delta$, then

$$\int_A |f_m - f_n| \, d\mu \leq \varepsilon/2 \qquad \forall m,n \geq n_o .$$

Choose $\delta_o \leq \delta$ such that if $\mu(A) \leq \delta_o$, then

$$\int_A |f_n| \, d\mu \leq \varepsilon/2 \qquad \text{for } n=1,2,\ldots,n_o-1$$

Then if $\mu(A) \leq \delta_o$ we have

$$\int_A |f_n| \, d\mu \leq \varepsilon \qquad \forall n \in N$$

completing the proof.

By the <u>weak topology</u> of a Banach space X we shall always mean the topology $\sigma(X,X^*)$. A subset $K \subseteq X$ is called <u>weakly relatively compact</u> if its weak closure is compact in the weak topology.

<u>Corollary 9.6</u>. The space $L_1(S,\mathcal{B},\mu)$ is sequentially complete for the weak topology. Moreover, a sequence $\{f_n\} \subseteq L_1(S,\mathcal{B},\mu)$ converges weakly to an element f_o iff

$$\lim_{n\to\infty} \int_A f_n \, d\mu \quad \text{exists for every } A \in \mathcal{B} .$$

Before characterizing the weakly relatively compact subsets of $M(S)$ we recall the following important theorem.

Theorem 9.7. (Eberlein-Smulian.) Let K be a subset of a Banach space X. Then the following statements are equivalent:

(1) K is weakly relatively compact.

(2) Every sequence in K has a subsequence converging weakly to an element of X.

(3) Every countable infinite subset of K has a limit point (in X) for the weak topology.

Theorem 9.8. Let $\mu \in M(S)$ with $\mu \geq 0$, and let $K \subseteq L_1(S, \mathcal{B}, \mu)$. Then K is weakly relatively compact iff

(a) K is bounded (in norm).

(b) For every $\varepsilon > 0$ there exists $\delta > 0$ such that $\mu(A) \leq \delta$ implies

$$\left| \int_A f \, d\mu \right| \leq \varepsilon \qquad \forall \, f \in K.$$

Remark. Condition (b) is equivalent to the corresponding condition with

$$\int_A |f| \, d\mu \leq \varepsilon .$$

Proof. If K is weakly relatively compact, then by the principle of uniform boundedness K is norm-bounded. If (b) did not hold, we could find an $\varepsilon_o > 0$, $\{f_n\} \subseteq K$ and $\{A_n\} \subseteq \mathcal{B}$ with $\mu(A_n) \leq \frac{1}{n}$ such that

$$\left| \int_{A_n} f_n \, d\mu \right| \geq \varepsilon_o \qquad n \in \mathbb{N}.$$

Then Theorem 9.5 (4) implies that no subsequence of f_n can converge weakly, contradicting the Eberlein-Smulian theorem. Hence the conditions are necessary.

We now show they are sufficient. Let $K \subseteq L_1(S,\mathcal{B},\mu)$ satisfy (a) and (b). The natural embedding $f \rightarrow \hat{f}$ of L_1 into $L_1^{**} = L_\infty^*$ is a homeomorphism, when L_1 has the $\sigma(L_1,L_\infty)$ topology and $\hat{L}_1$ has the relative topology from the $\sigma(L_\infty^*, L_\infty)$ topology on L_∞^*. By the Alouglu theorem it suffices to prove that the $\sigma(L_\infty^*, L_\infty)$ closure of $\hat{K}$ lies in $\hat{L}_1$.

Let θ lie in the $\sigma(L_\infty^*, L_\infty)$ closure of K. We prove that there exists a $g \in L_1(S,\mathcal{B},\mu)$ such that

$$\theta(h) = \int_S g\, h\, d\mu \qquad \forall\, h \in L_\infty(S,\mathcal{B},\mu).$$

By the Radon-Nikodym theorem, it is equivalent to show that the set function λ defined by

$$\lambda(E) = \theta(1_E) \qquad \forall\, E \in \mathcal{B}$$

is countably additive and μ-continuous.

Let $\varepsilon > 0$ be given and choose $\delta > 0$ according to (b). If $\mu(E) \leq \delta$ and $\gamma > 0$, then there exists $f \in K$ such that

$$|\theta(1_E) - \int_S 1_E\, f\, d\mu| \leq \gamma \,.$$

Hence

$$|\lambda(E)| = |\theta(1_E)| \leq |\theta(1_E) - \int_E f\, d\mu| + |\int_E f\, d\mu|$$
$$\leq \gamma + \varepsilon \,.$$

Since $\gamma > 0$ was arbitrary, we obtain

$$\mu(E) \leq \delta \implies |\lambda(E)| \leq \varepsilon \,.$$

Hence if E_n are Borel sets such that $E_n \downarrow \emptyset$, then $\mu(E_n) \downarrow 0$, since μ is countable additive, so λ is countable additive and μ-continuous.

Theorem 9.9. Let $\mu \in M(S)$ with $\mu \geq 0$, and let $\{f_n\}$ be a bounded sequence in $L_\infty(S,\mathcal{B},\mu)$ converging to g μ- a.e. Then

$$\lim_{n\to\infty} \int_S f_n h \, d\mu = \int_S g h \, d\mu$$

uniformly for h in any weakly compact subset of $L_1(S,\mathcal{B},\mu)$.

Proof. Let K be a weakly compact subset of $L_1(S,\mathcal{B},\mu)$, and let M be a constant such that

$$\sup_n \|f_n\|_\infty \leq M \quad \text{and} \quad \sup_{h\in K} \|h\|_1 \leq M.$$

Let $\varepsilon > 0$ be given. Then by Theorem 9.8. there exists $\delta > 0$ such that if $\mu(B) < \delta$, then

$$\int_B |h| \, d\mu \leq \varepsilon \qquad \forall \, h \in K.$$

By Egoroff's Theorem there exists a compact set F with $\mu(S \setminus F) \leq \delta$ and such that f_n converges uniformly to g on F. Hence there exists an $n_0 \in \mathbb{N}$ such that

$$|f_n(s) - g(s)| \leq \varepsilon \qquad \forall \, n \geq n_0, \qquad \forall \, s \in F.$$

We now have

$$\left|\int_S (f_n - g) h \, d\mu\right| \leq \int_{S\setminus F} |f_n - g| \, |h| \, d\mu + \int_F |f_n - g| \, |h| \, d\mu$$

$$\leq 2M \int_{S\setminus F} |h| \, d\mu + \|h\|_1 \sup_{s\in F} |f_n(s) - g(s)|$$

$$\leq 2M\varepsilon + M\varepsilon = 3M\varepsilon \qquad \forall \, n \geq n_0, \quad \forall \, h \in K$$

proving the theorem.

Remark. Let μ be a positive measure in M(S) and let

$$H = \{\lambda \in M(S) \mid \lambda \text{ is } \mu\text{-continuous}\} .$$

Then by the Radon-Nikodym theorem the correspondence $\lambda \to h$, where h is the Radon-Nikodym derivate of λ with respect to μ, is an isometry of H onto $L_1(S,\mathcal{B},\mu)$.

Theorem 9.10. For a bounded subset K of M(S) the following statements are equivalent:

(1) K is weakly relatively compact.

(2) For each uniformly bounded sequenxe $\{f_n\}$ of Borel functions which converges pointwise to a function g one has

$$\lim_{n\to\infty} \int_S f_n d\mu = \int_S g\, d\mu$$

uniformly for μ in K.

(3) For each bounded sequence $\{f_n\} \subseteq C(S)$ converging pointwise to zero one has

$$\lim_{n\to\infty} \int_S f_n\, d\mu = 0$$

uniformly for μ in K.

