Notes on weak convergence (MAT4380 - Spring 2006)

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1 Weak convergence

In what follows, let U denote an open, bounded, smooth subset of $\mathbb{R}^N$ with $N \geq 2$. We assume $1 \leq p < \infty$ and let p' be the conjugate exponent, i.e.,

$$\frac{1}{p} + \frac{1}{p'} = 1$$

($p' := \infty$ when $p = 1$).

A sequence $\{u_n\}_{n\geq 1} \subset L^p(U)$ **converges weakly** to $u \in L^p(U)$, in which case we write

$$u_n \rightharpoonup u \quad \text{in } L^p(U),$$

if

$$\int_U u_n v \, dx \to \int_U uv \, dx, \qquad \forall v \in L^{p'}(U).$$

When $p = \infty$, we say that a sequence $\{u_n\}_{n\geq 1} \subset L^p(U)$ **converges weakly -** $\star$ to $u \in L^\infty(U)$, and we write

$$u_n \overset{\star}{\rightharpoonup} u \quad \text{in } L^\infty(U),$$

if

$$\int_U u_n v \, dx \to \int_U uv \, dx, \qquad \forall v \in L^1(U).$$

We remark that when Ω is bounded the weak - $\star$ convergence of u_n in $L^\infty(\Omega)$ to some $u \in L^\infty(\Omega)$ implies weak convergence of u_n to u in any $L^p(\Omega)$, $1 \leq p < \infty$.

Theorem 1.1 (boundedness of weakly converging sequences). *Suppose* $1 \leq p < \infty$ *and*

$$u_n \rightharpoonup u \quad \text{in } L^p(\Omega) \ (\overset{\star}{\rightharpoonup} \text{ in } L^\infty(\Omega) \text{ if } p = \infty).$$

Then

$$u_n \text{ is bounded in } L^p(\Omega)$$

and

$$\|u\|_{L^p(\Omega)} \leq \liminf_{n\uparrow\infty} \|u_n\|_{L^p(\Omega)}$$

We have the following compactness theorem:

Theorem 1.2 (Weak convergence in L^p). *Suppose $1 < p < \infty$ and the sequence $\{u_n\}_{n\geq 1}$ is bounded in $L^p(U)$. Then there is a subsequence, still denoted by $\{u_n\}_{n\geq 1}$, and a function $u \in L^p(U)$ such that*

$$u_n \rightharpoonup u \quad \text{in } L^p(U).$$

If $p = \infty$, the result still holds with $\rightharpoonup$ replaced by $\overset{\star}{\rightharpoonup}$.

Theorem 1.2 is false for $p = 1$ since $L^1(U)$ is not the dual of $L^\infty(U)$. But a good substitute result exists by regarding $L^1(U)$ as a subset[1] of the space of (signed) Radon measures on U with finite mass, a space which we denote by $\mathcal{M}(U)$, and using the weak - $\star$ topology on $\mathcal{M}(U)$. Let $C_c(U)$ the denote the space of continuous functions on U with compact support. If $\mu \in \mathcal{M}(U)$, then

$$\langle \mu, v \rangle = \int_U v \, d\mu, \qquad \forall v \in C_c(U).$$

Recall that $\mu \in \mathcal{M}(U)$ if and only if

$$\left| \langle \mu, v \rangle \right| \leq C \, \|v\|_{L^\infty(U)}, \qquad \forall v \in C_0(U).$$

We define

$$\|\mu\|_{\mathcal{M}(U)} = \sup \left\{ \left| \langle \mu, v \rangle \right| : v \in C_c(U), \|v\|_{L^\infty(U)} \leq 1 \right\}.$$

The space $\left(\mathcal{M}(U), \|\cdot\|_{\mathcal{M}(U)}\right)$ is a Banach space and it is isometrically isomorphic to the dual space of $\left(C_c(U), \|\cdot\|_{L^\infty(U)}\right)$.

A sequence $\{\mu_n\}_{n\geq 1} \subset \mathcal{M}(U)$ **converges weakly -** $\star$ to $\mu \in \mathcal{M}(U)$, in which case we write

$$\mu_n \overset{\star}{\rightharpoonup} \mu \quad \text{in } \mathcal{M}(U),$$

if

$$\int_U v \, d\mu_n \to \int_U v \, d\mu, \qquad \forall v \in C_c(U).$$

Theorem 1.3. *Suppose*

$$\mu_n \overset{\star}{\rightharpoonup} \mu \quad \text{in } \mathcal{M}(\Omega).$$

Then

$$\limsup_{n\uparrow\infty} \mu_n(K) \leq \mu(K),$$

[1] We recall that each $u \in L^1(U)$ defines a linear functional on $C_c(U)$ via

$$v \mapsto \int_U uv \, dx, \qquad v \in C_c(U).$$

for each compact set $K \subset \Omega$, and

$$\mu(O) \leq \liminf_{n\uparrow\infty} \mu_n(O),$$

for each open set $O \subset \Omega$.

We have the following compactness theorem for measures:

Theorem 1.4 (Weak compactness in $\mathcal{M}(U)$). *Suppose the sequence $\{\mu_n\}_{n\geq 1}$ is bounded in $\mathcal{M}(U)$. Then there is a subsequence, still denoted by $\{\mu_n\}_{n\geq 1}$, and a measure $\mu \in \mathcal{M}(U)$ such that*

$$\mu_n \overset{\star}{\rightharpoonup} \mu \quad \text{in } \mathcal{M}(U).$$

Theorem 1.5 (Characterizations of weak convergence in L^p). *Let $u_n : \Omega \to \mathbb{R}$ be a sequence in $L^p(\Omega)$ and $u \in L^p(\Omega)$, $1 < p < \infty$. Suppose the sequence u_n is equibounded in $L^p(\Omega)$ Then the following statements are equivalent:*

1. *$u_n \rightharpoonup u$ in $L^p(\Omega)$.*
2. *$u_n \overset{\star}{\rightharpoonup} u$ in $\mathcal{M}(\Omega)$.*
3. *$u_n \to u$ in $\mathcal{D}'(\Omega)$.*
4. *For any Borel set $E \subset \Omega$, $|E| > 0$,*

$$(u_n)_E := \frac{1}{|E|}\int_E u_n\,dx \to (u)_E := \frac{1}{|E|}\int_E u\,dx.$$

5. *For any cube $Q \subset \Omega$, $|Q| > 0$,*

$$(u_n)_E := \frac{1}{|E|}\int_E u_n\,dx \to (u)_E := \frac{1}{|E|}\int_E u\,dx.$$

Remark 1.1. *Similar equivalences hold also if $p = \infty$, replacing in (1) weak convergence with weak - $\star$ convergence in $L^\infty(\Omega)$.*

Remark 1.2. *Condition (5) expresses the intuitive idea of weak convergence as convergence of mean values.*

The next lemma is simple but quite useful in a number of situations.

Lemma 1.1 (products of weak-strong converging sequences). *Let $1 < p < \infty$, $u_n : \Omega \to \mathbb{R}$ be a sequence in $L^p(\Omega)$, and $u \in L^p(\Omega)$. Let $v_n : \Omega \to \mathbb{R}$ be a sequence in $L^{p'}(\Omega)$, $\frac{1}{p} + \frac{1}{p'} = 1$ (or $p' = \frac{p}{p-1}$). Suppose*

$$u_n \rightharpoonup u \text{ in } L^p(\Omega),$$
$$v_n \to u \text{ in } L^{p'}(\Omega).$$

Then

$$u_n v_n \rightharpoonup uv \text{ in } L^1(\Omega).$$

2 Some typical behaviors of weakly converging sequences

Relevant for our study of nonlinear partial differential equations and calculus of variation, we will illustrate some typical behaviors of sequences that converge weakly but not strongly.

2.1 Oscillations

Sequences of rapidly oscillating functions provide examples of weakly – but not strongly – converging sequences.