(4) For each sequence $\{O_n\}$ of disjoint open sets one has

$$\lim_{n\to\infty} \mu(O_n) = 0$$

uniformly on K.

(5) For each $\varepsilon > 0$ and compact set F S, there exists an open set V with $F \subseteq V$ such that

$$|\mu|(V \setminus F) \leq \varepsilon \qquad \forall\, \mu \in K .$$

Proof. (1) $\Rightarrow$ (2). Let K be weakly relatively compact and suppose (2) is false. Then, passing to a subsequence, we get that there exists an $\varepsilon > 0$ and a uniformly bounded sequence $\{g_n\}$ of

Borel functions converging pointwise to a function g, together with a sequence of measures $\{\mu_n\}$ from K such that

$$(*) \qquad \left|\int_S (g_n - g)\,d\mu_n\right| \geq \varepsilon, \qquad \forall\, n \in \mathbb{N}.$$

Define a positive measure $\vartheta \in M(S)$ by

$$\vartheta = \sum_1^\infty |\mu_n|/2^n.$$

Then each μ_n is ϑ-continuous, so let h_n denote the Radon-Nikodym derivate of μ_n with respect to ϑ. By the preceding Remark and Theorem 9.8 we get that the sequence $\{h_n\}$ is weakly relatively compact in $L_1(S,\mathcal{B},\vartheta)$. Since $\{g_n\} \subseteq L_\infty(S,\mathcal{B},\vartheta)$, Theorem 9.9 implies

$$\lim_{n\to\infty} \int_S g_n\, h_m\, d\vartheta = \int_S g\, h_m\, d\vartheta$$

uniformly in m.. But by definition of h_m this means

$$\lim_n \int_S g_n\, d\mu_m = \int_S g\, d\mu_m$$

uniformly in m, contradicting $(*)$.

(2) $\Longrightarrow$ (3). Trivial.

(3) $\Longrightarrow$ (4). Suppose (4) is false. Then there exists an $\varepsilon > 0$ and, passing to a subsequence, a sequence $\{U_n\}$ of open disjoint sets, together with a sequence $\{\mu_n\}$ of measures from K such that

$$|\mu_n(U_n)| > \varepsilon, \qquad \forall\, n \in \mathbb{N}.$$

By regularity, for each n there exists $f_n \in C(S)$ satisfying

$$0 \leq f_n \leq 1_{U_n}$$

and such that

$$\left|\int_S f_n \, d\mu_n\right| > \varepsilon .$$

But since f_n converges pointwise to zero, this contradicts condition (3).

(4) $\Longrightarrow$ (5). If (5) is false there exists a compact subset F of S and an $\varepsilon > 0$ such that if V is any open neighbourhood of F then there exists a $\mu \in K$ such that

$$|\mu|(V \setminus F) > \varepsilon .$$

We make an inductive construction. Let $V_1 = S$ and choose $\mu_1 \in K$ such that

$$|\mu_1|(V_1 \setminus F) > \varepsilon .$$

By regularity of $|\mu_1|$ there exists an open set O_1 with

$$\bar{O}_1 \subseteq V_1 \setminus F \quad \text{and} \quad |\mu_1(O_1)| > \varepsilon/4.$$

Let $V_2 = \bar{O}_1{}^c$. Then V_2 is an open neighbourhood of F so we can find $\mu_2 \in K$ such that

$$|\mu_2|(V_2 \setminus F) > \varepsilon .$$

Again using regularity we can find an open set O_2 with

$$\bar{O}_2 \subseteq V_2 \setminus F \quad \text{and} \quad |\mu_2(O_2)| > \varepsilon/4.$$

Now let $V_3 = \bar{O}_1{}^c \cup \bar{O}_2{}^c$ and continue inductively.
At the n'th step let

$$V_n = \bigcup_1^{n-1} \bar{O}_i{}^c$$

and pick a measure $\mu_n \in K$ and an open set O_n such that

$$\bar{O}_n \subseteq V_n \setminus F \quad \text{and} \quad |\mu_n(O_n)| > \varepsilon/4.$$

In this way we obtain a sequence $\{O_n\}$ of open sets and a corresponding sequence $\{\mu_n\}$ of measures from K such that

$$|\mu_n(O_n)| > \varepsilon/4.$$

Since the sets $\{O_n\}$ are clearly disjoint we contradict condition (4).

(5) $\Rightarrow$ (1). By the Eberlein-Smulian theorem it suffices to prove that each countable subset of K has a weak limit point. Hence we may suppose K is countable, i.e.

$$K = \{\mu_n\}.$$

Defining a positive measure $\vartheta \in M(S)$ by

$$\vartheta = \sum_1^{\infty} |\mu_n|/2^n$$

the Remark on p. IX.11. allows us to consider K as a subset $\{f_n\}$ of $L_1(S,\mathcal{B},\vartheta)$, and it is easily seen that it is enough to show that K is weakly relative compact in $L_1(S,\mathcal{B},\vartheta)$. By Theorem 9.8 and the regularity of ϑ it is enough to show that given $\varepsilon > 0$, then there exists a $\delta > 0$ such that if U is any open set with $\vartheta(U) \leq \delta$, then

$$\int_U |f_n| d\vartheta \leq \varepsilon, \qquad \forall\, n \in \mathbb{N}.$$

Suppose this is false. Then we can find $\varepsilon > 0$ and a sequence $\{U_n\}$ of open sets and a subsequence $\{g_n\}$ of $\{f_n\} = K$ such that

$$\vartheta(U_n) \leq \frac{1}{2^{n+1}} \quad \text{and} \quad \int_{U_n} |g_n| d\vartheta > \varepsilon, \qquad n \in \mathbb{N}.$$

Let

$$V_n = \bigcap_{i=1}^{n} U_i.$$

Then V_n are decreasing and

$$\lim_{n\to\infty} \vartheta(V_n) = 0.$$

Using condition (5) on the compact sets $\{V_n^c\}$ we obtain a sequence $\{F_n\}$ of compact sets such that

$$F_n \subseteq V_n \quad \text{and} \quad \int_{V_n \setminus F_n} |g_n| \, d\nu \leq \varepsilon/2^{n+1}, \qquad \forall n \in \mathbb{N}.$$

Let

$$H_n = \bigcap_{i=1}^{n} F_n.$$

Then, since $\{V_n\}$ decreases

$$V_n \setminus H_n = \bigcup_{i=1}^{n} (V_n \setminus F_i) \subseteq \bigcup_{i=1}^{n} (V_i \setminus F_i),$$

so

$$\int_{V_n \setminus H_n} |g_n| \, d\nu \leq \sum_{i=1}^{n} \int_{V_i \setminus F_i} |g_n| \, d\nu \leq \sum_{i=1}^{n} \varepsilon/2^{i+1} \leq \varepsilon/2.$$

Thus for all n

$$\begin{aligned} \int_{H_n} |g_n| \, d\nu &= \int_{V_n} |g_n| \, d\nu - \int_{V_n \setminus H_n} |g_n| \, d\nu \\ &\geq \int_{U_n} |g_n| \, d\nu - \varepsilon/2 \\ &\geq \varepsilon/2. \end{aligned}$$

Consequently H_n is non-empty. By compactness

$$F = \bigcap_{n=1}^{\infty} H_n = \bigcap_{n=1}^{\infty} F_n$$

is a non-empty compact subset of S, and since $F_n \subseteq V_n$ we have

$$\nu(F) = 0.$$

Now by condition (5) there exists an open set U containing F such that

$$\int_U |f| \, d\sigma < \varepsilon/2, \qquad \forall f \in K.$$

However, since H_n decreases we can find an $n_0 \in \mathbb{N}$ such that $H_{n_0} \subseteq U$, but then

$$\int_U |g_{n_0}| \, d\sigma \geq \int_{H_{n_0}} |g_{n_0}| \, d\sigma \geq \varepsilon/2.$$

This contradiction completes the proof.

Corollary 9.11. A bounded set $K \subseteq M(S)$ is weakly relatively compact iff the set

$$|K| = \{ |\mu| \mid \mu \in K \}$$

is weakly relatively compact.

Proof. This follows immediately from Theorem 9.10 (5).

Theorem 9.12. The following conditions (6) and (7) are also each necessary and sufficient for a bounded set $K \subseteq M(S)$ to weakly relatively compact.

(6) If $\{E_n\}$ is a decreasing sequence of Borel sets such that $\bigcap_{n=1}^{\infty} E_n = \emptyset$, then

$$\lim_{n \to \infty} \mu(E_n) = 0$$

uniformly for μ in K.

(7) There exists a positive measure $\lambda \in M(S)$ such that given $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$|\mu(E)| \leq \varepsilon, \qquad \forall \mu \in K$$

if $\lambda(E) < \delta$.

Proof. (2) $\Rightarrow$ (6). Trivial.

(6) $\Rightarrow$ (4). If $\{O_n\}$ is a sequence of disjoint open sets, then

$$\mu(O_n) = \mu\left(\bigcup_{i=n}^{\infty} O_i\right) - \mu\left(\bigcup_{i=n+1}^{\infty} O_i\right)$$

and by (6) each of the terms on the right side converges to zero uniformly on K.

(7) $\Rightarrow$ (4). Obvious.

(1) $\Rightarrow$ (7). To prove this implication it suffices to prove that when K is weakly relatively compact in $M(S)$, then there exists a positive measure λ such that all measures in K are λ-continuous. This will imply that the natural image of K in $L_1(S,\mathcal{B},\lambda)$ will be weakly relatively compact, and the uniform λ-continuity then follows from Theorem 9.8.

Clearly we can suppose that all measures in K are positive We show first:

(#) For each $\varepsilon > 0$ there exists a finite set $\nu_1,\dots,\nu_n$ in K and a $\delta > 0$ such that if

$$\nu_i(E) \leq \delta \qquad i = 1,2,\dots,n,$$

then

$$\mu(E) \leq \varepsilon \qquad \forall \mu \in K.$$

Suppose not. Then there exists $\varepsilon_o > 0$ such that for this ε_o (#) can not be satisfied. Let $\mu_1 \in K$ be arbitrary. Then there exists $E_1 \in \mathcal{B}$ and $\mu_2 \in K$ such that

$$\mu_1(E_1) < \tfrac{1}{2}, \qquad \mu_2(E_1) \geq \varepsilon_o.$$

Again there exists $E_2 \in \mathcal{B}$ and $\mu_3 \in K$ such that

$$\mu_1(E_2) < \frac{1}{2^2}, \quad \mu_2(E_2) < \frac{1}{2^2}, \quad \mu_3(E_2) \geq \varepsilon_o.$$

Continuing inductively, we get sequences $\{\mu_n\} \subseteq K$ and $\{E_n\} \subseteq \mathcal{B}$ such that

$$\mu_i(E_n) < \frac{1}{2^n}, \qquad i = 1,2,\dots,n$$
$$\mu_{n+1}(E_n) \geq \varepsilon_0 .$$

If

$$F_n = \bigcup_{i=n}^{\infty} E_i,$$

then F_n is a decreasing sequence of sets and

$$\mu_{n+1}(F_n) \geq \varepsilon_0.$$

Now if $n > i$, then

$$\mu_i(F_n) \leq \sum_{j=n}^{\infty} \mu_i(E_j) \leq \sum_{j=n}^{\infty} \frac{1}{2^j} \leq \frac{1}{2^{n-1}},$$

hence

$$\lim_{n\to\infty} \mu_i(F_n) = 0, \qquad \forall\ i \in \mathbb{N}.$$

Let

$$F = \bigcap_{i=1}^{\infty} F_n \quad \text{and} \quad G_n = F_n \setminus F.$$

Then

$$\mu_i(F) = 0, \qquad \forall\ i \in \mathbb{N}$$

and G_n is a decreasing sequence of sets with $\bigcap_{n=1}^{\infty} G_n = \emptyset$. Moreover, we have

$$\lim_{n\to\infty} \mu_i(G_n) = \lim_{n\to\infty} \mu_i(F_n) = 0, \qquad \forall\ i \in \mathbb{N}$$

and

$$\mu_{n+1}(G_n) = \mu_{n+1}(F_n) \geq \varepsilon_0 \qquad \forall\ n \in \mathbb{N}$$

contradicting (6). This proves (#).

Now for all $n \in \mathbb{N}$, let $\delta_n > 0$ and $\vartheta_1^n, \ldots, \vartheta_{m_n}^n \in K$ be chosen such that

$$\vartheta_i^n (E) \leq \delta_n, \qquad i = 1,2,\ldots,m_n,$$

implies

$$\mu(E) \leq \frac{1}{n}, \qquad \forall \mu \in K,$$

and define

$$\lambda = \sum_{n=1}^{\infty} \frac{1}{2^n} \left(\sum_{i=1}^{m_n} \frac{1}{2^i} \vartheta_i^n \right).$$

Then $\lambda(E) = 0$ implies $\mu(E) = 0$ for all $\mu \in K$, so all measures in K are λ-continuous. This completes the proof.

We record a well known theorem which follows from the results so far.

Theorem 9.13. (Vitali-Hahn-Saks.) Let $\{\mu_n\} \subseteq M(S)$ and suppose

$$\lim_{n\to\infty} \mu_n(E) = \theta(E), \qquad \forall\, E \in \mathcal{B}.$$

Then θ is a countable additive set function, and $E_n \downarrow \emptyset$ implies

$$\lim_{k\to\infty} |\mu_n|(E_k) = 0$$

uniformly in $n \in \mathbb{N}$. Moreover if all the μ_n are ϑ-continuous, where ϑ is a positive measure in $M(S)$, then θ is ϑ-continuous, and to every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$|\mu_n|(E) \leq \varepsilon, \qquad \forall\, n \in \mathbb{N}$$

if $\vartheta(E) < \delta$.

Proof. Define the positive measure ν by

$$\nu = \sum_{n=1}^{\infty} \frac{|\mu_n|}{2^n \|\mu_n\|} .$$

Then

$$\mu_n(E) = \int_E f_n \, d\nu \qquad \forall \; n \in \mathbb{N},$$

where $f_n \in L_1(S, \mathcal{B}, \nu)$. Now apply Theorem 9.5 .

Remark. The elements of l_1 are sequences $x = \{\xi_n\}$ with

$$\|x\| = \sum_{n=1}^{\infty} |\xi_n| < \infty .$$

If $S = \mathbb{N} \cup \{\infty\}$ is the one point compactification of the natural numbers, then each $x \in l_1$ defines a measure on S, if we set

$$x(E) = \sum_{n \in E} \xi_n, \qquad \text{if } E \subseteq \mathbb{N}$$

$$x(\{\infty\}) = 0.$$

Thus we can regard l_1 as a closed subspace of co-dimension one in $M(S)$.

Corollary 9.14. In l_1 a sequence converges weakly iff it converges in the norm topology. Thus the weakly compact and the norm compact sets coincide in l_1.

Proof. It suffices to show that if $x_n \to 0$ weakly, then $\|x_n\| \to 0$. Since the sequence $\{x_n\}$ is weakly relatively compact so is the sequence $\{y_n\}$, where

$$x_n = \{\xi_i^n\} \qquad \text{and} \qquad y_n = \{|\xi_i^n|\}.$$

Let

$$E_k = \{ j \in \mathbb{N} \mid j \geq k \}.$$

Then

$$\bigcap_{k=1}^{\infty} E_k = \emptyset,$$

so by Theorem 9.12 (6), we have

$$\lim_{k\to\infty} |y_n|(E_k) = \lim_{k\to\infty} \sum_{i=k}^{\infty} |\xi_i^n| = 0$$

uniformly in $n \in \mathbb{N}$.