Letting

$$u_n(x) = \sin(nx), \qquad x \in (0, 2\pi), \qquad n = 1, 2, \ldots,$$

one can easily check that

$$u_n \rightharpoonup u := 0 \text{ in } L^p(0,2\pi)\ \forall p \geq 1\ (\overset{\star}{\rightharpoonup} \text{ in } L^\infty(0,2\pi) \text{ if } p = \infty),$$

but

$$\|u_n\|_{L^p(0,2\pi)} = C(p) > 0.$$

Hence u_n does not converge strongly in $L^p(0, 2\pi)$ for any $1 \leq p \leq \infty$. Furthermore, we have

$$\|u\|_{L^p(0,2\pi)} < \liminf_{n\uparrow\infty} \|u_n\|_{L^p(0,2\pi)}.$$

Recall that if $u_n \rightharpoonup u$ in L^p, then by the weak lower semicontinuity of the L^p norms we have always

$$\|u\|_{L^p(0,2\pi)} \leq \liminf_{n\uparrow\infty} \|u_n\|_{L^p(0,2\pi)}. \tag{1}$$

If $u_n \to u$ in L^p, then we have instead (trivially) equality in (1), but be aware that we can have this equality under mere weak convergence, as the the next examples shows.

Let

$$u_n(x) = 1 + \sin(nx), \qquad x \in (0, 2\pi), \qquad n = 1, 2, \ldots,$$

then one can easily check that

$$u_n \rightharpoonup u := 1 \text{ in } L^p(0,2\pi)\ \forall p \geq 1\ (\overset{\star}{\rightharpoonup} \text{ in } L^\infty(0,2\pi) \text{ if } p = \infty)$$

and

$$\begin{cases} \displaystyle\int_0^{2\pi} |u_n|\ dx = \int_0^{2\pi} |u|\ dx = 2\pi, & \forall n, \\ \displaystyle\int_0^{2\pi} |u_n - u|\ dx = \int_0^{2\pi} |\sin(nx)|\ dx = 4, & \forall n, \end{cases}$$

so that

$$u_n \text{ converges weakly – but strongly – to } u \text{ in } L^1(0, 2\pi).$$

On the other hand, we have

$$\int_0^{2\pi} |u|^p \, dx < \liminf_{n\uparrow\infty} |u_n|^p \, dx, \qquad \forall p > 1,$$

since otherwise, passing to a subsequence if necessary, we would have for all $p > 1$

$$u_n \rightharpoonup u \text{ in } L^p(0, 2\pi) \text{ and } \|u_n\|_{L^p(0,2\pi)} \to \|u\|_{L^p(0,2\pi)},$$

which implies (via Brezis-Lieb's refinement of Fatous's lemma)

$$u_n \to u \text{ in } L^p(0, 2\pi).$$

More generally, the following theorem about weak limits of rapidly oscillating periodic functions is well known:

Theorem 2.1. *Let $1 \le p \le \infty$ and u be a Y periodic function in $L^p(Y)$. For $n = 1, 2, \ldots,$ set*

$$u_n(x) := u(nx).$$

Then, if $1 \le p < \infty$, as $n \uparrow \infty$,

$$u_n \rightharpoonup \frac{1}{|Y|} \int_Y u(y) \, dy \ \text{ in } L^p(O) \text{ for any bounded open set } O \subset \mathbb{R}^N.$$

If $p = \infty$, as $n \uparrow \infty$,

$$u_n \overset{\star}{\rightharpoonup} \frac{1}{|Y|} \int_Y u(y) \, dy \ \text{ in } L^\infty(\mathbb{R}^N).$$

2.2 Concentrations

For example, Dirac masses arise by concentration. But concentration phenomena arise also in the context of functions.

Define $u_n : (-1, 1) \to \mathbb{R}$ by

$$u_n(x) = \begin{cases} n, & \text{if } x \in [0, 1/n], \\ 0, & \text{otherwise} \end{cases}$$

Then once can easily check that

$$u_n \overset{\star}{\rightharpoonup} \delta_0 \text{ in } \mathcal{M}(-1, 1). \tag{2}$$

One can also smooth out u_n and (2) still holds. Observe here also that while $u_n \overset{\star}{\rightharpoonup} 0$ in $\mathcal{M}(0, 1)$, we do not have that u_n converges weakly to 0 in $L^1(0, 1)$ (in view of Dunford-Pettis there is a lack of equiintegrability).