Let $\varepsilon > 0$ be given. Then there exists k_o such that

$$\sum_{i=k_o}^{\infty} |\xi_i^n| \leq \varepsilon/2, \qquad \forall\ n \in \mathbb{N},$$

and then an $n_o \in \mathbb{N}$ such that

$$\sum_{i=1}^{k_o-1} |\xi_i^n| \leq \varepsilon/2 \qquad \text{for} \quad n \geq n_o.$$

Hence

$$\|x_n\| = \sum_{i=1}^{\infty} |\xi_i^n| \leq \varepsilon \qquad \text{if} \quad n \geq n_o.$$

Theorem 9.15. Let $K \subseteq M(S)$ and suppose

$$\sup_{\mu\in K} |\mu(G)| < \infty \qquad \text{each open } G \in \mathcal{B}.$$

Then

$$\sup_{\mu\in K} \|\mu\| < \infty.$$

Proof. We suppose that K is unbounded, and prove that then there exists an open $G \in \mathcal{B}$ such that

$$\sup_{\mu\in K} |\mu(G)| = \infty.$$

We must consider two cases.

Case A. For each $s \in S$ there is an open neighbourhood V_s such that

$$\sup_{\mu \in K} |\mu|(V_s \setminus \{s\}) < \infty .$$

Then we can cover S by $V_{s_1}, \dots, V_{s_n}$. The deleted neighbourhoods $V_{s_i} \setminus \{s_i\}$ cover $S \setminus \{s_1, \dots, s_n\}$, so

$$\sup_{\mu \in K} |\mu|(S \setminus \{s_1, \dots, s_n\}) < \infty .$$

But K is unbounded, so for some i

$$\sup_{\mu \in K} |\mu(\{s_i\})| = \infty .$$

Then

$$\sup_{\mu \in K} |\mu(V_{s_i})| = \infty$$

and we are done.

Case B. There exists $s_o \in S$ such that for every open neighbourhood V of s_o we have

$$\sup_{\mu \in K} |\mu|(V \setminus \{s_o\}) = \infty .$$

In this case we choose $\mu_1 \in K$, and by regularity an open set G_1 with

$$\overline{G_1} \subseteq S \setminus \{s_o\} \quad \text{and} \quad |\mu_1(G_1)| \geq 1.$$

If

$$\sup_{\mu \in K} |\mu(G_1)| = \infty$$

we are done. Otherwise let $U_1 = S$ and by regularity of $|\mu_1|$ choose an open set U_2 with

$$s_o \in U_2, \quad \overline{U_2} \cap \overline{G_1} = \emptyset$$

$$|\mu_1|(U_2 \setminus \{s_o\}) \leq 1.$$

Now there exists $\mu_2 \in K$ and G_2 open with

$$\overline{G_2} \subseteq U_2 \setminus \{s_o\},$$

and

$$|\mu_2(G_2)| > 2 + |\mu_2(G_1)|.$$

If

$$\sup_{\mu \in K} |\mu(G_2)| = \infty$$

we are done; otherwise continue inductively. Then either we find an open set G_n with

$$\sup_{\mu \in K} |\mu(G_n)| = \infty$$

and we are done, or we construct sequences $\{G_n\}$ and $\{U_n\}$ of open sets satisfying

(a) $\overline{G_i} \cap \overline{G_j} = \emptyset$ for $i \neq j$

(b) $s_o \in U_n$ and $U_{n+1} \subseteq \overline{U_{n+1}} \subseteq U_n$, $\forall n \in \mathbb{N}$

(c) $\overline{G_n} \subseteq U_n \setminus \overline{U_{n+1}}$ $\forall n \in \mathbb{N}$

(d) $|\mu_{n+1}(G_{n+1})| \geq n + 1 + \sum_{i=1}^{n} |\mu_{n+1}(G_i)|,$ $\forall n \in \mathbb{N}$

(e) $|\mu_n|(U_{n+1} \setminus \{s_o\}) \leq 1.$ $\forall n \in \mathbb{N}$

Let

$$G_o = \bigcup_{n=1}^{\infty} G_n.$$

Then G_o is open and

$$|\mu_n(G_o)| \geq |\mu_n(G_n)| - |\mu_n(\bigcup_{i=1}^{n-1} G_i)| - |\mu_n(\bigcup_{i=n+1}^{\infty} G_i)|$$

$$\geq |\mu_n(G_n)| - \sum_{i=1}^{n-1} |\mu_n(G_i)| - |\mu_n(U_{n+1} \setminus \{s_o\})|$$

$$\geq n-1, \qquad n = 2, 3, \ldots ,$$

so

$$\sup_{\mu\in K} |\mu(G_o)| = \infty$$

and the proof is complete.

<u>Theorem 9.16</u>. A sequence $\{\mu_n\}$ in $M(S)$ converges weakly iff $\lim_{n\to\infty}\mu_n(G)$ exists for every open set G.

<u>Proof</u>. Suppose $\{\mu_n\}$ is sequence of measures such that $\lim_{n\to\infty}\mu_n(G)$ exists for each open set G. We show that $\{\mu_n\}$ is weakly relatively compact. This implies that $\{\mu_n\}$ has weak cluster points. Since any two clusterpoints of $\{\mu_n\}$ agree on all open sets they must agree on all Borel sets. Thus $\{\mu_n\}$ has a unique cluster point to which it must converge.

From the preceding theorem it follows that the sequence $\{\mu_n\}$ is bounded in norm. Now let $\{G_n\}$ be any sequence of disjoint open sets. We prove

$$\lim_{j\to\infty}\mu_n(G_j) = 0$$

uniformly in n. By Theorem 9.10 this will imply that $\{\mu_n\}$ is weakly relatively compact.

For $\mu\in M(S)$ define $\tilde{\mu}\in l_1$ by

$$\tilde{\mu}(j) = \mu(G_j).$$

The sequence $\{\tilde{\mu}_n\}$ is a weak Cauchy sequence in l_1, since for each subset $E\subseteq N$,

$$\tilde{\mu}_n(E) = \sum_{j\in E}\tilde{\mu}_n(j) = \mu_n(\bigcup_{j\in E} G_j)$$

has a limit as $n\to\infty$. (Use the Remark on p. 128. and apply Corollary 9.6). Hence the sequence $\{\tilde{\mu}_n\}$ is weakly relatively compact in l_1 so again by the Remark and Corollary 9.11, the sequence $\{|\tilde{\mu}_n|\}$ is weakly relatively compact. Now let

$$E_k = \{ i \in \mathbb{N} \mid i \geq k \}.$$

Then applying Theorem 9.12 (6) we get

$$|\mu_n(G_k)| \leq \sum_{j=k}^{\infty} |\mu_n(G_j)| = \sum_{j=k}^{\infty} |\tilde{\mu}_n(j)|$$

$$\leq |\tilde{\mu}_n|(E_k) \longrightarrow 0$$

uniformly in n. That completes the proof.

SECTION 10

Phillips' theorem and applications.

Let S be any discrete set and let $\beta(S)$ be the Stone-Čech compactification of S. Then we have a canonical identification og $l_\infty(S)$ with $C(\beta(S))$ (section 2 , Theorem 2.14) and hence of $l_\infty(S)^*$ with $M(\beta(S))$. There is another useful representation of $l_\infty(S)^*$, not involving $\beta(S)$ explicitly.