Here is another example. Define $u_n : (-1,1) \to \mathbb{R}$ by

$$u_n(x) = \begin{cases} -n, & \text{if } x \in (-1/n, 0), \\ n, & \text{if } x \in (0, +1/n), \\ 0, & \text{otherwise.} \end{cases}$$

Then once can check that

$$u_n \overset{\star}{\rightharpoonup} u := 0 \text{ in } \mathcal{M}(-1,1),$$

but no subsequence of u_n converges weakly in $L^1(-1,1)$ (again there is a lack of equiintegrability). Indeed, for $\varphi = \chi_{(0,1)} \in L^\infty(-1,1)$, we have

$$\int_{-1}^{1} u_n \varphi \, dx = \int_0^1 u_n \, dx = 1, \quad \text{while} \quad \int_{-1}^{1} u\varphi \, dx = 0.$$

In other words, the sequence is not precompact in weak topology of $L^1(-1,1)$, although

$$\int_{-1}^{1} |u_n| \, dx = 2 \quad \forall n,$$

i.e., the sequence u_n is uniformly bounded (equibounded) in $L^1(-1,1)$. Observe also that there is a "*loss of energy*":

$$0 = \|u\|_{L^1(-1,1)} < \liminf_{n\uparrow\infty} \|u_n\|_{L^1(-1,1)} = 2.$$

Here the energy disappears as a measure. Finally, note that $\|u_n\|_{L^p(-1,1)} \to \infty$ for any $p > 1$.

Of course, concentration phenomena show up also in L^p spaces with $p > 1$. For $p > 1$, define $u_n : (-1,1) \to \mathbb{R}$ by

$$u_n(x) = \begin{cases} n^{\frac{1}{p}}, & \text{if } x \in [0, 1/n], \\ 0, & \text{otherwise.} \end{cases}$$

Then we have

$$\|u_n\|_{L^p(-1,1)} = 1 \text{ for all } n \text{ and } u_n \rightharpoonup 0 \text{ in } L^p(-1,1),$$

but u_n does not converge strongly in $L^p(-1,1)$ since the sequence $\|u_n\|_{L^p(-1,1)}$ concentrates. But we have

$$u_n \rightharpoonup 0 \text{ in } L^q(-1,1) \text{ for any } 1 \le q < p.$$

Indeed, we have

$$\int_{-1}^{1} |u_n|^q \, dx = n^{\frac{q}{p}-1} \to \quad \text{as } \frac{q}{p} < 1.$$

Hence the sequence the sequence $\|u_n\|_{L^p(-1,1)}$ does not concentrate.

2.3 Nonlinearity destroys weak convergence

If u_n converges weakly – but not strongly – to u, then for a generic nonlinear function the sequence $f(u_n)$ does NOT converge weakly to $f(u)$. Here is an example. Let

$$u_n(x) = \sin(nx), \qquad x \in (0, 2\pi), \qquad f(\xi) = \xi^2 \text{ for } \xi \in \mathbb{R}.$$

Then

$$u_n \rightharpoonup u := 0 \text{ in } L^p(0, 2\pi)\ \forall p \geq 1\ (\overset{\star}{\rightharpoonup} \text{ in } L^\infty(0, 2\pi) \text{ if } p = \infty),$$

but

$$f(u_n) = \sin^2(nx) = \frac{1}{2}(1 - \cos(2nx)) \rightharpoonup \frac{1}{2} \neq f(u) = 0.$$

Observe here also that $f(u_n)$ is uniformly bounded (equibounded) in $L^\infty(0, 2\pi)$.

Another example is

$$f(\xi) = \max(0, \xi), \qquad \xi \in \mathbb{R}.$$

Then

$$f(u_n) \rightharpoonup \frac{1}{\pi} \neq f(u) = 0 \text{ in } L^1(0, 2\pi).$$

If

$$f(\xi) = |\xi|, \qquad \xi \in \mathbb{R},$$

then

$$f(u_n) \rightharpoonup \frac{2}{\pi} \neq f(u) = 0 \text{ in } L^1(0, 2\pi).$$

Here is a final example. Let f be any continuous (nonlinear) function and select two number $a < b$ such that

$$f\left(\frac{a+b}{2}\right) \neq \frac{f(a) + f(b)}{2}.$$

Define a sequence of functions $u_n : (0, 1) \to \mathbb{R}$ by

$$u_n(x) = \begin{cases} a, & \text{if } x \in \left[i/n, (i + \frac{1}{2})/n\right],\ i =, 0, \ldots, n-1, \\ b, & \text{otherwise.} \end{cases}$$

Then one can easily check that

$$u_n \overset{\star}{\rightharpoonup} u := \frac{a+b}{2} \text{ in } L^\infty(0, 1) \text{ and hence weakly in any } L^p(0, 1),\ 1 \leq p < \infty,$$

but

$$f(u_n) \overset{\star}{\rightharpoonup} \overline{f} := \frac{f(a) + f(b)}{2} \neq f(u) = f\left(\frac{a+b}{2}\right).$$

3 Weak compactness in L^1

Let us now turn to the more delicate issue of weak compactness in L^1. We start with a definition.

Definition 3.1 (equiintegrability). *Let $\Omega \subset \mathbb{R}^N$ and $\mathcal{U} \subset L^1(\Omega)$ be a family of integrable functions. We say that $\mathcal{U}$ is an equiintegrable family if the following two conditions hold:*

1. *For any $\varepsilon > 0$ there exists a measurable set A with $|A| < \infty$ such that*
$$\int_{\Omega \setminus A} |u| < \varepsilon,$$
for all $u \in \mathcal{U}$. (This condition is trivially satisfied if $|\Omega| < \infty$, since then we can take $A = \Omega$.)

2. *For any $\varepsilon > 0$ there exists $\delta > 0$ such that for every measurable set E with $|E| < \delta$ there holds*
$$\int_E |u| < \varepsilon$$
for all $u \in \mathcal{U}$.

We have the following three equivalent formulations of the equiintegrability property.

Lemma 3.1. *Let $\Omega \subset \mathbb{R}^N$ and $\mathcal{U} \subset L^1(\Omega)$ be a family of integrable functions.*

1. *Then $\mathcal{U}$ is equiintegrable if and only if for any sequence of measurable sets E_n with $E_n \downarrow \emptyset$ there holds*
$$\lim_{n\uparrow\infty} \sup_{u\in\mathcal{U}} \int_{E_n} |u| \; dx = 0.$$

2. *If $|\Omega| < \infty$ and $\mathcal{U}$ is bounded in $L^1(\Omega)$, then $\mathcal{U}$ is equiintegrable if and only if*
$$\mathcal{U} \subset \left\{ u \in L^1(\Omega) : \int_\Omega \Psi(|u|)\, dx \le 1 \right\},$$
for some increasing function $\Psi : [0,\infty) \to [0,\infty]$ satisfying
$$\lim_{\xi\uparrow\infty} \frac{\Psi(\xi)}{\xi} \to \infty.$$

3. *If $|\Omega| < \infty$ and $\mathcal{U}$ is bounded in $L^1(\Omega)$, then $\mathcal{U}$ is equiintegrable if and only if*
$$\lim_{\xi\uparrow\infty} \sup_{u\in\mathcal{U}} \int_{\{|u|>\xi\}} |u| \; dx = 0.$$

Here is a restatement of (2). If $|\Omega| < \infty$ and $\mathcal{U}$ is bounded in $L^1(\Omega)$, then the family $\mathcal{U}$ is equiintegrable if and only if

$$\sup_{u \in \mathcal{U}} \int_\Omega \Psi(|u|)\, dx < \infty,$$

for some increasing function $\Psi : [0, \infty) \to [0, \infty]$ satisfying

$$\lim_{\xi \uparrow \infty} \frac{\Psi(\xi)}{\xi} \to \infty.$$

Let us give an example illustrating how to use this restatement in the context of sequences of functions. Let $|\Omega| < \infty$ and $u_n : \Omega \to \mathbb{R}$ be a sequence of functions that are equibounded (uniformly bounded) in $L^1(\Omega)$, i.e.,

$$\int_\Omega |u_n|\ dx \le C, \qquad \forall n.$$

Then a sufficient condition for the sequence u_n to be equiintegrable is that there exists a constant C, independent of n, such that

$$\int_\Omega |u_n|^{1+\theta}\ dx \le C,$$

for some $\theta > 0$.