Definition 10.1. Let Σ denote the class of all subsets of S. By ba(S,Σ) we will mean the class of all finitely additive set functions μ defined on Σ satisfying

$$\|\mu\| = \sup \sum_{i=1}^{n} |\mu(E_i)| < \infty,$$

where the supremum is taken over all disjoint decompositions $S = \bigcup_{i=1}^{n} E_i$, where $E_i \in \Sigma$. For fixed $E \in \Sigma$ we define the total variation $|\mu|$ of μ by

$$|\mu|(E) = \sup \sum_{i=1}^{n} |\mu(E \cap E_i)|,$$

where the supremum again is taken over all disjoint decompositions $S = \bigcup_{i=1}^{n} E_i$, $E_i \in \Sigma$. Clearly $|\mu|$ also belongs to bs(S,Σ) and μ and $|\mu|$ has the same norm, since

$$\|\mu\| = |\mu|(S) = \| |\mu| \|.$$

Remark. It is easy to see that ba(S,Σ) is complete. For let $\{\mu_n\}$ be a Cauchy sequence; then clearly

$$\mu_o(E) = \lim_{n\to\infty} \mu_n(E)$$

exists for every $E \in \Sigma$. Suppose $\varepsilon > 0$ and

$$\|\mu_n - \mu_m\| < \varepsilon \quad , \qquad \forall\ m,n \geq n_o.$$

Then for any partition $S = \bigcup_{i=1}^{p} E_i$, we have

$$\sum_{i=1}^{p} |\mu_n(E) - \mu_o(E)| = \lim_{m\to\infty} \sum_{1}^{p} |\mu_n(E) - \mu_m(E)|$$

$$< \varepsilon \qquad \text{for } n \geq n_o.$$

Thus $\mu_n - \mu_o \in ba(S,\Sigma)$, so $\mu_o \in ba(S,\Sigma)$ and $\|\mu_n - \mu_o\| \to 0$ as $n \to \infty$.

Regarding the elements of $l_\infty(S)$ as bounded functions on S, then for each $\mu \in ba(S,\Sigma)$ we can define <u>the integral</u> $\int_S f d\mu$ as follows. If f is a simple function

$$f = \sum_{i=1}^{n} \alpha_i 1_{E_i}, \qquad E_i \text{ disjoint,}$$

we define

$$\int_S f\, d\mu = \sum_{i=1}^{n} \alpha_i \mu(E_i).$$

Clearly

$$\left| \int_S f\, d\mu \right| \leq \|f\|_\infty \|\mu\|.$$

If f is an arbitrary bounded function on S, then we can find a sequence $\{f_n\}$ of simple functions converging uniformly to f. Now

$$\left| \int_S (f_m - f_n) d\mu \right| \leq \|f_n - f_m\|_\infty \|\mu\|$$

so we may define

$$\int_S f\, d\mu = \lim_{n\to\infty} \int_S f_n\, d\mu .$$

If $\{g_n\}$ is another sequence of simple functions converging uniformly to f, then

$$\left| \int_S (f_n - g_n) d\mu \right| \leq \|f_n - g_n\|_\infty \|\mu\|.$$

Since the right side converges to zero as $n \to \infty$, we see that the limit is independent of the choice of $\{f_n\}$. Clearly the integral is linear and

$$\left| \int_S f \, d\mu \right| \leq \|f\|_\infty \|\mu\|$$

for all $f \in l_\infty(S)$ and $\mu \in ba(S, \Sigma)$.

We now identify $ba(S, \Sigma)$ with the dual of $l_\infty(S)$.

Theorem 10.2. For each $x^* \in l_\infty(S)^*$ there exists a unique $\mu \in ba(S, \Sigma)$ such that

$$x^*(f) = \int_S f \, d\mu , \qquad \forall f \in l_\infty(S).$$

The correspondence $x^* \to \mu$ is linear, isometric and onto

Proof. Let $x^* \in l_\infty(S)^*$ and define

$$\mu(E) = x^*(1_E), \qquad \forall E \in \Sigma .$$

Then clearly μ is finitely additive, and given any disjoint partition $S = \bigcup_{i=1}^{n} E_i$, $E_i \in \Sigma$, then

$$\sum_{i=1}^{n} |\mu(E_i)| = \sum_{i=1}^{n} |x^*(1_{E_i})| = x^*\left(\sum_1^n \alpha_i \, 1_{E_i} \right)$$
$$\leq \|x^*\|$$

so $\mu \in ba(S, \Sigma)$ and $\|\mu\| \leq \|x^*\|$. Since the formula

$$x^*(f) = \int_S f \, d\mu$$

holds for simple functions, it follows by uniform density that it holds for all $f \in l_\infty(S)$. Since

$$|x^*(f)| = \left| \int_S f \, d\mu \right| \leq \|f\|_\infty \|\mu\|$$

we have $\|x^*\| \leq \|\mu\|$, so $\|x^*\| = \|\mu\|$. Clearly the correspondence $x^* \rightarrow \mu$ is linear, and since every $\mu \in ba(S,\Sigma)$ defines a linear functional $x^* \in l_\infty(S)^*$ by

$$x^*(f) = \int_S f \, d\mu \qquad \forall f \in l_\infty(S),$$

the correspondence is onto.

We prove now a fundamental theorem concerning $ba(S,\Sigma)$ due to R.S.Phillips, and develope some consequences.

Theorem 10.3. (Phillips.) Let $\{\mu_n\} \subseteq ba(S,\Sigma)$ and suppose

$$\lim_{n\to\infty} \mu_n(E) = 0, \qquad \forall \; E \in \Sigma .$$

Then

$$\lim_{n\to\infty} \mu_n(A) = 0$$

uniformly on all finite set $A \in \Sigma$.

Proof. Suppose the theorem is false. Then there exists an $\varepsilon > 0$ and a subsequence $\{\nu_n\}$ of $\{\mu_n\}$ together with a sequence $\{A_n\}$ of finite sets such that

$$|\nu_n(A_n)| > 2\varepsilon , \qquad \forall n \in \mathbb{N}.$$

The proof now proceeds by construction of certain further subsequences of $\{\mu_n\}$ and sequences of sets, to obtain a contradiction.

Step 1. There exists a subsequence $\{\theta_n\}$ of $\{\nu_n\}$ and a sequence $\{B_n\}$ of disjoint finite sets such that

(i) $\qquad |\theta_n(B_n)| > \varepsilon \qquad \forall n \in \mathbb{N}$

(ii) $\qquad |\theta_n|\left(\bigcup_{j=1}^{n-1} B_j\right) < \varepsilon/4, \qquad \forall n \in \mathbb{N}.$

Before we start the construction, we note that if A is any finite set, then

$$(\#) \qquad \lim_{n\to\infty} |\vartheta_n|(A) = 0.$$

Now let $B_1 = A_1$ and $\theta_1 = \vartheta_1$. Then by (#) there exists an integer $n_2 > 1$ such that

$$|\vartheta_{n_2}(B_1)| < \varepsilon/4.$$

Then

$$|\vartheta_{n_2}(A_{n_2} \setminus B_1)| \geq |\vartheta_{n_2}(A_{n_2})| - |\vartheta_{n_2}(B_1)|$$

$$> 2\varepsilon - \varepsilon/4 > \varepsilon .$$

Define $B_2 = A_{n_2} \setminus B_1$ and $\theta_2 = \vartheta_{n_2}$. Now suppose $\theta_1, \ldots, \theta_k$ and $B_1, \ldots, B_k$ have been chosen to satisfy (i) and (ii), and suppose $\theta_j = \vartheta_{n_j}$ and $n_j < n_{j+1}$. Again by (#) we can find $n_{k+1} > n_k$ such that

$$|\vartheta_{n_{k+1}}|\left(\bigcup_{j=1}^{k} B_j\right) < \varepsilon/4.$$

Define

$$B_{k+1} = A_{n_{k+1}} \setminus \left(\bigcup_{j=1}^{k} B_j\right) \quad \text{and} \quad \theta_{k+1} = \vartheta_{n_{k+1}}.$$

Then we have

$$|\theta_{k+1}(B_{k+1})| > \varepsilon .$$

This establishes Step 1.

Step 2. There exists a subsequence $\{\phi_n\}$ of $\{\theta_n\}$ and sequences $\{W_n\}$ and $\{C_n\}$ of sets, where the C_n are disjoint finite sets and the W_n are infinite sets, satisfying

(a) $W_1 \supseteq W_2 \supseteq \cdots$

(b) $C_j \subseteq W_j \setminus W_{j+1}$, $\forall\ j \in \mathbb{N}$

(c) $|\phi_j(C_j)| > \varepsilon$, $\forall\ j \in \mathbb{N}$

(d) $|\phi_j|(W_{j+1}) < \varepsilon/4$, $\forall\ j \in \mathbb{N}$

(e) $|\phi_j|(\bigcup_{p=1}^{j-1} C_p) < \varepsilon/4$, $\forall\ j \in \mathbb{N}$.

We call the disjoint finite sets B_i of Step 1 "blocks". Set

$$W_1 = \bigcup_{i=1}^{\infty}, \quad C_1 = B_1 \quad \text{and} \quad \phi_1 = \theta_1.$$

Since ϕ_1 is of bounded variation, there exists an infinite union W_2 of the remaining blocks $B_2, B_3, \ldots$ in W_1 such that

$$|\phi_1|(W_2) < \varepsilon/4.$$

Select a block $C_2 = B_{n_2}$ in W_2 and let $\phi_2 = \theta_{n_2}$. Now suppose we have constructed ϕ_j, C_j and W_j satisfying the conditions (a)-(e) for $1 \leq j \leq k$, and for each j, C_j is one of the blocks in W_j. There exists an infinite union W_{k+1} of the remaining blocks in W_k such that

$$|\phi_k|(W_{k+1}) < \varepsilon/4.$$

Let $C_{k+1} = B_{n_{k+1}}$ be a block in W_{k+1} and set $\phi_{k+1} = \theta_{n_{k+1}}$. Then (a), (b), and (d) are clearly satisfied. Condition (c) follows from (i) in Step 1, and from (ii) in Step 1, we get

$$|\phi_{k+1}|(\bigcup_{j=1}^{k} C_j) \leq |\theta_{n_{k+1}}|(\bigcup_{l=1}^{n_{k+1}-1} B_l) < \varepsilon/4,$$

so (e) is satisfied. This proves Step 2.

Now define

$$Q = \bigcup_{k=1}^{\infty} C_k.$$

Then for each j

$$|\theta_j(Q)| = |\theta_j(\bigcup_{k=1}^{j-1} C_k) + \theta_j(C_j) + \theta_j(\bigcup_{k=j+1}^{\infty} C_k)$$

$$\geq |\theta_j(C_j)| - |\theta_j|(\bigcup_{k=1}^{j-1} C_k) - |\theta_j|(W_{j+1})$$

$$\geq \varepsilon - \varepsilon/4 - \varepsilon/4 = \varepsilon/2,$$

contradicting the assumption that $\lim_{j\to\infty} \theta_j(E) = 0$ for each $E \in \Sigma$.

Corollary 10.4. Let $\{\mu_n\} \subseteq ba(S,\Sigma)$ and suppose

$$\lim_{n\to\infty} \mu_n(E) = 0, \qquad \forall E \in \Sigma.$$

Then

$$\lim_{n\to\infty} \sum_{s\in S} |\mu_n(\{s\})| = 0.$$

Proof. For each $n \in \mathbb{N}$ we have

$$\sum_{s\in S} |\mu_n(\{s\})| \leq 4 \sup\{|\mu_n(A)| \mid A \subseteq S, \ A \text{ finite}\}$$

and by Theorem 10.3 the right side converges to zero as $n \to \infty$.

Remark. For a sequence $\{\mu_n\} \subseteq ba(S,\Sigma)$ the conclusions of Theorem 10.3 and Corollary 10.4 are equivalent.

Remark. If $E \in \Sigma$, let $\bar{E}$ denote the closure of E in $\beta(S)$. Then $1_{\bar{E}}$ is the unique continuous extension to $\beta(S)$ of the function 1_E on S. Thus the isomorphism between $l_\infty(S)$ and $C(\beta(S))$ induces an isomorphism between $ba(S,\Sigma)$ and $M(\beta(S))$ where if $\mu \in ba(S,\Sigma)$ and ν is the corresponding countably additive measure in $M(\beta(S))$ we have

$$\mu(E) = \nu(\bar{E}), \qquad E \in \Sigma .$$

Thus $\mu(A) = \nu(A)$ for all finite sets A. Consider the splitting

$$\nu = \nu|_S + \nu|_{\beta(S) \sim S} = \nu_1 + \nu_2 .$$

Then by regularity of ν_1,

$$\nu_1(E) = \sum_{s \in E} \nu(\{s\}) = \sum_{s \in E} \mu(\{s\}),$$

while

$$\nu_2(E) = \nu_2(\bar{E} \sim E), \qquad E \in \Sigma ,$$

so ν_2 vanishes on all finite sets in Σ. Correspondingly we have the splitting

$$\mu = \mu_1 + \mu_2 .$$

But

$$\mu_1(E) = \sum_{s \in E} \mu(\{s\}) = \nu_1(E), \qquad E \in \Sigma ,$$

so μ_1 is countably additive. We call μ_1 and μ_2 the <u>countably additive</u> and <u>purely finitely additive</u> parts of μ. Note

$$\|\mu\| = \|\mu_1\| + \|\mu_2\|,$$

since this holds for the splitting of ν.

There is another way of looking at this splitting of $ba(S,\Sigma)$. Consider the natural imbeddings

$$\widehat{c_o(S)} \subseteq c_o(S)^{**} = l_\infty(S)$$
$$\widehat{l_1(S)} \subseteq l_1(S)^{**} = l_\infty(S)^* = ba(S,\Sigma).$$

If $\kappa : c_o(S) \to \widehat{c_o(S)}$ is the imbedding map for $c_o(S)$, then

$$ba(S,\Sigma) = \widehat{l_1(S)} \oplus c_o(S)^\perp$$

and κ^* is the corresponding projection of $ba(S,\Sigma)$ onto $\widehat{l_1(S)}$. The subspace $\widehat{l_1(S)}$ consists precisely of the countably additive measures in $ba(S,\Sigma)$, and $c_o(S)^\perp$ is just those measures which are purely finitely additive. We leave these verifications for the reader. We can now state Phillips theorem in these terms.

Corollary 10.5. Suppose $\{\mu_n\} \subseteq ba(S,\Sigma) = l_\infty(S)^*$ is a sequence converging for the weak star topology. Then their countably additive parts converge in the norm topology.

Proof. If $\mu_n \to \mu_o$ weak star, then clearly

$$\lim |\mu_n(E) - \mu_o(E)| = 0, \qquad E \in \Sigma,$$

so

$$\lim_{n\to\infty} \sum_{s\in S} |(\mu_n - \mu_o)(\{s\})| = 0.$$

Remark. Note that in Theorem 10.3 we have not assumed that the measures μ_n are uniformly bounded in norm. Actually this boundedness is implied by the existence of the limit $\lim_{n\to\infty} \mu_n(E)$ for each E. However, we shall not pursue this point.

We now develop some applications of Phillips theorem.

Theorem 10.6. There exists no bounded projection of $l_\infty(S)$ onto the closed subspace $c_o(S)$.