The next theorem gives a necessary and sufficient condition (namely equiboundedness and equiintegrability) for compactness with respect to the weak convergence in L^1.

Theorem 3.1 (Dunford-Pettis). *Let $u_n : \Omega \to \mathbb{R}$ be a sequence in $L^1(\Omega)$. Suppose*

1. *the sequence u_n is equibounded in $L^1(\Omega)$, i.e.,*

$$\sup_n \|u_n\|_{L^1(\Omega)} < \infty,$$

2. *the sequence u_n is equiintegrable.*

Then there exists a subsequence of u_n that converges weakly in $L^1(\Omega)$. Conversely, if u_n converges weakly in $L^1(\Omega)$, then 1. and 2. hold.

The following theorem is the L^1 analog of Theorem 1.5.

Theorem 3.2 (Characterizations of weak convergence in L^1). *Let $u_n : \Omega \to \mathbb{R}$ be a sequence in $L^1(\Omega)$ and $u \in L^1(\Omega)$. Suppose the sequence u_n is equibounded in $L^1(\Omega)$ and equiintegrable. Then the following statements are equivalent:*

1. *$u_n \rightharpoonup u$ in $L^1(\Omega)$.*

2. $u_n \overset{\star}{\rightharpoonup} u$ *in* $\mathcal{M}(\Omega)$.

3. $u_n \to u$ *in* $\mathcal{D}'(\Omega)$.

4. *For any Borel set* $E \subset \Omega$, $|E| > 0$,

$$(u_n)_E \to (u)_E.$$

5. *For any cube* $Q \subset \Omega$, $|Q| > 0$,

$$(u_n)_Q \to (u)_Q.$$

The next lemma is simple but still very useful.

Lemma 3.2 (products of weak-strong converging sequences). *Let* $u_n, u, v_n, v : \Omega \to \mathbb{R}$ *be measurable functions.*

1. *If* $u_n \to u$ *a.e. in* Ω, $\|u_n\|_{L^\infty(\Omega)} \leq C$ *for all* n, *and* $v_n \rightharpoonup v$ *in* $L^1(\Omega)$, *then*

$$u_n v_n \rightharpoonup uv \text{ in } L^1(\Omega).$$

2. *If* $u_n \to u$ *in* $L^1(\Omega)$, $\|u_n\|_{L^\infty(\Omega)} \leq C$ *for all* n, *and* $v_n \rightharpoonup v$ *in* $L^1(\Omega)$, *then*

$$u_n v_n \rightharpoonup uv \text{ in } L^1(\Omega).$$

The next lemma is useful in many applications.

Lemma 3.3 (Vitali). *Let* $\Omega \subset \mathbb{R}^N$ *be bounded and* $u_n : \Omega \to \mathbb{R}$ *be a sequence in* $L^1(\Omega)$. *Suppose*

1. $\lim_{n \uparrow \infty} u_n(x)$ *exists and is finite for a.e.* $x \in \Omega$,

2. *the sequence* u_n *is equiintegrable.*

Then

$$\lim_{n \uparrow \infty} \int_\Omega u_n(x)\, dx = \int_\Omega \lim_{n \uparrow \infty} u_n(x)\, dx.$$

A typical application of Vitali's lemma is provided by the next simple lemma.

Lemma 3.4. *Let* $\Omega \subset \mathbb{R}^N$ *be bounded and* $u_n : \Omega \to \mathbb{R}$ *be a sequence in* $L^1(\Omega)$. *Suppose*

1. $u_n \to u$ *a.e. in* Ω,

2. *the sequence* u_n *is bounded in* $L^p(\Omega)$ *for some* $p > 1$.