Proof. Suppose $c_o(S)$ is complemented in $l_\infty(S)$ and $P: l_\infty(S) \to c_o(S)$. Let $\{s_n\}$ be any sequence of distinct points in S. Consider the elements $\delta_{s_n} \in l_1(S)$ ($\delta_{s_n}(s) = 1$ if $s = s_n$, zero otherwise). Then the sequence $\{\delta_{s_n}\}$ converges to zero weak star as functionals on $c_o(S)$. Now $P^*: l_1(S) \to ba(S,\Sigma)$ and

$$(P^*\delta_{s_n})(f) = \delta_{s_n}(Pf) \to 0$$

for each $f \in l_\infty(S)$, since $Pf \in c_o(S)$. Thus

$$\lim_{n\to\infty} \sum_{s\in S} |(P^*\delta_{s_n})\{s\}| = 0.$$

But all terms of this sum are zero except for $s = s_n$ and there

$$(P^*\delta_{s_n})(\{s_n\}) = \delta_{s_n}(P\,\delta_{s_n}) = \delta_{s_n}(\delta_{s_n}) = 1$$

where we regard δ_{s_n} also as an element of $c_o(S)$. This contradiction completes the proof.

Next we give an application of Phillips' theorem to convergence of measures on Stonian spaces.

Theorem 10.7. Let S be compact and stonian. A sequence $\{\mu_n\}$ in $M(S)$ converges weakly iff it converges weak star.

Proof. Suppose that

$$\lim_{n\to\infty} \int_S f\,d\mu_n = 0, \qquad \text{each } f \in C(S).$$

It suffices to show the sequence $\{\mu_n\}$ is weakly relatively compact. By Theorem 9.10 it is enough to show that for every sequence $\{U_n\}$ of disjoint open sets one has

$$\lim_{n\to\infty} \mu_m(U_n) = 0$$

uniformly for $m = 1,2,\ldots$. Thus if the sequence fails to be weakly relatively compact, there exists an $\epsilon > 0$ and a sequence of disjoint open sets $\{O_n\}$ and subsequence $\{\nu_n\}$ of $\{\mu_n\}$ such that

$$|\nu_i(O_i)| > \epsilon \quad , \qquad i = 1,2,\dots .$$

By regularity we can suppose all the sets O_i are open and closed.

For each i define the element $\beta_i \in ba(N,\Sigma)$ by

$$\beta_i(B) = \nu_i(\overline{\bigcup_{j\in B} O_j}), \qquad B \subseteq N .$$

The finite additivity follows from the fact that if $B_1 \cap B_2 = \emptyset$, then

$$\overline{\bigcup_{j\in B_1\cup B_2} O_j} = (\overline{\bigcup_{j\in B_1} O_j}) \cup (\overline{\bigcup_{k\in B_2} O_k}).$$

Clearly $\|\beta_i\| \leq \|\nu_i\|$ and

$$\lim_{i\to\infty} \beta_i(B) = \lim \nu_i(\overline{\bigcup_{j\in B} O_j}) = 0$$

for each $B \subseteq N$, since $\nu_i \to 0$ weak star. (It is here that we use the hypothesis that S is stonian.) However, by Phillips theorem:

$$|\nu_i(O_i)| = |\beta_i(\{i\})| \leq \sum_{j=1}^{\infty} |\beta_j(\{i\})| \rightarrow 0$$

as $i \to \infty$, contradicting the fact that $|\nu_i(O_i)| > \epsilon$ for all i.

For the last application of Phillips theorem we shall need the following notion:

<u>Definition 10.7</u>. Let S be compact, and X be a closed subspace of $C(S)$. A closed set $F \subseteq S$ will be called an <u>interpolation set</u> for X if

$$X|F = C(F).$$

The property of being an interpolation set is a very special In general for a closed set F, $X|F$ will not even be dense, or closed in $C(F)$. The next theorem gives a useful criterion due to Glicksberg for a set F to be an interpolation set for a subspace. If $X \subseteq C(S)$, then $X^{\perp}$ is the set of all measures μ in $M(S)$ for which

$$\int_S f\,d\mu = 0, \qquad \text{all } f \in X.$$

Theorem 10.8. Let X be a closed subspace of $C(S)$, S compact, and let F be a closed subset of S. Then $X|F = C(F)$ iff there exists t, $0 < t \leq 1$ such that

(*)
$$\|\mu_{F^c}\| \geq t\|\mu\|, \qquad \text{all } \mu \in X^{\perp}.$$

Proof. Let $Tf = f|F$, $f \in C(S)$. Then $T: X \to C(F)$. Since $X^* = M(S)/X^{\perp}$ and $C(F)^* = M(F)$ it follows that $T^*: M(F) \to X^*$ is given by

$$T^*\nu = \nu + X^{\perp}, \qquad \nu \in M(F).$$

Suppose T maps X onto $C(F)$. Then T^* is one to one with closed range [12] p. 487. Thus there exists $C > 0$ such that

$$C\|\nu\| \leq \|T^*\nu\| = \|\nu + X^{\perp}\|, \qquad \nu \in M(F).$$

Now if $\mu \in X^{\perp}$,

$$\|\mu\| - \|\mu_{F^c}\| \leq \|\mu - \mu_{F^c}\| = \|\mu_F\|$$

$$\leq C\|\mu_F - X^{\perp}\| \leq C\|\mu_F - \mu\|$$

$$= C\|\mu_{F^c}\|,$$

so

$$\|\mu_{F^c}\| \geq \frac{1}{C+1}\|\mu\|, \qquad \mu \in X^{\perp},$$

so (*) holds.

Conversely, suppose (*) holds. Then

$$t\|\mu_F\| \leq t\|\mu\| \leq \|\mu_{F^c}\|, \qquad \mu \in X^{\perp}.$$

Now let $\nu \in M(F)$, $\mu \in X^{\perp}$. Since $t \leq 1$,

$$\|\nu - \mu\| = \|(\nu-\mu)_F\| + \|(\nu-\mu)_{F^c}\|$$
$$= \|\nu - \mu_F\| + \|\mu_{F^c}\|$$
$$\geq t\|\nu - \mu_F\| + \|\mu_{F^c}\|$$
$$\geq t\|\nu\| - t\|\mu_F\| + \|\mu_{F^c}\|$$
$$\geq t\|\nu\| - \|\mu_{F^c}\| + \|\mu_{F^c}\| = t\|\nu\|.$$

Thus $\|\nu - X^{\perp}\| \geq t\|\nu\|$ for all $\nu \in M(F)$. It follows that T^* is one to one with closed range. Hence T has closed range $X|F$. However $X|F$ is dense in $C(F)$, since if $\nu \in M(F)$ and $\nu(X|F) = 0$, i.e. $\nu \in X^{\perp}$, then (*) shows $\nu = 0$, since $\nu_{F^c} = 0$. Thus $X|F = C(F)$.

Now consider a locally compact, σ-compact space S. Let $C_o(S)$ be the subspace of functions vanishing at infinity on S. We shall show that $C_o(S)$ is uncomplemented in $C(\beta(S))$. However, we shall show much more than this.

Suppose $C_o(S)$ is complemented, i.e. there is a closed subspace X of $C(\beta(S))$ such that

$$C(\beta(S)) = C_o(S) \oplus X.$$

Then clearly

$$(*) \qquad X \mid \beta(S) \sim S = C(\beta(S) \sim S).$$

Thus the compact set $\beta(S) \sim S$ will be an interpolation set for X. We show this is impossible by showing that <u>any</u> closed subspace X of $C(\beta(S))$ satisfying (*) must be so large that it contains non zero functions in $C_o(S)$. We show in fact that there is necessarily a closed neighbourhood V of $\beta(S) \sim S$ in $\beta(S)$ for which

$$X \mid V = C(V).$$

In other words, if $\beta(S) \sim S$ is an interpolation set for a subspace X, there must be an open set U with compact closure in S such that $V = U^c$ is also an interpolation set for X.

Theorem 10.9. Let S be locally compact and σ-compact. Let X be a closed subspace of $C(\beta(S))$ such that

$$X|\beta(S)\sim S = C(\beta(S)\sim S).$$

Then there exists a closed neighbourhood V of $\beta(S)\sim S$ such that $X|V = C(V)$.

Proof. Write $Q = \beta(S)\sim S$. Let $\{W_n\}$ be open in $\beta(S)$, $\overline{W_{n+1}} \subseteq W_n$, $Q = \bigcap W_n$. Since $X|Q = C(Q)$, by Theorem 10.8 there exists t, $0 < t \leq 1$ such that

$$|\mu|(S) = |\mu|(Q^c) \geq t\|\mu\|, \qquad \mu \in X^\perp.$$

Suppose the theorem is false, i.e. if V is any closed neighbourhood of Q, then $X|V \neq C(V)$. Hence for each $\epsilon > 0$ there is $\mu \in X^\perp$, $\|\mu\| = 1$, $|\mu|(V^c) < \epsilon$.

Let $V_1 \subseteq W_1$ be a closed neighbourhood of Q, choose $\mu_1 \in X^\perp$, $\|\mu_1\| = 1$ such that

$$|\mu_1|(V_1{}^c) < \frac{t}{2};$$

then

$$|\mu_1|(V_1\sim Q) = |\mu_1|(Q^c) - |\mu_1|(V_1{}^c) > t - \frac{t}{2} = \frac{t}{2}.$$

Now select a closed neighbourhood V_2 of Q, $V_2 \subseteq W_2 \cap \operatorname{int}(V_1)$ such that

$$|\mu_1|(V_2\sim Q) < \frac{t}{2}.$$

Hence

$$|\mu_1|(V_1\sim V_2) = |\mu_1|(V_1\sim Q) - |\mu_1|(V_2\sim Q) > \frac{t}{2} - \frac{t}{4} = \frac{t}{4}.$$

Now select $\mu_2 \in X^\perp$, $\|\mu_2\| = 1$, such that

$$|\mu_2|(V_2{}^c) < \frac{t}{2^2}.$$

Continuing inductively, construct sequences $\{\mu_n\} \subseteq X^\perp$ and $\{V_n\}$ such that $\mu_n \in X^\perp$, $\|\mu_n\| = 1$, V_n is a closed neighbourhood of Q with

$$V_{n+1} \subseteq W_{n+1} \cap \operatorname{int}(V_n)$$
$$|\mu_n| (V_n^c) < \frac{t}{2^n}$$
$$|\mu_n| (V_n \sim V_{n+1}) > \frac{t}{4}.$$

Now select functions $y_n \in C_o(S)$ (the continuous functions vanishing at infinity on S) such that $\operatorname{supp}(y_n) \subseteq V_n \sim V_{n+1}$, $0 \leq y_n \leq 1$ and

$$|\mu_n (y_n)| > \frac{t}{16}.$$

We use the y_n to define a subspace W of $C(\beta(S))$ isometric to $l_\infty(N)$. If $\{\alpha_n\} \in l_\infty(N)$ define the continuous function **z** on S by

$$z(s) = \sum \alpha_n y_n(s), \qquad s \in S.$$

Then z has a unique continuous extension to $\beta(S)$, since $\bigcap_{n=1}^{\infty} V_n = Q$.

Consider the measures μ_n as linear functionals on W and define elements β_n of $ba(N,\Sigma)$ by

$$\beta_n(E) = \mu_n(\sum_{j \in E} y_j), \qquad E \subseteq N.$$

If we show

$$\lim_{n\to\infty} \mu_n(f) = 0, \qquad f \in W$$

it will follow from Phillips theorem that

$$\lim_{n\to\infty} \beta_n(E) = 0, \qquad E \subseteq N$$

and hence that

$$\lim_{n\to\infty} |\mu_n(y_n)| = \lim_{n\to\infty} |\beta_n(\{n\})|$$
$$\leq \lim_{n\to\infty} \sum_{j=1}^{\infty} |\beta_n(\{j\})| = 0,$$

contradicting the fact that

$$|\mu_n(y_n)| > \frac{t}{16}, \qquad n = 1, 2, \ldots .$$

Let $f \in W$. Since $X|Q = C(Q)$ we may write $f = g + h$, where $g \in C_o(S)$ and $h \in X$. Thus since $\mu_n \in X^\perp$

$$\begin{aligned} |\mu_n(f)| &= |\mu_n(g)| \\ &\leq \left| \int_{V_n^c} g \, d\mu_n \right| + \left| \int_{V_n \sim Q} g \, d\mu_n \right| \\ &\leq \|g\|_\infty |\mu_n(V_n^c)| + \sup_{s \in V_n \sim Q} |g(s)| \\ &\leq \frac{t\|g\|_\infty}{2^n} + \sup_{s \in W_n} |g(s)| \to 0 \end{aligned}$$

and we are done.

Corollary 10.10. If S is locally compact and σ-compact, then there exists no bounded projection of $C(\beta(S))$ onto $C_o(S)$.

SECTION 11

Bibliographic Comments.

In the brief comments below we indicate where various theorems in the text can be found. We have attempted to give references for all results not included in the treatise of Dunford and Schwarts [12]. There has been no attempt to trace the historical background of the theorems.

We wish to call particular attention to the survey articles [34] and [35] of Semadeni, where many references to results on C(S) can be found.

Section 1.

Theorem 1.6 and related results are in Phelps [30]. Theorem 1.8 is from [3]. The proof of 1.12 is taken from [12].

Section 2.

Theorem 2.2 is due to Arens [1]. See also Kelley and Vaught [24]. The classical Theorem 2.13 is from [13] where a different proof is given. Kakutani's Theorem 2.18 was proved in [20]. The present proof follows Kelley and Namioka [23].

Section 3.

The material of this section is mostly taken from Pełczyński's monograph [29], where references to earlier work can be found. For Theorem 3.11 see Dugundji [10] and Borsuk [5]. See also the notes of Semadeni [36].

Section 4.

This material is again taken from Pełczyński [29]. Theorem 4.9 is due to Yoshizawa [39]. References for Theorem 4.13 are Milutin [26] and [27], and Pełczyński [29]. The present proof is due to Ditor [8].

Section 5.

Again the references are [26], [27], and [29]. For Theorem 5.9 see Sobczyk [37].

Section 6.

For results related to this section see Arens [2], Isbell and Semadeni [19] and Pełczyński [29]. The basic Lemma 6.7 and the results which follow from it are due to Ditor [8].

Section 7.

The material through Theorem 7.14 is drawn from Gleason [14] and Rainwater [32]. Corollary 7.15 is due to Grothendieck [17]. Theorem 7.17 is from Stone [38]. Theorem 7.20 covers work of Nachbin [28], Goodner [16], and Kelley [22]. Theorem 7.28 and the remaining results of this section are from Cohen [6]. For a penetrating discussion of the p_λ spaces and further developments see Lindenstrauss [25].

Section 8.

The results are all drawn from Dixmier [9]. See also Sakai [33], and Grothendieck [17].

Section 9.

The material on weak compactness in L_1 is from [12]. The original theorems are due to Dunford and Pettis [11]. Theorem 9.10 is from Grothendieck [18], while 9.12 - 9.14 are again from [12]. Theorem 9.15 is due to Dieudonné [7]. For Theorem 9.16 see Grothendieck.

Section 10.

Phillips' Theorem 10.3 is from [31] as is the application 10.6. Theorem 10.8 is from Glicksberg [15], and Theorem 10.9 is due to Bade [4].

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