Then

$$u_n \to u \text{ in } L^r(\Omega) \text{ for all } 1 \leq r < p.$$

Proof. By a previous theorem,

$$u \in L^p(\Omega) \text{ (and also } u_n \rightharpoonup u \text{ in } L^p(\Omega)).$$

Define

$$v_n = |u - u_n|^r, \qquad r < p.$$

Then

$$v_n \to 0 \text{ a.e. in } \Omega$$

and

$$v_n \text{ is bounded in } L^{\frac{p}{r}}(\Omega) \text{ and } p/r > 1.$$

Hence the sequence v_n is equiintegrable, so that by Vitali's lemma

$$\lim_{n\uparrow\infty} \int v_n \, dx = 0,$$

that is $u_n \to u$ in $L^r(\Omega)$. □

The next lemma recall the well known fact that convex (concave) functions are lower (upper) semicontinuous with respect to the weak convergence.

Lemma 3.5 (weak lower semicontinuity of convex functions). *If $F : \mathbb{R} \to \mathbb{R}$ is convex and*

$$u_n \rightharpoonup u \quad \text{in } L^1,$$

then

$$\int F(u) \, dx \le \liminf_{n\uparrow\infty} \int F(u_n) \, dx.$$

If $F : \mathbb{R} \to \mathbb{R}$ is concave and

$$u_n \rightharpoonup u \quad \text{in } L^1,$$

then

$$\int F(u) \, dx \ge \limsup_{n\uparrow\infty} \int F(u_n) \, dx.$$

4 Additional reading - collection of some results

Definition 4.1 (convergence in measure). *Let $u_n, u : \Omega \to \mathbb{R}$ be measurable functions. We say that*

$$u_n \to u \text{ in measure}$$

if

$$\lim_{n\uparrow\infty} |\{x \in \Omega : |u_n(x) - u(x)| > \varepsilon\}| = 0, \qquad \forall \varepsilon > 0.$$

The a.e. convergence and convergence in measure can be easily compared. Indeed, the following statements hold:

1. If $|\Omega| < \infty$ and $u_n \to u$ a.e., then $u_n \to u$ in measure.
2. If $u_n \to u$ in measure, then a subsequence of u_n converges a.e. to u.

We also remark that if

$$u_n \to u \text{ in } L^p(\Omega),$$

then a subsequence of u_n converges a.e. to u. This is trivial for $p = \infty$, and follows for $1 \le p < \infty$ by Chebyshev's inequality, stating that

$$|\{x \in \Omega : g(x) > \varepsilon\}| \le \frac{1}{\varepsilon} \int_\Omega g \, dx, \qquad \forall \varepsilon > 0, \quad 0 \le g \in L^1(\Omega),$$

which implies immediately that

$$u_n \to u \text{ in measure},$$

and hence (2) applies.

The following lemma exploits convergence of the L^p norm to get strong convergence from a.e. convergence.

Lemma 4.1. *Let $u_n : \Omega \to \mathbb{R}$ be a sequence in $L^p(\Omega)$, $1 \le p < \infty$, and suppose*

1. *$u_n \to u$ a.e. in Ω,*
2. *$\|u_n\|_{L^p(\Omega)} \to \|u_n\|_{L^p(\Omega)}$.*

Then

$$u_n \to u \text{ in } L^p(\Omega).$$

Let us now look at a refinement of Fatou's lemma. Suppose

$$\begin{cases} u_n \rightharpoonup u \text{ in } L^p(\Omega), \\ u_n \to u \text{ a.e. in } \Omega. \end{cases} \tag{3}$$

As explained before, concentration phenomena will arise when we have the weak convergence in (3) simultaneously with the a.e. convergence in (3). From the a.e. convergence and Fatou's lemma,

$$\|u\|_{L^p(\Omega)} \le \liminf_{n \uparrow \infty} \|u_n\|_{L^p(\Omega)} .$$

But we know already this from the weak convergence. Brezis and Lieb have analyzed this situation more carefully and proved the following sharpened assertion:

Theorem 4.1 (Brezis-Lieb). *Suppose* (3) *holds with* $1 \leq p < \infty$*. Then*

$$\lim_{n\uparrow\infty}\left(\|u_n\|^p_{L^p(\Omega)} - \|u_n - u\|^p_{L^p(\Omega)}\right) = \|u\|_{L^p(\Omega)}\,.$$

The message is that u_n in the limit (as measured in the L^p norm) decouples into $u_n - u$ and u. Note that the case $p = 2$ is immediate and does need the a.e. convergence in (3). Indeed, if

$$u_n \rightharpoonup u \text{ in } L^2(\Omega),$$

then we have

$$\begin{aligned}
&\lim_{n\uparrow\infty}\left(\int_\Omega |u_n|^2\,dx - \int_\Omega |u_n - u|^2\,dx\right)\\
&= \lim_{n\uparrow\infty}\left(\int_\Omega |u_n|^2 - |u_n|^2 + 2u_n u - |u|^2\,dx\right)\\
&= \lim_{n\uparrow\infty}\left(\int_\Omega 2u_n u - |u|^2\,dx\right) = \int_\Omega |u|^2\,dx.
\end{aligned}$$

If $|\Omega| < \infty$, then a.e. convergence is equivalent to uniform convergence, up to arbitrarily small sets. This is the content of Egoroff's theorem.

Theorem 4.2 (Egoroff). *Suppose* $|\Omega| < \infty$ *and that a sequence of measurable functions* $u_n : \Omega \to \mathbb{R}$ *converges to* u *a.e. in* Ω*. Then for any* $\varepsilon > 0$ *there exists a measurable set* Ω_ε *such that* $|\Omega \setminus \Omega_\varepsilon| < \varepsilon$ *and*

$$u_n \to u \textit{ uniformly in on } \Omega_\varepsilon.$$

We recall next some deeper results showing that under additional assumptions the convergence of a sequence in L^p can be improved. The first result says that convergence a.e. or in measure implies weak convergence if the norms are uniformly bounded.

Theorem 4.3. *Let* $u_n : \Omega \to \mathbb{R}$ *be a sequence in* $L^p(\Omega)$*,* $1 < p < \infty$*, converging a.e. or in measure to* u*, with*

$$\|u_n\|_{L^p(\Omega)} \leq C, \qquad \forall n.$$

Then $u \in L^p(\Omega)$ *and*

$$u_n \rightharpoonup u \textit{ in } L^p(\Omega).$$

Using the uniform convexity of the L^p spaces, $1 < p < \infty$, it is not hard to prove the following result:

Theorem 4.4 (Radon-Riesz). *With* $1 < p < \infty$*, let* $u_n : \Omega \to \mathbb{R}$ *be a sequence in* $L^p(\Omega)$ *converging weakly to* $u \in L^p(\Omega)$ *and*

$$\|u_n\|_{L^p(\Omega)} \to \|u\|_{L^p(\Omega)}\,.$$

Then

$$u_n \to u \textit{ in } L^p(\Omega).$$

This theorem shows that if a weakly converging sequence in L^p, $1 < p < \infty$, does not converge strongly in L^p, then there is must be a loss in the p - norm energy.

As a corollary of the previous theorem or by a direct proof, here is a weaker form of the Radon-Riesz theorem.

Corollary 4.1. *Let $u_n : \Omega \to \mathbb{R}$ be a sequence in $L^p(\Omega)$ and $u \in L^p(\Omega)$, $1 < p < \infty$. Suppose*

$$u_n \to u \text{ a.e. in } \Omega,$$
$$\|u_n\|_{L^p(\Omega)} \to \|u\|_{L^p(\Omega)}.$$

Then

$$u_n \to u \text{ in } L^p(\Omega).$$

Final remarks

A large part of this section is based on Evans' lecture notes [1], see also Evans' book [2].

References

[1] L. C. Evans. *Weak convergence methods for nonlinear partial differential equations*, volume 74 of *CBMS Regional Conference Series in Mathematics*. Published for the Conference Board of the Mathematical Sciences, Washington, DC, 1990.

[2] L. C. Evans. *Partial differential equations*. American Mathematical Society, Providence, RI, 1998.